UNDERSTANDING ELEMENTARY MATHEMATICS

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Understanding Elementary Mathematics

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About this Book

Understanding Elementary Mathematics, a series of hands-on Workbook Modules, was written especially for prospective elementary school teachers in a way that they would actively learn the structures, concepts and foundations of elementary mathematics.

The format of the book is different from most college textbooks. The pages are not bound, but instead are on loose-leaf standard-sized paper, and is 3-hole punched. Each module is self-contained, and can easily be placed into a standard 3-pronged portfolio folder, preferably with pockets on each side. Each module has a front and back cover on colored paper, and is labeled as Module 1, Module 2, etc. Within each module, there is a table of contents, exercise sets, review exercises and solutions. A typical page number looks like this: 4-33— the number before the dash is the module number; the number after the dash is the page number of that module. So, 4-33 is on page 33 of Module 4.

The text was written informally, in order to be user-friendly and easy to read and understand. There is not a lot of text, and then a lot of exercises, as in traditional textbooks. It is all mixed together with explanation, exploration, examples, exercises, etc. It was written for students to learn in a small collaborative group setting, where students read and work together, helping each other to learn the material. Ideally, the instructor would not necessarily lecture, but would act more as a facilitator, and be available to clarify ideas. Although the intention is that there would be time to work on some exercises in class, most of the reading and work will have to be done out of class, due to the fact that most will take this as a college course, with minimal amount of in-class instructional time.

As previously stated, it is preferable if students work together in a small group setting both in and out of class as they read through each module, and work through the exercises. Because it may take a few hours to complete each exercise set, it isn't always possible for group members to do all of each set together. Therefore, although some exercises refer to discussing concepts with others in a group, an individual can work on the exercises alone as well. This material can also be learned in a self-paced or individual format if the student is highly motivated and has a strong background in both mathematics and English.

Each module is self-contained, and there is space left in the booklet to do all of the work for each module. Ideally, all of the work should be done and shown in the workbook. The advantage to putting each module in a portfolio folder in an unbound format is that extra pages can easily be inserted. Almost all of the solutions to all of the exercises are included.

A separate packet of *Material Cards*, containing manipulatives needs to be purchased. Students use the material cards to make their own set of manipulatives in order to do hands-on activities found in most of the exercise sets. Manipulatives should be prepared ahead of time, with each type of manipulative kept in its own baggie. The manipulatives to be used in a given exercise set are listed at the beginning of that set. Students should also have handy some extra paper, a calculator, a small stapler, scissors, tape, colored pencils, pens, markers, highlighters and/or crayons.

Good Luck on your journey to really Understanding Elementary Mathematics!

Julie Harland

Please call or write me if you have any comments, suggestions or corrections to share. I can be reached at 760-757-2121 ext. 6387; e-mail: jharland@miracosta.edu

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Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing****.*

Acknowledgments

I especially want to warmly thank and acknowledge **Julian Weissglass**, University of California, Santa Barbara. It was his book, *Exploring Elementary Mathematics: A Small-Group Approach for Teaching*, that inspired me to teach using the small-group approach, and to eventually write my own book. He was a pioneer of the collaborative approach to teaching mathematics to prospective elementary school teachers. In his approach, students worked cooperatively as they learned together in small groups using manipulatives to understand the concepts. Unlike most mathematics majors, I took the two course mathematics sequence designed for prospective elementary teachers (because it looked fun and interesting — not because it was a requirement). In fact, I used Dr. Weissglass's book when I took the classes at U.C.S.B. in the late 1970's as an undergraduate. Many of the ideas and exercises in my modules came about from the ones presented in his book. He has been gracious in encouraging me to pursue writing my own modules. I owe him a debt of gratitude.

I gratefully acknowledge the MiraCosta College Governing Board for granting the sabbatical leave that allowed me to begin the long term project of writing this book. I also thank my many colleagues who supported my decision to take some time off from teaching in order to work on this book.

Many students over the years encouraged (even begged) me to write my own book — and now that I have begun, they have continuously asked me to finish it! I feel particularly indebted to those students who help me improve this book as they inadvertently edit it by finding typos and other mistakes, and give suggestions for improvement. Their patience in being amongst the first to use it, in its not so polished form, is greatly appreciated!

Dedication

This book

is lovingly dedicated

to

John Browne and Barbara Browne

and

John Harland

My parents, John and Barbara Browne, have always been there for me. They were my first teachers. They have supported me and stood behind me in all of my endeavors. They taught me that I could do anything and they always believed I really could. Their faith in me has been unfailing. I have the deepest gratitude, love and affection for them and thank them for these most important gifts they have given me. Thank you, Mom and Dad. I love you.

My husband, John, is the wind beneath my wings. He has patiently helped me as I have worked on this book and gives me overwhelming support and love as I strive to complete this project. He inspires and encourages me. I have truly been blessed to have found my soul mate and companion for life. Thank you, John. I will always love you.

I also want to thank my three adorable children, Jakob, Ian and Kira, for being the lights of my life. I love you so very much.

Preface

A Big Thank You to many of my Former Teachers!

I want to thank some of my own mathematics teachers (and one special English teacher) I've had over the years. In elementary school, I was not inspired by the mathematics curriculum nor the way arithmetic was taught (although I did like flash cards). I went to junior high and high school in Vista, California. It was when I finally got into junior high school that I began to enjoy mathematics. I was placed in the B math class in seventh grade. Mr. **Malcolm Blakeslee** was my 7th grade instructor, and I was happy to start earning A's. In eighth grade, I had to practically beg the administration to move me from the A- class into the A math class. I wanted to be in with the "smarter" kids. Mrs. **Marie McElliott** was a tough, demanding, dedicated teacher. There was no fooling around, and I was lucky to earn a B. In ninth grade, I was placed in Algebra I, and that is when math really got fun. I loved Algebra, and Mr. **Ted Norris** was not only nice, young and good-looking, but let us do challenging extra credit problems if we finished our tests early. Although he wasn't a mathematics teacher, Mr. **George Calhoun** was a truly inspiring teacher, and I learned most of what I know about English and the humanities from him. He was my ninth grade English teacher. I want to acknowledge and thank him because he was a real pioneer. In my sophomore year, I took Plane/Solid Geometry with Mr. John **(Pat) Murphy**. He was a forward-looking teacher who let us sit where we wanted, and preferred that we call him Pat. That was hard for me to get used to, but I liked it. The class was hard, but I made it through and ended up being a T.A. for him as a senior. In eleventh grade, I took Algebra II/Trig with Mrs. **Polly Willard**. She was a great teacher. I really liked the class, she had perfect control, and always showed respect for her students. She was matter-of-fact about everything, and I liked her style even though she was strict. Mrs. **Eunice Fraley** taught Math Analysis/Analytical Geometry, and Physics my senior year. There were twenty-one of us for both classes (the first two periods each day) and we had a fantastic year. We appreciated her and she enjoyed our class — we actually took a class picture and she hosted a party at her house for us. I was an instructional aide for her my first semester of college. By the end of high school, I knew I wanted to be a mathematics teacher, and I credit that decision to having good math teachers in junior and senior high school.

As a mathematics major at U.C. Santa Barbara, I had the good fortune of taking geometry from a fantastic professor, Mr. **Paul Kelly**. I spent many hours in his office, where he gave me a lot of encouragement and helped me really understand how to do proofs. I took everything I could from him. After receiving my B.A. in Mathematics at U.C.S.B., I entered the secondary teaching credential program there and worked under Dr. **Vern Cotter**. He pretty much left me alone to do my own thing in the classroom, which was wonderful. I was confident, competent and energetic, and happy to be spending time doing what I had always wanted to do since fourth grade — be a teacher. It was during my tenure as a student teacher that I began rewriting a textbook I was using for the Algebra II class I was teaching. Dr. Cotter encouraged me to write my own independent materials in the future. After earning my teaching credential to teach mathematics at the secondary level, I taught at my Alma Mater, Vista High School, for two years, alongside some of my 9th-12th grade teachers. I then went to graduate school at U.C. San Diego, where I was fortunate enough to get to know the professor I credit for getting me through grad school, **Gill Williamson**. Because of him, I was introduced to the world of Combinatorics. It was in this class that I excelled, and learned what a great experience it is to put in a lot of effort, then succeed, and be at the top of the class. I saved the paper where he did the curve for the final grades — I was practically off the charts. What a great sense of accomplishment I felt. He was supportive throughout my two years at U.C.S.D., and he spent many hours counseling and encouraging me. I took every class and seminar I could from him. I also taught a section for his undergraduate combinatorics class. At the graduation ceremony, he handed me my diploma. I owe him a lot!

I want to thank and acknowledge all of these teachers, who helped me along in my journey to becoming a mathematics teacher. My teaching has been shaped by their influences. I appreciate them and hope they know they made a difference. That, I believe, is what teaching is all about.

CHAPTER OVERVIEW

1: Set Theory

The Manipulatives listed below are to be used with this workbook. You will be using the material cards to make your own manipulatives in order to do hands on activities in some of the exercise sets. You should prepare them ahead of time, and keep each type of manipulative in its own baggie. The manipulatives to be used in an exercise set are listed at the beginning of that set. You should also have handy a calculator, scissors, tape, colored pencils, pens, markers, highlighters and/or crayons.

Everything you always wanted to know about sets but were afraid to ask . . .

My mom and best girlfriend, Barbara Browne, taught me to ask questions. She taught me not to be afraid to ask what, how or why. I learned to be inquisitive and to form my own beliefs. This carries over to my philosophy about learning and understanding mathematics. It's not enough to believe something simply because a teacher or parent tells you it's true. You must be able to prove to yourself the validity of any argument. You must yourself be convinced before you can pass on the ideas to others. You must really understand and own the concepts. That is why I believe in a hands–on approach to learning by discovering the concepts through investigation and exploration. Mom gave me the foundation that started me on my own journey of learning.

Thank you, Mom.

1.1: The Basics of Sets
1.2: Venn Diagrams
1.3: More about Sets
1.4: Survey Problems
1.5: Homework

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1.1: The Basics of Sets

You will need: ***Coins*** *(Material Card 1),* ***A–Blocks*** *(Material Cards 2A–2E)*

? Exercise 1

Have you ever collected anything? If so, what?

? Exercise 2

Name some things people might collect, including anything people in your group collect.

? Exercise 3

Suppose someone collected 25 coins from the years 1964 –1969.

a. What different types of coins might be in their collection?

b. Name one way you might sort the coins into groups.

c. Can you think of a different way to sort them into groups? Explain.

d. How do you think a child would sort them?

Let's assume these are the 25 coins that were collected:

1966 penny, 1967 nickel, 1966 quarter, 1967 penny, 1965 penny, 1966 half dollar, 1967 quarter, 1965 dime, 1967 dime, 1968 quarter, 1964 dime, 1966 nickel, 1965 nickel, 1967 half dollar, 1966 dime, 1964 nickel, 1969 quarter, 1969 half dollar, 1965 half dollar, 1968 penny, 1968 dime, 1964 quarter, 1965 quarter, 1969 dime, 1968 nickel

To simplify writing each coin out, let's abbreviate 1966 penny by $6P$, and 1967 nickel by $7N$, etc. So in our collection, we have the following:

$6P, 7N, 6Q, 7P, 5P, 6H, 7Q, 5D, 7D, 8Q, 4D, 6N, 5N, 7H, 6D, 4N, 9Q, 9H, 5H, 8P, 8D, 4Q, 5Q, 9D \text{ and } 8N$

A physical model for these coins is found on Material Card 1. If you haven't already done so, cut out a set of coins from this Material Card and use them to do several of the following exercises.

? Exercise 4

Sort the coins into five groups; pennies, nickels, dimes, quarters and half dollars. Using the abbreviations on each coin, list the coins that go into each group (such as $6P, 7N, 6Q, 7P, 5P$, etc. each coin must be listed with **a number *and* a letter**. You can't just write a letter or a number, like 6 or 5 or P. Each coin is made up of one of each! It's like writing the initials of a first and last name instead of the full name.)

a. pennies: ___ (did you list 4 pennies?)

b. dimes: ___ (did you list 6 dimes?)

c. half dollars: ___ (did you list 4 of them?)

d. Now sort the coins into groups by year. How many groups are there? List two different groups and which coins go into each group. Remember to list each coin individually each of the 25 coins has a digit followed by a letter.

The collection of coins we have been working with is called a ***set***. A ***set*** is simply a collection of objects. That is the definition of a set (without the word "simply").

Exercise 5

Define ***set***: A ***set*** is

In exercises e and f, you grouped the ***elements*** or ***members*** of the set into *different* ***subsets***. The ***elements*** of a set are the objects in the set. In this case, the ***elements*** of the set are the 25 different coins.

Exercise 6

Define ***elements***: The ***elements*** of a set are

A ***subset*** is also a set. When we talk about ***subsets***, it is in reference to another set. For example, the set of pennies in the collection above is a ***subset*** of the collection of coins.

The following is a formal definition of ***subset***:

A set ***B*** is a ***subset*** of a set ***C*** if every element of set ***B*** is an element of set ***C***.

*To show that one set is **not** a subset of another set, **there would have to be an element in the first set that is** **not** **in the second set.***

There is one thing about sets that confuses a lot of people. By definition, we say that a set is a collection of objects. But in reality, a set can have no elements in it. A set containing no elements is called the ***empty set*** or ***null set***. Look at the definition of a subset very carefully. Is the null set is a subset of every set?

I hope this was your conclusion: ***The null set is a subset of every set.***

Exercise 7

Define ***null set*** : ___

Note that in my definition of a subset on the previous page, I used the letters B and C for my sets. I did not have to use these letters. They are just dummy variables much like when we use x, y and n in algebra!! Here is another perfectly fine way to define subset: A set ***R*** is a ***subset*** of a set ***Q*** if every element of set ***R*** is an element of set ***Q***.

Exercise 8

Use different letters than just used to define ***subset***:

We usually use a lot of abbreviations in math so that it isn't so cumbersome writing everything out. Many times, we pick a capital letter to represent a set or subset and *define* the set by that letter. I will define some sets now: The set of 25 coins we've been working with will be defined as **C**. The subsets of pennies, nickels, dimes, quarters and half dollars will be denoted by **P**, **N**, **D**, **Q** and **H**, respectively. Notice I gave them meaningful letters so it is easy to remember which letter stands for which subset.

One way to describe a set is to list its objects, if possible. (Later, we will discuss why this is not always practical or possible.) When we list the objects, we do so *inside of braces using commas to separate the elements*. The order in which the elements are listed is irrelevant. Using the listing method, one way to write our coin collection, set **C**, is

$$\mathbf{C} = \{6P, 7N, 6Q, 7P, 5P, 6H, 7Q, 5D, 7D, 8Q, 4D, 6N, 5N, 7H, 6D, 4N, 9Q, 9H, 5H, 8P, 8D, 4Q, 5Q, 9D, 8N\}$$

Exercise 9

Let **S** be the subset of coins from 1964, **V** from 1965, **W** from 1966, **X** from 1967, **Y** from 1968, **Z** from 1969 and **T** from 1970. Are **P**, **N**, **D**, **Q**, **H**, **S**, **V**, **W**, **X**, **Y**, **Z** and **T** all subsets of **C**? ___ Using the correct notation, write out these 12 sets using the listing method by listing the elements in each set. *The only elements in **C** are those listed above this exercise. Note that 6P is one element in **P** because 6P is a particular penny in **C** but 6 is not an element and P is not an element.* The difference between how you write the answers here and how you did it in exercise 4 is that you are using proper notation with the braces, and commas in between the elements.

N = ___	W = ___
Q = ___	Y = ___
S = ___	T = ___

The way to express that an element, like 4D, is in set **C** is with the symbol which means “is an element of”. We write: 4D (read "4D is an element of **C**").

Let **A** represent the one dollar coins in our set **C**. Notice that set **A** is empty. We can show that **A** contains no elements by writing **A = {∅}**. The null or empty set is also written ∅. So, we can also write **A** = ∅. In fact, **{ }** = ∅ always. It is correct to write the null or empty set either way. But note that it is ***not correct*** to write the null set like this: {∅}. That would be a set that ***does*** contain at least one element the null set! See how this can be a bit confusing?

Now to some questions about subsets...

Exercise 10

a. Is there *any* element in **N** that is not in **D**? ___ If so, name *one*: ___

b. From your work in part a, answer this question: Is **N** a subset of **D**? ___

c. Is there *any* element in **A** that is not in **C**? ___ If so, name *one*: ___

d. From your work in part a, answer this question: Is **A** a subset of **C** ? ___

The symbol ⊆ means "is a subset of" and so the way to express that **P** is a **subset** of **C** is by writing . Since **N** is also a subset of **C**, we can write **N** ⊆ **C**. If the first set has **fewer** elements than the second set, then the first set is a **proper subset** of the second set. In this case, the symbol ⊆ may be used in place of ⊆. So, in the case of the coins, it is also correct to write P ⊂ C and N ⊂ C. A first set is **not** a subset of a second set if there is *at least one element in the first set* that is *not in the second set*. Think again about why the null set has to be a subset of every set. For it not to be a subset, there would have to be *at least one element in it* that wasn't in the second set. But, the null set is empty!

Exercise 11

P is not a subset of **N**. To show that **P** is not a subset of **N**, we must name at least one element in **P** that is not in **N**. Name an element in **P** that is not in **N**: ___

Think about and discuss the subtle differences between the following statements: $6P \in C$, $\{6P\} \subset C$, $D \subseteq D$ and $P \subset C$ are correct whereas $6P \subset C$, $\{6P\} \in C$, $D \subset D$ and $P \in C$ are **not** correct!

Exercise 12

Decide which of the following statements are true and which ones are false:

a. _______ $\varnothing \subset A$	b. _______ $\varnothing \subseteq \varnothing$	c. _______ $S \subseteq C$
d. _______ $C \subseteq C$	e. _______ $B \subset B.$	f. _______ $7P \in P$
g. _______ $T \subseteq \varnothing$	h. _______ $9D \in Z.$	i. _______ $5H \in N$

Exercise 13

Make up three true statements and three false statements using the newly defined subset and element symbols. Use the null set in at least one true and one false statement.

True Statement	False Statement

Something else to consider when working with sets is how many elements are in a given set. If we count the number of elements in a set, we call that number the **cardinality** or **cardinal number** of that set. To denote the cardinality of set **C**, we write n(**C**) = 25. Think of the *n* as the ***number** of objects in the set*. Do you remember any functional notation from algebra like f(x) = 2x + 3 implies f(5) = 2(5) + 3 = 10 + 3 = 13 – that is, when you plug 5 into the function f, out pops 13? Well, when working with the cardinality of a set, it is the same kind of thing and we use similar notation. In this case, when you plug **C** into the function n, out pops the number 25!!! When you are asked about the cardinality of a set, the answer is a number. We say that two sets are **equivalent** if they each have the *same **number** of elements* which is to say they have the *same cardinality*. We use the symbol, ~, to denote equivalence. For instance, since **P** and **H** each contain four elements, we can write **P** ~ **H**. Two sets are **equal** if they have the ***exact same elements***.

Exercise 14

Compute the following, using the information from exercise 9.

a. n(**P**) = ____ b. n(**N**) = ____ c. n(**Q**) = ____ d. n(**D**) = ____

e. n(**Z**) = ____ f. n(**H**) = ____ g. n(**Y**) = ____ h. n(**T**) = ____

i. n(**S**) = ____ j. n(**W**) = ____ k. n(**X**) = ____ l. n(**V**) = ____

m. n(**C**) = ____ n. n(**A**) = ____ (**A** represents the one dollar coins in our set **C**)

o. Consider the subsets of **C** we have defined so far. Using the equivalence symbol, express which pairs of subsets are equivalent to each other.

What if you were asked which coins were pennies and ***also*** from the year 1965? In set **C**, there is only one coin that satisfies both criteria and that is $5P$. When trying to find the elements that are common to two sets, in this case, **P** (pennies) and **V** (coins from 1965), we use the word **intersection**. A formal definition of **intersection** follows:

The **intersection** of two sets, **A** and **B**, written A ∩ B, is the set of elements which are found in both **A** and **B**.

Using the correct notation, we could now write P ∩ V = {5P}

Exercise 15

Complete the following, using correct notation:

a. = ____ X ∩ Y	b. = ____ H ∩ Z	c. = ___ N ∩ S
d. = ____ Q ∩ W	e. = ____ D ∩ P	

Exercise 16

Two sets are ***disjoint*** if their intersection is empty. For instance, **D** and **Q** are disjoint because $D \cap Q = \varnothing$. Name three other pairs of sets that are disjoint.

Exercise 17

Define ***disjoint*** sets: ___

Exercise 18

What if you were asked which coins were either pennies **or** from the year 1965? List which coins would satisfy one or both of these conditions:

Did you get eight different coins this time? ___ Any coin having a 5 or P or both would do it. You just found the ***union*** of the two sets **P** and **V**. A formal definition of union follows:

The ***union*** of two sets, **A** and **B**, written $A \cup B$, is the set of elements that are *either* in **A** *or* **B** *or* **both**.

Using correct notation, we write P ∪ $= 6P, 7P, 5P, 5D, 5N, 5H, 8P, 5Q$ Remember that the order in which the elements are written is irrelevant! ***IMPORTANT:*** *If an element is in both sets, it is only written down* ***once*** *in the union.*

? Exercise 19

Complete the following, using correct notation:

a. N ∪ S = ____

b. Q ∪ W = ____

c. D ∪ P = ____

d. X ∪ Y = _____

? Exercise 20

Write the definition for ***union*** and then the definition for ***intersection***. Use different letters than A and B.

? Exercise 21

Make up one new problem involving the coins using the intersection symbol and one new problem using the union symbol. Write the solutions to the problems.

? Exercise 22

Make up one true and one false statement involving the coins using the intersection and union symbols. State which statement is true and which is false.

Put aside the coin set for now. Let's work with another set. When working with sets, we need to be clear about what we are talking about. Here is another definition for you.

Mathematicians often refer to the set under consideration as the ***universe*** or ***universal set,*** and usually use the letter **U** to represent the universe.

In our coin problem, the ***universe*** wasn't specified. It could have been the *25* coins in the collection, all of the coins made between 1960 and 1970 or maybe all the coins in the world. Without knowing the universe, there are some things we couldn't answer in set theory. Now that we have that out of the way, here are two more important definitions.

The ***complement*** of a set **A**, written **A**^{c} or $\bar{A}$ or **A',** is the set of elements in the universal set that are not in **A**. We read **A**^{c} as "**A complement"** or "**not *A".*** *The universe must be specified in order to compute the complement.* For example, if **U**={1, 2, 3, 4, 5} and **H** = {3, 4}, then H' = {1, 2, 5).

The ***difference*** of two sets **A** and **B**, written **A – B (or A – B) ,** is the set of elements of **A** that are ***not also*** in **B**. (Some people list the elements of **A** and then cross off any that are in **B** to get the answer for **A – B** . So if **F**= {1, 2, 3 ,4} and **G** = {1, 3, 5, 7}, then to find **F – G**, list the elements of **F** and cross off any that are in G: {~~1~~,2,~~3~~,4}. The answer is: {2, 4)

? Exercise 23

Try the following problems in which the universe is the first 9 counting numbers. If U represents the universe, we have U = {1, 2, 3, 4, *5*, 6, 7, 8, 9} Three subsets of U are defined as follows: **A** = {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}

a. **A** ∩ **B** = _____	b. **A** ∩ **C** = ____
c. **B** ∩ **C** = _____	d. **A** ∪ **B** = ____
e. **A** ∪ **C** = _____	f. **B** ∪ **C** = ____
g. **A**^{c} = _____	h. **B**^{c} = ____

i. $\mathbf{C}^c$ = _____	j. $\mathbf{A} - \mathbf{B}$ = _____
k. $\mathbf{B} - \mathbf{A}$ = ____	1. $\mathbf{A} - \mathbf{C}$ = _____
m. $\mathbf{C} - \mathbf{A}$ = ____	n. $\mathbf{B} - \mathbf{C}$ = _____
o. $\mathbf{C} - \mathbf{B}$ = _____	

Did you notice that the order matters ***for*** *difference, but not for union or intersection?*

Continuation of exercise 23 where **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9}. The subsets are:
A= {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}

Try these “more involved” problems. These will take more than one step to do. You might need scratch paper.

p. $(\mathbf{A} \cup \mathbf{C}) - \mathbf{B}$ = ____	q. $(\mathbf{A} - \mathbf{C})^c$ = ____
r. $(\mathbf{A} \cap \mathbf{B}) \cup \mathbf{C}$ = ____	s. $(\mathbf{A} \cup \mathbf{C}) \cap \mathbf{B}$ = ____
t. $(\mathbf{A} \cup \mathbf{C})^c$ = ____	u. $\mathbf{A}^c \cap \mathbf{B}^c$ = ____
v. $A \bar{\cap} B$ = ____	w. $A \bar{\cup} B$ = ____
x. $\mathbf{A}^c \cup (\mathbf{B}^c \cap \mathbf{C}^c)$ = _____	

Answer true or false for the following:

y. $\varnothing \subset \mathbf{B}$ ___	z. $(\mathbf{A} - \mathbf{C}) \sim (\mathbf{C} - \mathbf{B})$ ___
aa. $5 \in (\mathbf{C} - (\mathbf{A} \cup \mathbf{B}))$ ___	bb. $n(\mathbf{A} \cup \mathbf{B}) = 9$ ___
cc. $n(\mathbf{A}) + n(\mathbf{B}) = 9$ ___	dd. $(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$ ___
ee. $(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c \cap \mathbf{B}^c$ ___	ff. $\mathbf{A} - \mathbf{B} = \mathbf{A} - (\mathbf{A} \cup \mathbf{B})$ ___
gg. **A** and **B** are disjoint ___	hh. **B** and **C** are disjoint ___
ii. $U - \mathbf{A} = \mathbf{A}^c$ ___	jj. $\{3, 4, 5, 6, 8\} \subset \{4, 5, 6, 7, 8, 9\}$ __

Make sure you can work through all of the problems above perfectly on your own. If you have trouble understanding any of them, check with your classmates or ask your teacher if something still needs clarifying!!! Then, make up some more problems of your own as instructed below before going on.

Make up four of your own “more involved problems” and show the solutions. Use the same sets (**U**, **A**, **B** and **C**) as in exercise 23.

1. ___
2. ___
3. ___
4. ___

? Exercise 24

Now it’s time for you to make up some original problems on your own and to provide the solutions. First, decide on a universal set, U, and define three subsets, D, E and F. Then, make up two original and nontrivial (thoughtful and involving two or more steps) problems using combinations of union, complement, intersection and difference

Remember when I mentioned that the listing method wasn’t always practical or possible? Consider a universal set that contained all of the whole numbers. These could not be listed. Another way of expressing this set would be by description. You could say **U** = (the whole numbers) This is a very formal way of writing this same set **U** = {x|x is a whole number} which is read "**U** is the set of all x such that x is a whole number". You may have seen this notation in algebra. We won't be so formal here. Another way to get around the listing problem is to use three consecutive dots which means "and so on". We would write **U** = {0,1,2,3,...}. Here is a way to express all the even numbers between 13 and 509: {14,16,18,...,508}. You could list them out (if you didn't have a life!), but you can see the advantage of using three dots!!!! Using the three dots doesn't always work either. Consider the set of all real numbers. They cannot be listed. You would have to write something like this: {all real numbers) or {x|x is a real number}.

? Exercise 25

Use correct set notation to express the following sets in one or more ways:

a. the letters of the alphabet

b. the even integers between and including 124 and 698

c. the whole numbers strictly in between 100 and 1000

d. the names of the past presidents of the United States up until 1995

e. the names of the presidents of the U.S. from 1981 - 1995

It's now time get out your **A-blocks**. They can be found on Material Cards 2A - 2E. The white cards are value label cards (which you won't use until Exercise Set 3), and the colored objects are the actual A-blocks. There are 24 objects in each set of A-blocks.

We will use the following abbreviations when referring to each of the 24 elements.

S = small	**L** = large	**Y** = yellow	**R** = red	**B** = blue	**G** = green	**Q** = square	**T** = triangle	**C** = circle

We'll use three letters to specify each object in this order - size, color, shape. The small blue square would be denoted by SBQ and the large red circle would be denoted by LRC. These abbreviations are on the objects that you cut out.

Let's define our universe of A-blocks by **A**.

? Exercise 26

Play with the A-blocks a little bit. How do you think children might play with them?

? Exercise 27

Arrange the A-blocks into some piles of subsets. What kinds of piles did you make and how many were in each pile?

? Exercise 28

Divide them up differently. This time, what kinds of piles did you make and how many were in each pile?

? Exercise 29

a. Select a shape and take all the blocks of that shape and put them on a blank piece of paper. Close your eyes and have a friend remove one of objects. Look at the remaining pieces and determine which piece was removed. Think about how you figured it out. How do you think a child would figure it out?

b. Repeat part (a), but this time select a color and put all the blocks of that color on a blank sheet of paper.

c. Repeat part (a), but this time select a size and put all the blocks of that size on a blank sheet of paper.

d. Which was the fastest for you to figure out — shape, color or size? Why do you think that was so?

e. **Extra Credit:** Work this activity with 2 young children. One should be very young (you decide) and the other several years older. Document your findings and share on the Forum along with a picture or video. Compare how they figured out the missing piece to how you did. Be sure to do the activity yourself first.

? Exercise 30

In this exercise, parts a through f are not related to each other. Start with the whole set of A-Blocks for each part and then proceed to follow the directions each time.

a. Make subsets of the blocks, dividing them by shape (disregard size)

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets of shapes. State which shape you picked: ____________

Using correct set notation, and abbreviations, list its elements

b. Make subsets of the blocks, dividing them by color.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which color you picked: _____________

Using correct set notation, and abbreviations, list its elements

c. Make subsets of the blocks, dividing them by size.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which size you picked: _____________

Using correct set notation, and abbreviations, list its elements

d. Make subsets of the blocks, dividing them by color shape.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which color and shape you picked: _____

Using correct set notation, and abbreviations, list its elements

e. Make subsets of the blocks, dividing them by color and size.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which color and size you picked: ______

Using correct set notation, and abbreviations, list its elements

f. Make subsets of the blocks, dividing them by size and shape.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which size and shape you picked: _____

Using correct set notation, and abbreviations, list its elements

A particular **shape**, **color** or **size** is called a **value**. Let each of the values of A-blocks be defined in terms of subsets **S**, **L**,**R**, **B**, **G**, **Y**, **Q**, **T**, **C** will define small, large, red, blue, green, yellow, square, triangle and circle, respectively.

? Exercise 31

a. Have a friend think of two values (like blue and square, for instance). Then have your friend put all the blocks having either of these values on a blank paper, but intentionally leaving one out. Try to figure out which block is missing. What do you have to figure out first?

b. Which values were chosen? ____ Which piece was missing?

Do this exercise two more times, with other values. Try it with all the blocks hidden one time and with all of them in clear sight another time.

c. Is it easier if all the blocks, including the missing block, are in plain view somewhere or if the missing block as well as all the rest are hidden?

d. Try this game by having your friend choose three values. Is this easier or harder? Try it more than once. List the values chosen and the piece missing for one of the games.

e. When making a subset that has either of two values as in part *a*, you are forming the ____ of those two sets. Here's a hint, *it isn't* complement. Using correct set notation, write the set formed by the union of the two values chosen in part a. _____ ∪ _____ = _______

? Exercise 32

Choose a value (a particular shape, color or size). Put all the blocks with that value in one pile. Pick a different value and put all the blocks with that value in a separate pile.

a. Which values were chosen? ____
b. Were there any pieces that needed to go in both piles? ____ If so, which pieces?
c. Suppose the values chosen were red and blue. Were there any blocks that needed to go in both piles? ____ If so, which ones?
d. Suppose yellow and large were chosen. Were there any blocks that needed to go in both piles? ____ If so, which ones? ____
e. Elements belonging in both piles must be both yellow AND large. When making a subset where the objects must have both values, you are forming the ____ of those two sets, Using set notation, we write $\mathbf{Y} \cap \mathbf{L}$ =____

Put your A-blocks away for now. Don't lose them!! Before we continue working with the A-blocks, we'll need to learn about and work with Venn diagrams. We will then come back and do more work with our A-blocks.

? Exercise 33

Let's consider a situation where a mother, Mary, has three children –Alicia, Brent and Carlos. Mary has red hair. The question we will attempt to answer is this: What possibilities exist regarding which, if any of her children, have red hair? List the possibilities you came up with:

Let's work through the above problem. First of all, let's break the problem down a little bit. It's possible none of her children have red hair, only one has red hair, two out of three have red hair or all three of them ended up as redheads. Within each of these four situations, let's write down each of the possibilities. We will list each possibility as a set, using the childrens' names as the possible elements.

Situation 1: None of the children are redheads: { }
Situation 2: One child is a redhead. This gives us the three possibilities listed below: {Alicia} {Brent} {Carlos}

Situation 3: Two out of three have red hair. You can also think of this as only one child not having red hair. In any case, that gives us the three possibilities below: {Brent, Carlos} {Alicia, Carlos} {Alicia, Brent}
Situation 4: All three children have red hair: {Alicia, Brent, Carlos}

We came up with eight different possibilities.

The above problem really amounts to a set theory question. Given a set M, where M = {A, B ,C}, list all possible subsets of M. First of all, remember that the null set is a subset of every set and the entire set is a subset of itself (it’s just not a proper subset). You did remember those two points, didn’t you?

Let’s begin by listing the possibilities according to how many elements might be in the subsets either no elements, one element, two elements or three elements. For each of these situations, list all possibilities:

No elements: { }	Two elements: {A, B}, {A, C}, {B, C}
One element: {A}, {B}, {C}	Three elements: {A, B, C}

Here is a list of all possible subsets: { }, {A}, {B}, {C}, {A, B}, {A, C}, {B, C}, {A, B, C}

WOW! Did you see how we came up with the same eight possibilities? This set theory stuff is just amazing, isn’t it?

? Exercise 34

List all possible subsets for each of the sets given.

a. ∅: ___
b. {P}: ___
c. {G, F}: ___
d. {X, Y, Z}: ___
e. {1, 2, 3, 4}: ___

? Exercise 35

Using the information you gained from exercise 34, answer the following question.
How many subsets are there for a set containing the given number of elements?

a. no elements: ___
b. one element: ___
c. two elements: ___
d. three elements: ___
e. four elements: ___
f. five elements: ___
g. n elements (use a formula): ___

There is one more operation we will define on sets. So far, you’ve worked with union, intersection, difference and complement. Now we will learn what it means to take the Cartesian product of two sets, A and B.

Cartesian product

The **Cartesian product** of set **A** with set **B**, which is written **A** × **B** and is read as '**A** cross **B**" is the set of all possible ordered pairs (a,b), where a ∈ **A** and b ∈ **B**.

If you are finding the Cartesian product, the answer is a set that contains ordered pairs. The order matters! The first element must come from the set written to the left of the × and the second element must come from the set written to the right of the ×.

Example 1

If A = {x, y, z} and B = {a, b}, find A × B and B × A.

Solution

A × B = {(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}

B × A = {(a, x), (b, x), (a, y), (b, y), (a, z), (b, z)}

Notice there were 3 elements in one set and two in the other. The Cartesian product has six elements in it, six ordered pairs.

Example 2

If E = {1, 2} and F = {2,3}, find E × F, F × E, E × E, and F × F

Solution

E × F= {(1, 2), (1.3), (2, 2), (2, 3)}	E × E = {(1, 1), (1, 2), (2, 1), (2, 2)}
F × E = {(2, 1), (3, 1), (2, 2), (3, 2)}	F × F = {(2, 2), (2, 3), (3, 2), (3, 3)}

Notice there were two elements in one set and two in the other. Each Cartesian product has four elements in it – four ordered pairs.

Example 3

Find {4, 5} × {7, 2, x, #}.

Solution

{4, 5} × {7, 2, x, #} = {(4, 7), (4, 2), (4, x), (4, #), (5, 7), (5, 2), (5, x), (5, #)}

Notice there were two elements in one set and four in the other. The Cartesian product has eight elements in it – eight ordered pairs.

In Example I, one set contained 3 elements and the other contained 2 elements, The Cartesian product contained 6 elements – 6 ordered pairs. In Example 2, one set contained 2 elements and the other contained 2 elements, The Cartesian product contained 4 elements, 4 ordered pairs. In Example 3, one set contained 2 elements and the other contained 4 elements. The Cartesian product contained 8 elements , 8 ordered pairs.

Exercise 36

If a set **B** contains 5 elements (in other words, n(**B**) = 5) and a set **C** contains 7 elements (or n(**C**) = 7), then how many elements (where each element is an ordered pair) are in the Cartesian product **B** × **C**?

Exercise 37

If a set **B** contains 5 elements (in other words, n(**B**) = 5) and a set **C** contains 7 elements (or n(**C**) = 7), then how many elements (where each element is an ordered pair) are in the Cartesian product **C** × **B**?

Exercise 38

In general, for any two sets, F and G, is n(F × G) = n(G × F)?

Exercise 39

Assume a set B contains b elements (in other words, n(B) = b) and a set C contains c elements (or n(C) = c). Use this information to compute the following, where you are asked how many elements (where each element is an ordered pair) are in a given Cartesian product.

a. n(B × C) = ____

b. n(B × B) = ____

c. n(C × C) = ____

Exercise 40

In general, if you take any two sets, A and B, is A × B = B × A? ____ If you answered yes, provide an example of two different sets, A and B, and show that A × B = B × A. If you answered no, provide an example of two different sets, A and B, and show that A × B ≠ B × A.

Example 4

This one is trickier: Find {(4, 3), 5) × {(3, 3),{4, 7, 5, 2}, 1}

Okay, take a breath...you can do this. Look carefully at the first set; it has two elements, the first element happens to be the ordered pair (4, 3) and the second element is the number

5. Look carefully at the second set; it has three elements, the first shown happens to be the ordered pair (3,3), the second happens to be set containing four elements, and the third is the element 1, So the first set contains 2 elements and the second set contains 3 elements. First, think about how many elements are in the Cartesian product. Did you get the idea to multiply 2 by 3? I hope so! There are six elements in the Cartesian product. Each of those elements is an ordered pair, and some of these ordered pairs will look kind of weird. See if you can understand the solution written below.

Solution

{(4, 3), 5} × {(3, 3), {4, 7, 5, 2}, l} = {((4, 3), (3, 3)), ((4, 3), {4, 7, 5, 2}), ((4, 3), 1), (5, (3 , 3)), (5, {4, 7, 5, 2}), (5, 1)}

Look at this solution carefully. There are six ordered pairs in the solution. The first three ordered pairs have (4, 3) as its first coordinate and the second three ordered pairs have 5 as its first coordinate. Then (3, 3) is the second coordinate for the first and fourth ordered pair, {4, 7, 5, 2} is the second coordinate for the second and fifth ordered pair and 1 is the second coordinate for the third and sixth coordinate. If you can follow this one, you should do great on the next exercise.

Exercise 41

Write the Cartesian product. Each answer is a set containing ordered pairs. Before doing each problem, think about how many ordered pairs will be in the answer. Make sure you write your answer using correct notation, put the ordered pairs in a set. There is a comma between each coordinate in each ordered pair and there is a comma between each element in the set. Use braces { } around the set. One of the answers is the null set!

a. {3, 4} × {2, 6} = ____

b. {6, 7, 8, 9} × {5} = ____

c. {r, s, t} × { } = ____

d. {a} × {a} = ____

e. {x, y} × {x, y} = ____

f. {1, 3, 5} × {1, 3, 5} = ____

g. {(9, 4), C)} × {D, {a, b, c}} = ___

h. $\{\{5, 6, 7, 8, 9\}\} \times \{g, \{4, 3\}\} =$ ____

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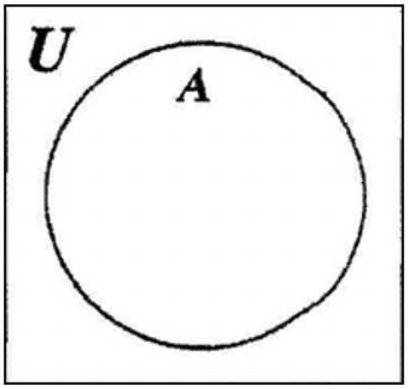
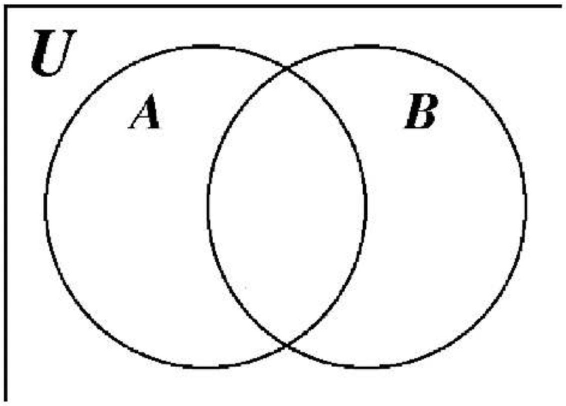
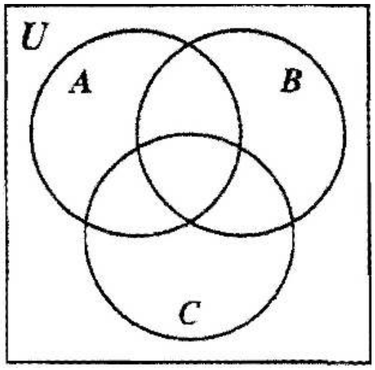
1.2: Venn Diagrams



This is a Venn diagram using only one set,
A



This is a Venn diagram Below using two sets,
A
and
B
.

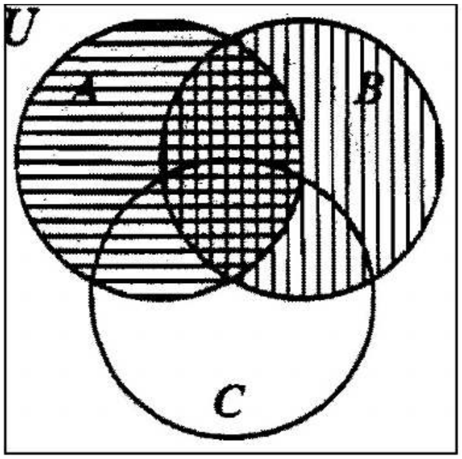


This is a Venn diagram using sets **A**, **B** and **C**.

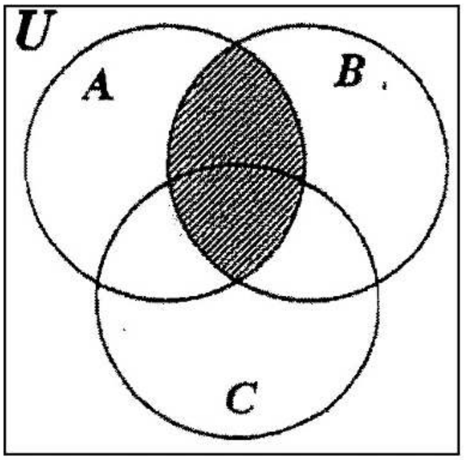
Study the Venn diagrams on this and the following pages. It takes a whole lot of practice to shade or identify regions of Venn diagrams. Be advised that it may be necessary to shade several practice diagrams along the way before you get to the final result.

We shade Venn diagrams to represent sets. We will be doing some very easy, basic Venn diagrams as well as several involved and complicated Venn diagrams.

To find the intersection of two sets, you might try shading one region in a given direction, and another region in a different direction. Then you would look where those shadings overlap. That overlap would be the intersection.



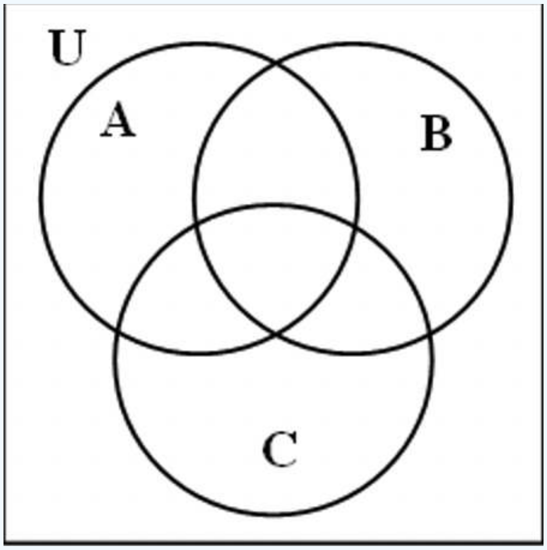
For example, to visualize $A \cap B$, shade A with horizontal lines and B with vertical lines. Then the overlap is $A \cap B$. The diagram on the left would be a first step in getting the answer. The shaded part on the diagram to the right shows the final answer.



Here are two problems for you to try. Only shade in the final answer for each exercise.

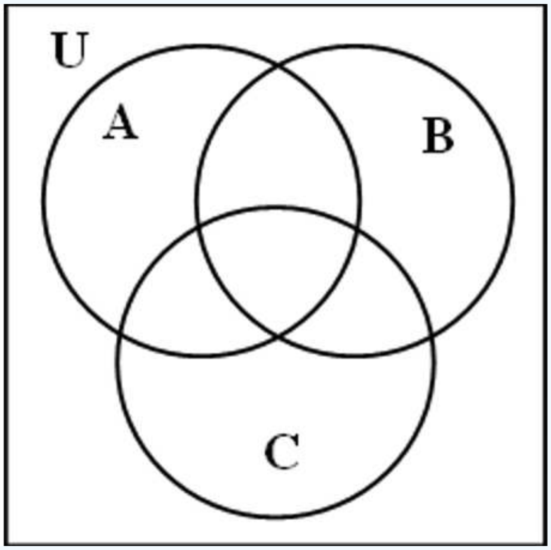
? Exercise 1

Shade the region that represents $A \cap C$

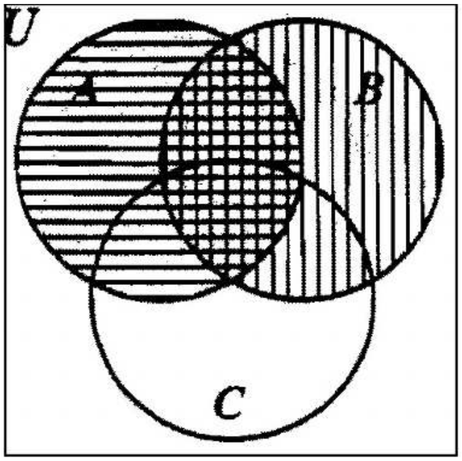


Exercise 2

Shade the region that represents $B \cap C$



To shade the union of two sets, shade each region completely or shade both regions in the same direction. Thus, to find the union of **A** and **B**, shade all of **A** and all of **B**.

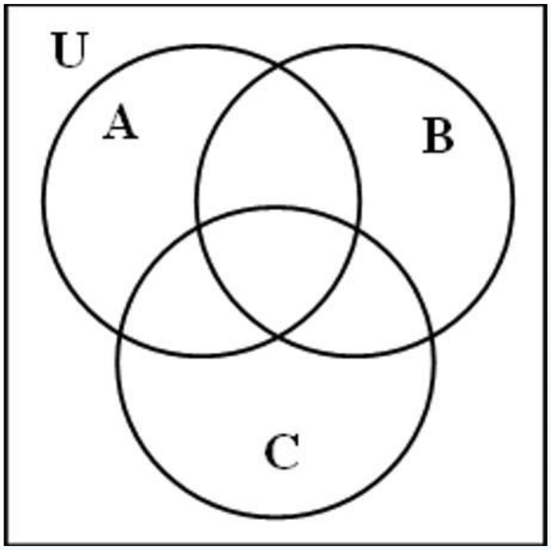


The final answer is represented by the shaded area in the diagram to the right.

Here are two problems for you to try. Only shade in the final answer for each exercise.

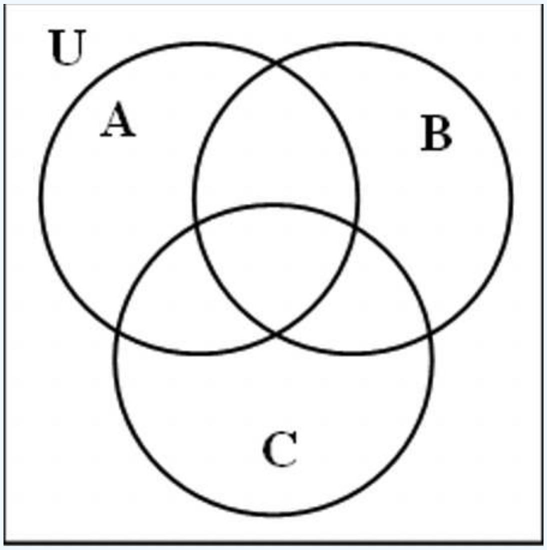
Exercise 3

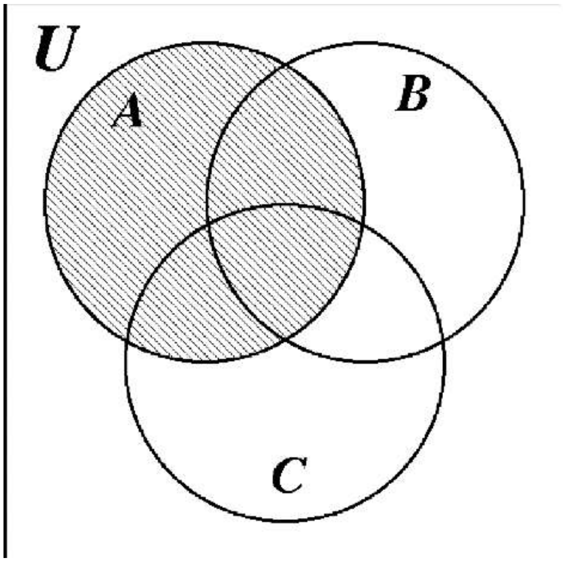
Shade the region that represents $A \cup C$



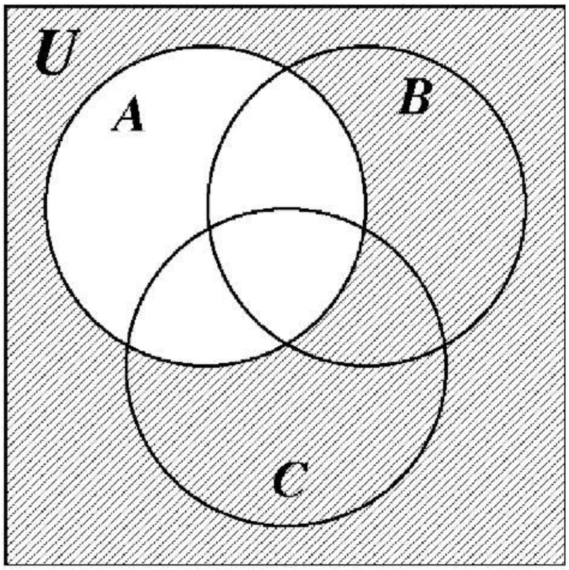
Exercise 4

Shade the region that represents $B \cup C$

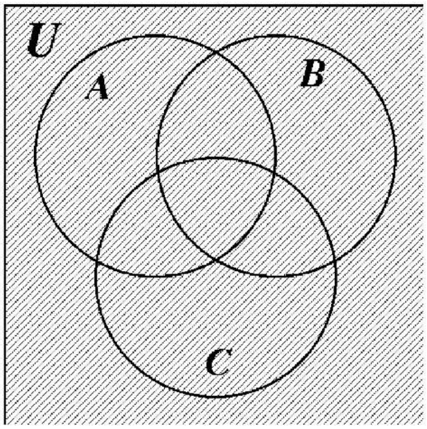




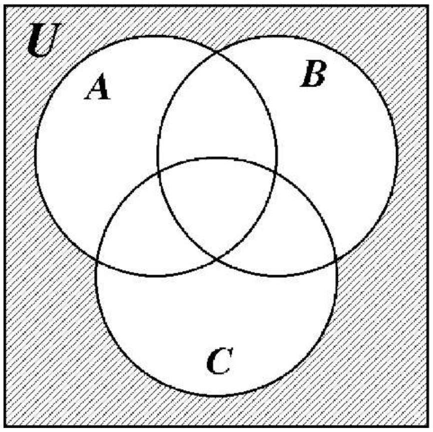
For the complement of a region, shade everything outside the given region. You can think of it as shading everything except that region. On the Venn diagram to the left, the shaded area represents **A**. On the Venn diagram to the right, the shaded area represents .



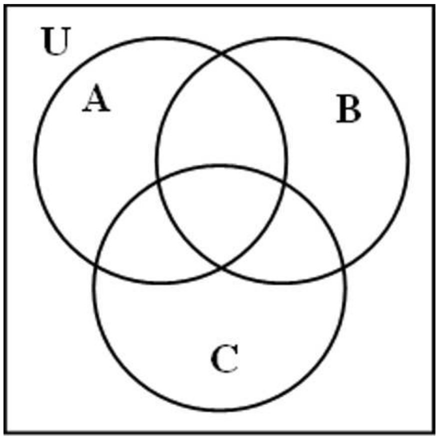
Many people are confused about what part of the Venn diagram represents the universe, **U**. The universe is the ***entire*** Venn diagram, including the sets **A**, **B** and **C**. The three Venn diagrams on the next page illustrate the differences between **U**, U^c and $(A \cup B \cup C)^c$. Carefully note these differences.



U



$(A \cup B \cup C)^c$

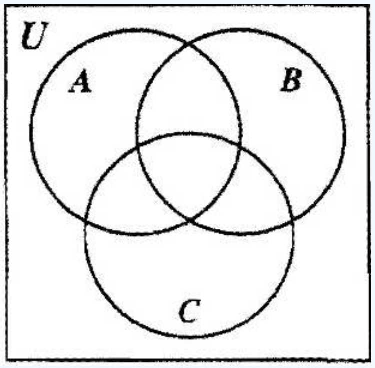


U^c

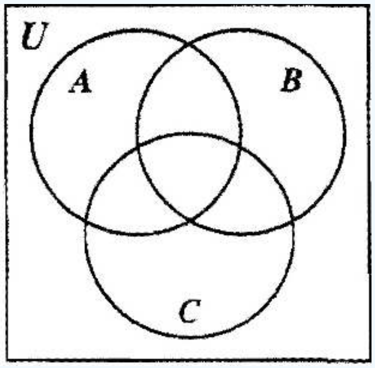
Usually, parentheses are necessary to indicate which operation needs to be done first. If there is only union or intersection involved, this isn't necessary as in $(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})^c$ above. Convince yourself that $((\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}) = (\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}))$. Similarly, convince yourself of the analogous fact for intersection by performing the following steps. On the first Venn diagram below, shade $\mathbf{A} \cap \mathbf{B}$ with horizontal lines and shade **C** with vertical lines. Then, the overlap is $((\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C})$. On the second Venn diagram, shade **A** with lines slanting to the right and $\mathbf{B} \cup \mathbf{C}$ with lines slanting to the left. Then the overlap is $(\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}))$. Check to see that the final answer, the overlap in this case, is the same for both. Shade the final answer in the third Venn diagram.

? Exercise 5

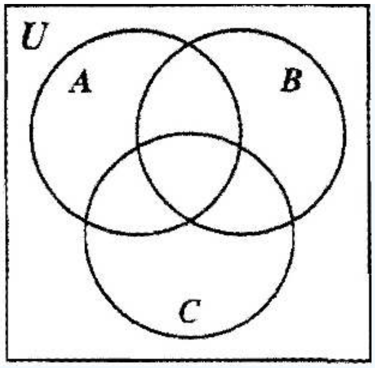
a. $(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$



b. $(\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}))$



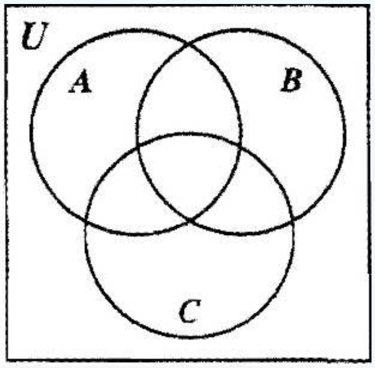
c. Shade final answer here.



Now, it's time for you to try a few more diagrams on your own. It may take more than one step to figure out the answer. You might need to do preliminary drawings on scratch paper first. The shadings you show here should be the final answer only, but you should be able to explain and support how you arrived at your answer. Compare your answers with other people in your class and make sure a consensus is reached on the correct answer. Do this for all the Venn diagrams throughout this exercise set. **Shade in the region that represents what is written above each of the six Venn diagrams on the following page**. Note that in cases involving more than one operation, it is necessary to use parentheses and follow order of operations. Exercises 10 and 11 illustrate why this is necessary.

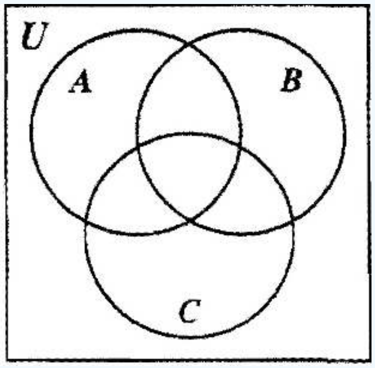
? Exercise 6

$\mathbf{B}^c$



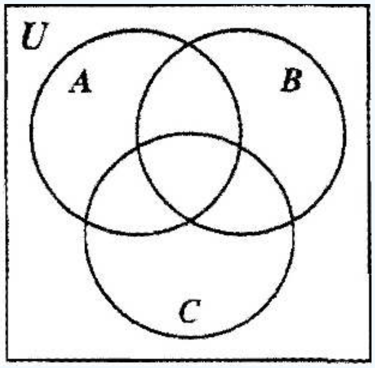
? Exercise 7

$(\mathbf{C} \cap \mathbf{A})^c$



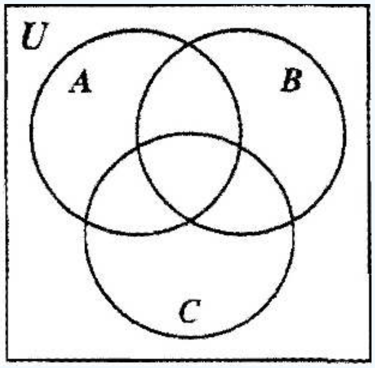
Exercise 8

$(\mathbf{B} \cup \mathbf{C})^c$



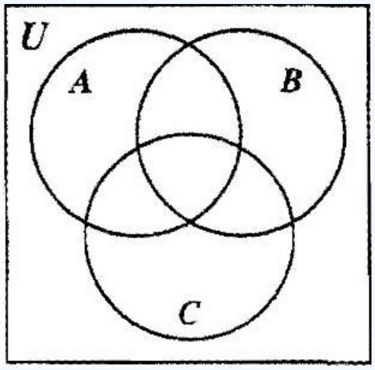
Exercise 9

$(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})^c$



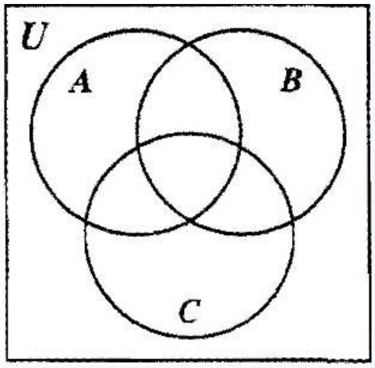
Exercise 10

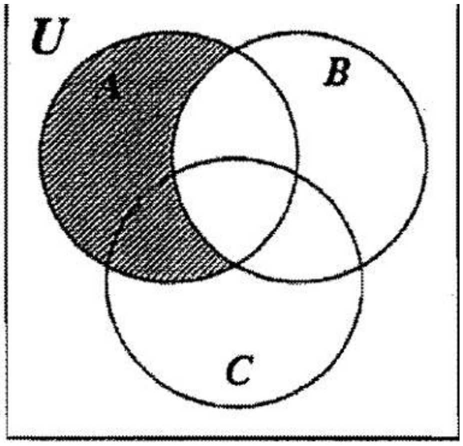
$(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$



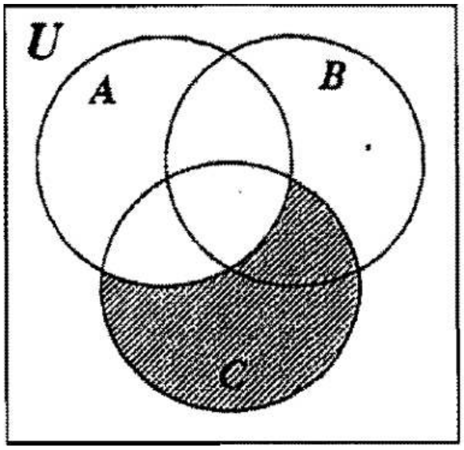
Exercise 11

$(\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})$





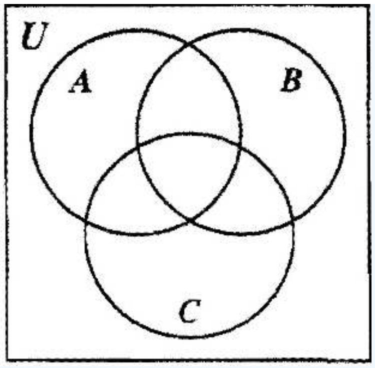
For difference, shade the region coming before the difference sign (**–**) *but don't include or shade* ***any*** *part of the region that follows the difference sign.* The Venn above represents **A–B** and the one below represents **C**–**A.**



Here are two problems for you to try. Only shade in the final answer for each exercise.

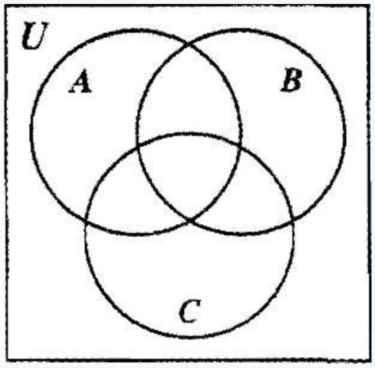
? Exercise 12

Shade the region that represents **A** – **C**

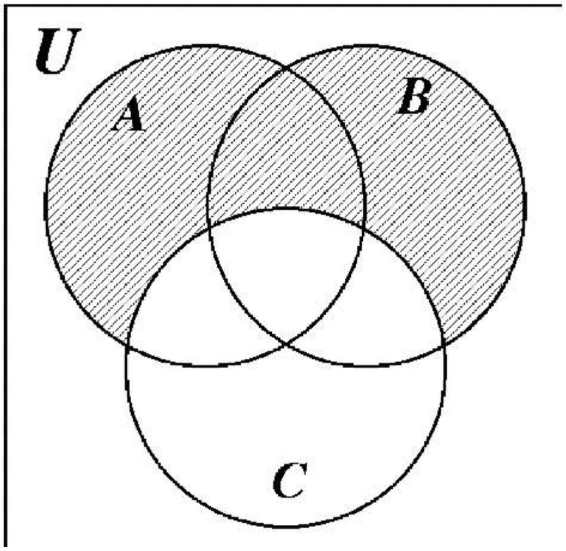


Exercise 13

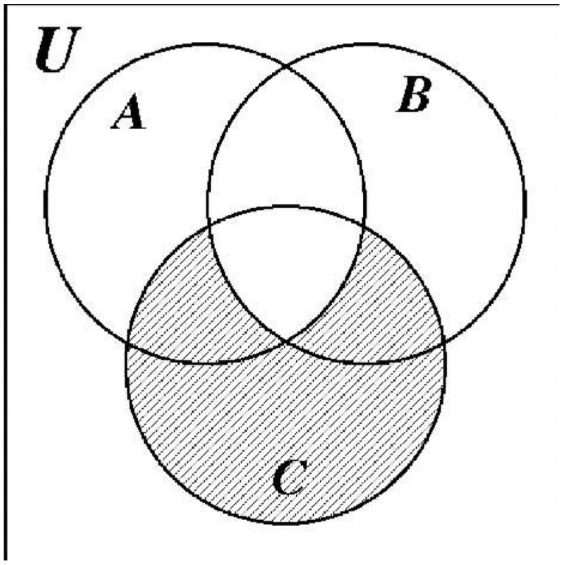
Shade the region that represents **B – C**



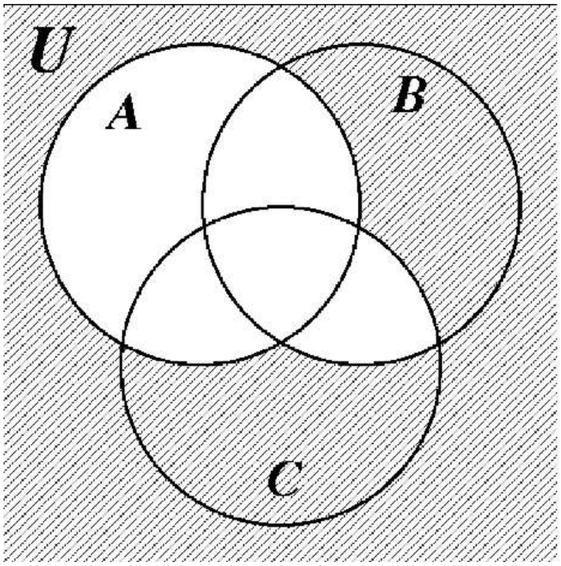
Study the following Venn diagrams. Make sure you understand how to get the answers.



(**A** ∪ **B**) – **C**



C – (**A** ∩ **B**)

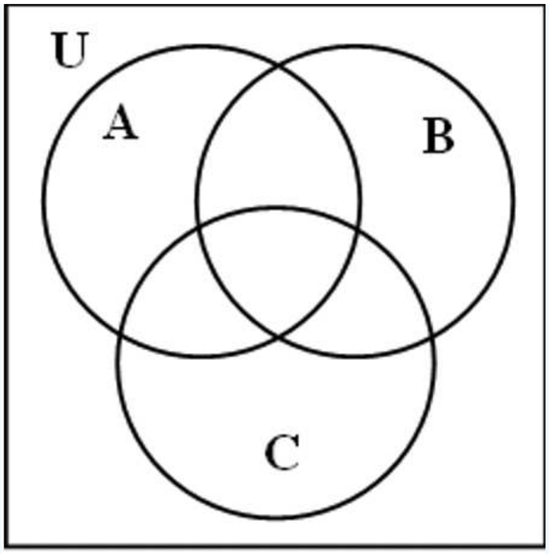


A^c – (**B** ∩ **C**)

It's your turn to shade in the region that represents what is written above each diagram.

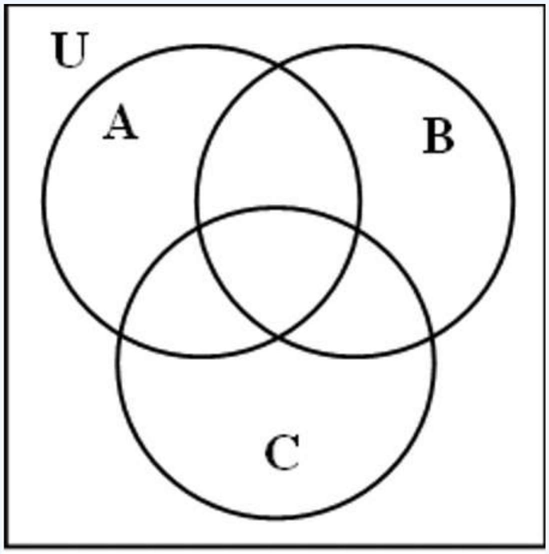
Exercise 14

(**A** ∩ **C**) – **B**



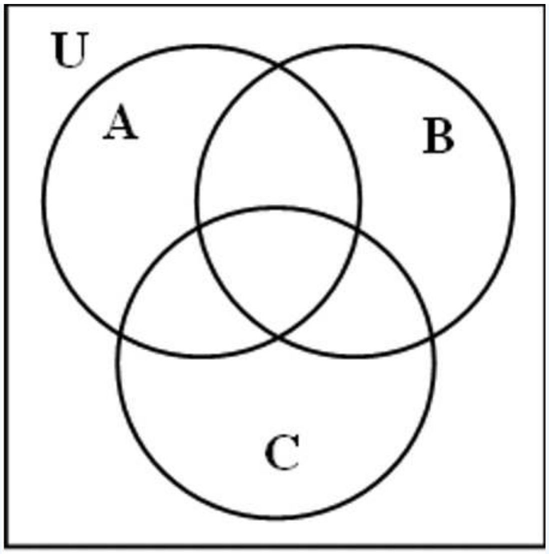
Exercise 15

B – (**A** ∩ **C**)



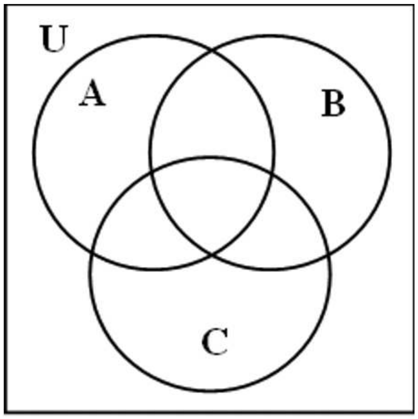
Exercise 16

(**A** – **C**) ∪ (**B** – **A**)

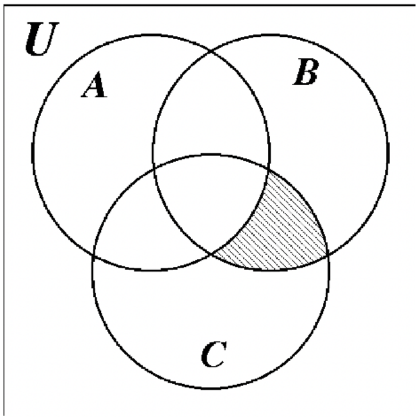


Suppose you wanted to find $((\mathbf{C} - \mathbf{A}) \cap \mathbf{B})^c$. This would probably take a few steps to get the answer. One approach to finding the correct shading is to notice that the final answer is the complement of $(\mathbf{C} - \mathbf{A}) \cap \mathbf{B}$. That means we would have to first figure out what $(C - \mathbf{A}) \cap \mathbf{B}$ looked like. In order to do that, we notice that this is the intersection of two things **C – A** and **B.** On the blank Venn diagram to the left below, shade C – **A** with horizontal lines and B with vertical lines. The overlap would be the intersection. The overlap on your drawing should match the shading shown on the Venn diagram in the middle. Does it? The last step would then be to take the complement of the shading shown on the middle diagram. This is shown on the Venn diagram on the far right. So, it took drawing three Venns to come up with the final answer for this problem. Someone else might be able to do it in fewer steps while someone else might take more steps.

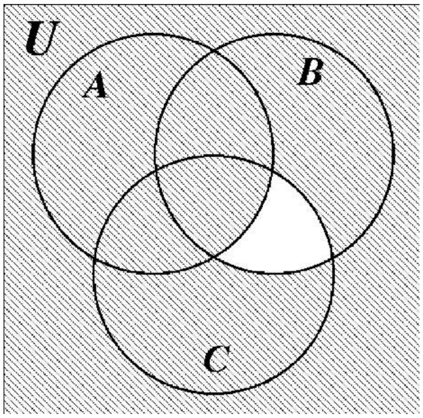
? Exercise 17



$(\mathbf{C} - \mathbf{A}) \cap \mathbf{B}$



$((\mathbf{C} - \mathbf{A}) \cap \mathbf{B})^c$



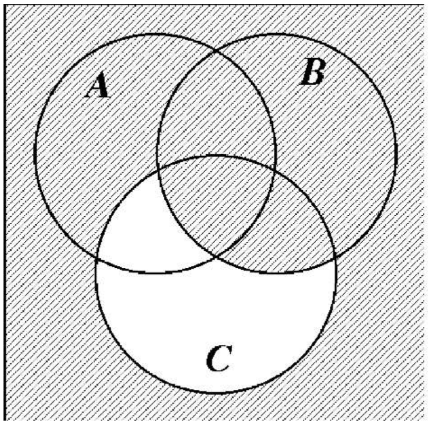
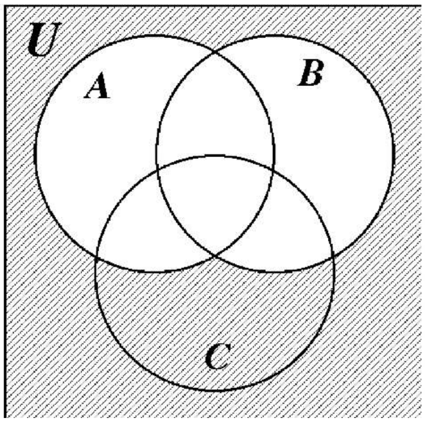
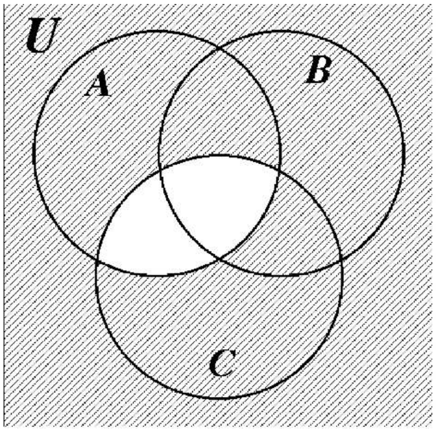
As mentioned previously, it takes a lot of practice to get good at shading Venn diagrams. It's even trickier to look at a Venn diagram and describe it, In fact, there is usually more than one way to describe a Venn diagram. For example, the shading for $((\mathbf{C} - \mathbf{A}) \cap \mathbf{B})^c$ shown on the previous page is the same as it is for $((\mathbf{C} \cap \mathbf{B}) - \mathbf{A})^c$. What does this mean? We're so used to only having one correct answer. Well, consider if someone asked you to write an arithmetic problem for which the answer was 2. There would be infinitely many possibilities. For example, 5 - 3 or 1 + 1 or 10/5 would all be acceptable answers. Granted, this kind of question on a test would be harder for a teacher to grade because each student's response would have to be checked to see if it would work. There isn't one pat answer. The same goes if a teacher asks you to look at a shading of a Venn diagram and describe it. On the other hand, if a description is given and you are asked to shade the Venn diagram, there is only one correct shading. It is much like being asked to compute an arithmetic problem. The answer to 10 - 8 is 2 and that is the only acceptable answer!

The point of all this is that to master shadings of Venn diagrams and descriptions of Venn diagrams by looking at the shadings takes lots and lots and lots of practice. Give yourself plenty of time to study and work on them and you will accomplish this feat!!!

On the next few pages, you are asked to shade several one, two and three set Venn diagrams. The correct shadings follow. Make sure you try these problems in earnest. Make sure you can explain the steps involved to arrive at the correct shading. After mastering the shadings, see if you can look at a shaded Venn diagram and come up with an accurate description. Again, remember there is more than one way to describe a given Venn diagram.

These Venn diagrams will be helpful when studying for a test. Go back and practice drawing the same Venn diagrams later. Use the answers to see if you can describe them by looking at the picture. Of course, remember that your description might not match exactly since there as more than one way to describe any given Venn diagram. If your description is different, make sure you go through the steps of shading a Venn with your description and see if your shading really matches the Venn diagram you were trying to describe.

Here are a few shaded Venn diagrams. See if you can look at the shadings and come up with a description. I've put some possible answers at the bottom of this page.

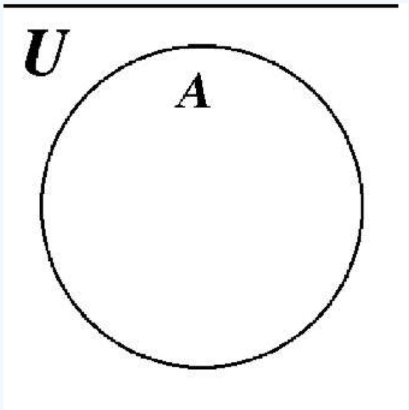
Here are some possible descriptions for the above Venn diagrams:

$(\mathbf{C} - \mathbf{B})^c$	$(\mathbf{A} \cup \mathbf{C})^c$	$(\mathbf{A} \cap \mathbf{C})^c$

Shade the region that represents what is written above each of the one and two set Venn diagrams below. You may need to draw preliminary drawings first for some of them.

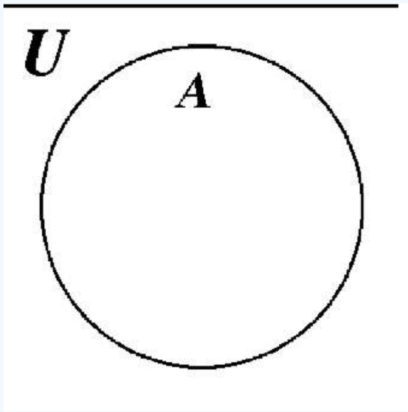
Exercise 18

A



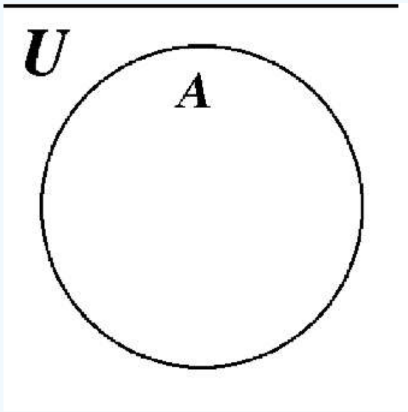
Exercise 19

A^c



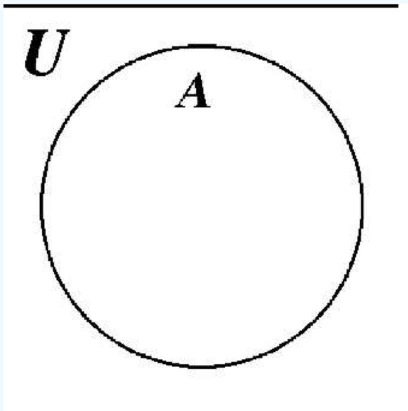
Exercise 20

U



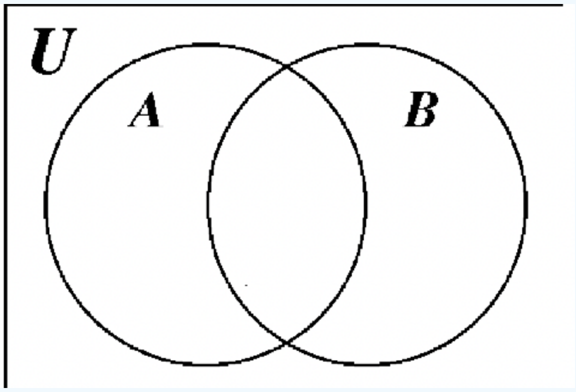
Exercise 21

U^c



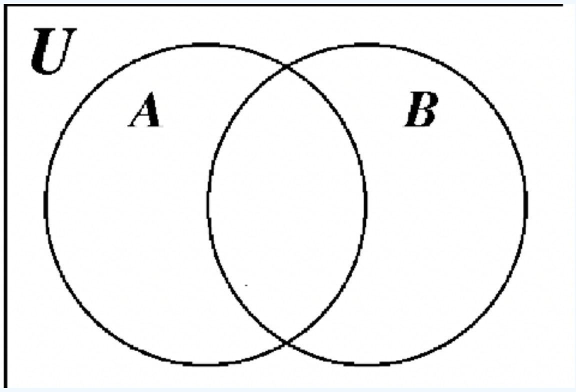
Exercise 22

A ∩ B



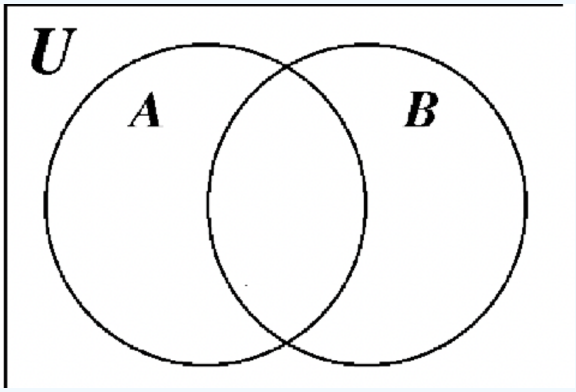
Exercise 23

A ∪ B



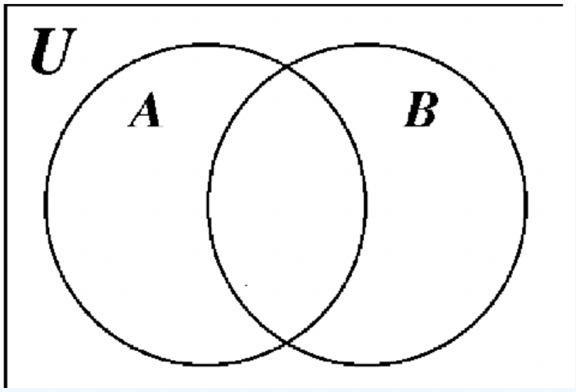
Exercise 24

$A \cup B^c$



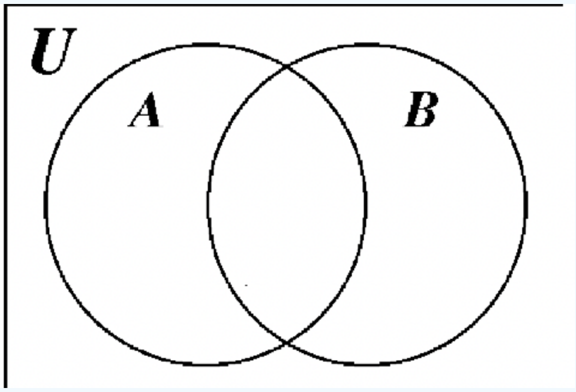
Exercise 25

$(A \cap B)^c$



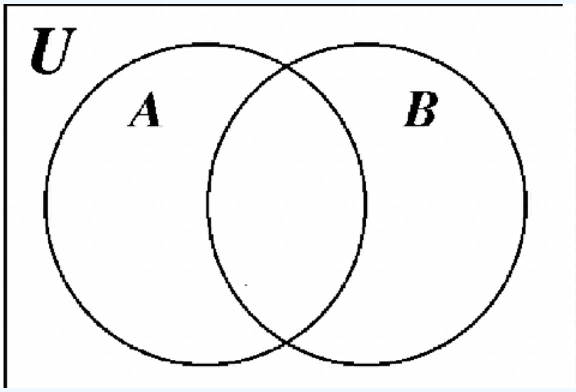
Exercise 26

$(A \cup B)^c$



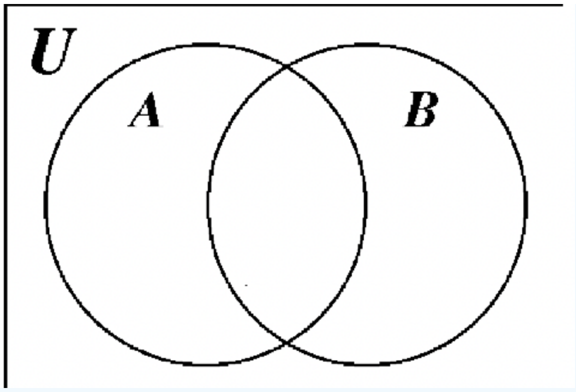
Exercise 27

$(\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{B} \setminus \mathbf{A})$



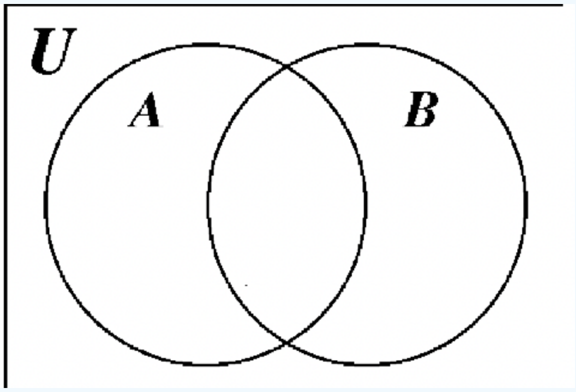
Exercise 28

$A^c \cup B^c$



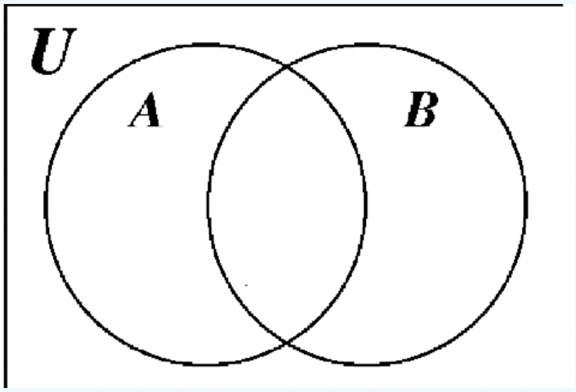
? Exercise 29

$A^c \cap B^c$



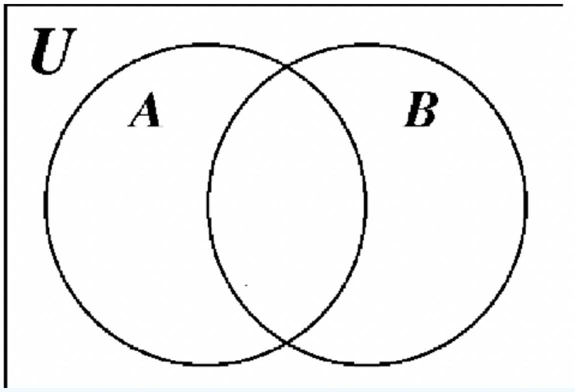
? Exercise 30

$(A \cup B)^c \cup (A \cap B)$



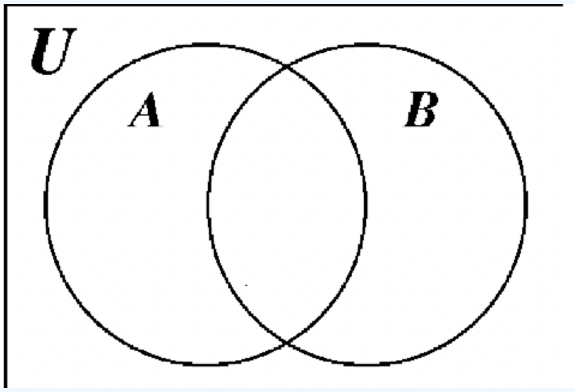
? Exercise 31

B



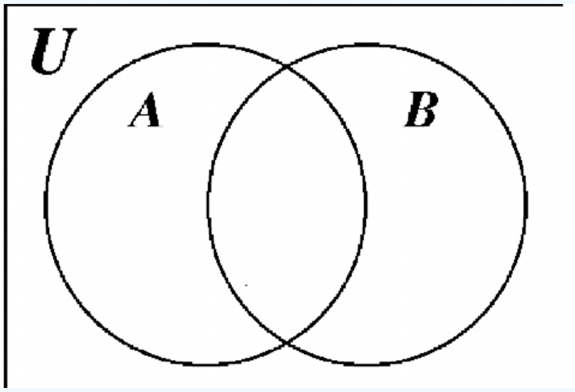
Exercise 32

B - A



Exercise 33

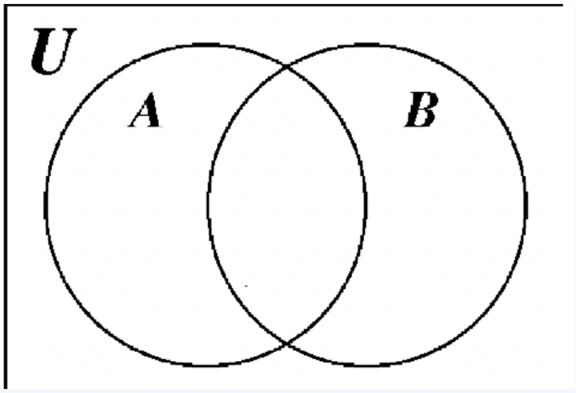
B^c



Shade the region that represents what is written above each of the one and two set Venn diagrams below. You may need to draw preliminary drawings first for some of them.

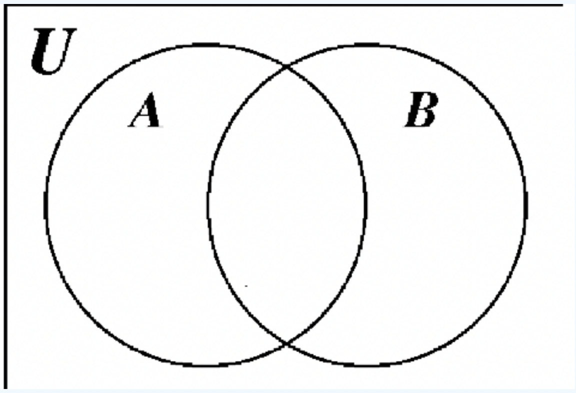
Exercise 34

(A ∩ B) – C



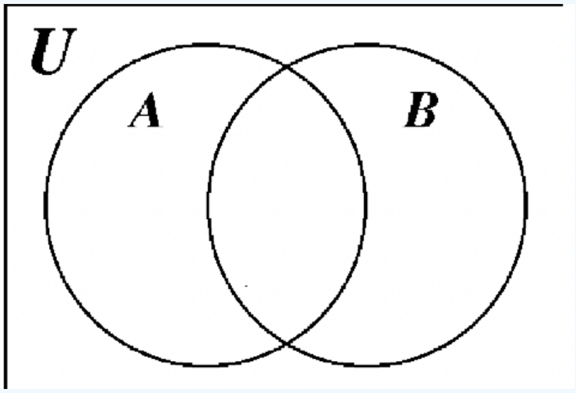
Exercise 35

(C ∪ B) – A



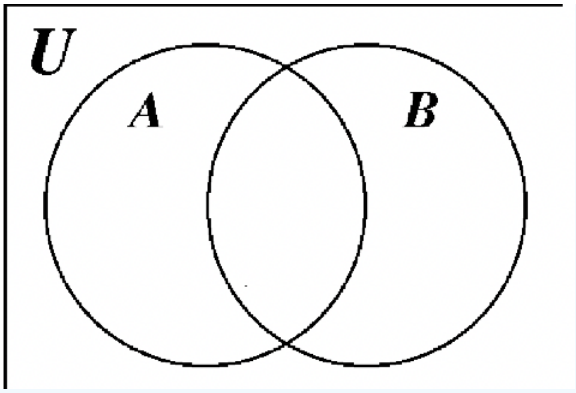
Exercise 36

(A ∩ B) ∪ C



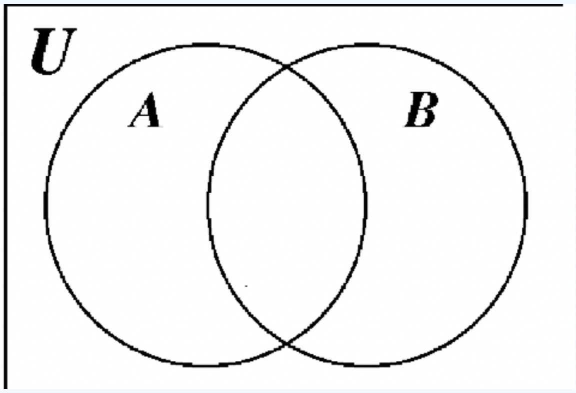
Exercise 37

(A ∪ B) ∩ C



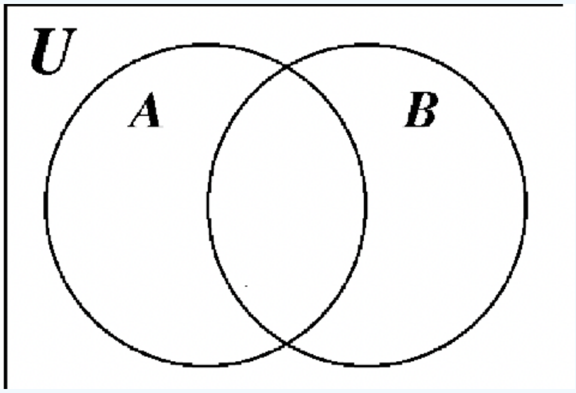
Exercise 38

$\mathbf{A^c - B}$



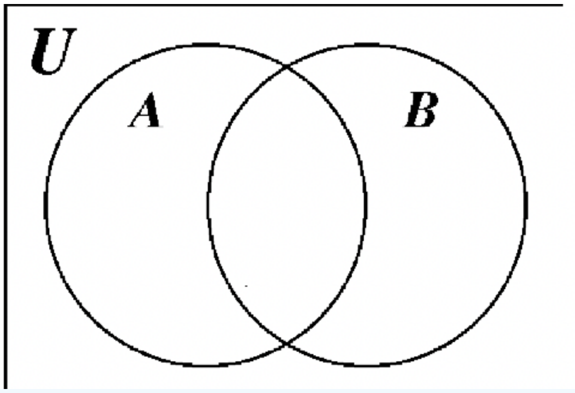
Exercise 39

A ∩ B ∩ C) – B



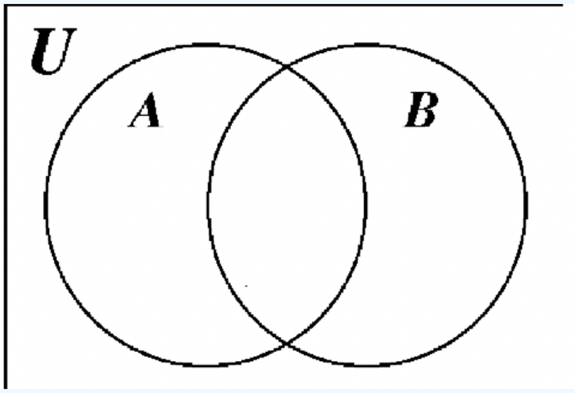
Exercise 40

B – (**A** ∪ **C**)



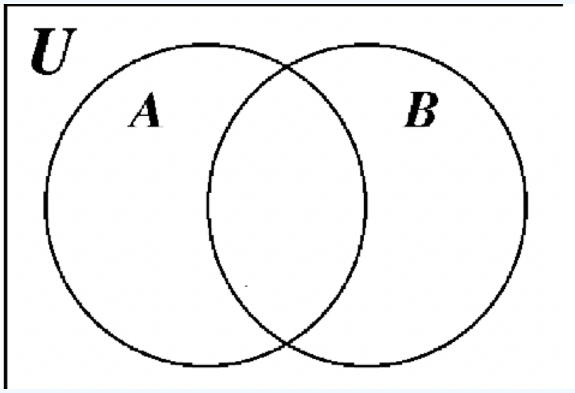
Exercise 41

C – (A ∩ B)



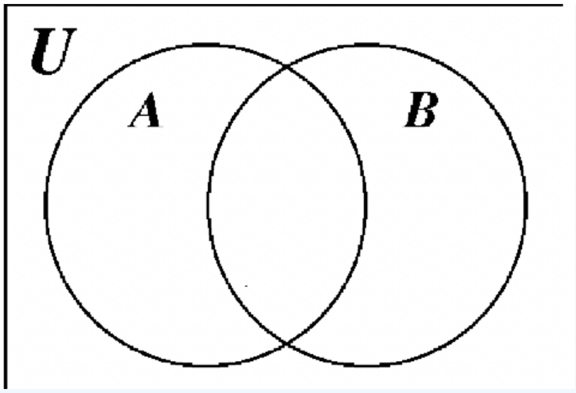
Exercise 42

(B – A) ∩ **(B – C)**



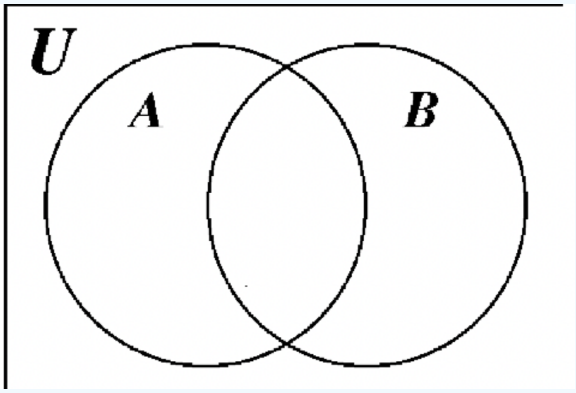
Exercise 43

(B – A) ∪ **(B – C)**



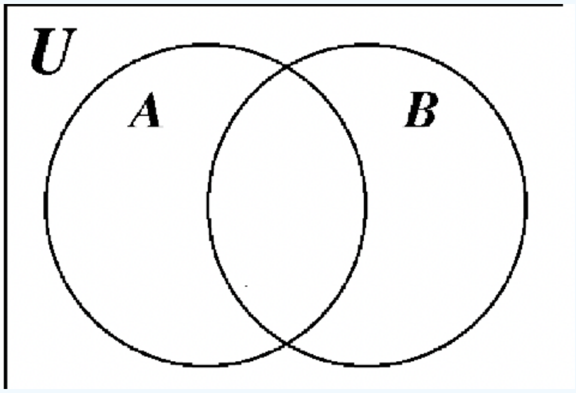
Exercise 44

$(\mathbf{A} \cup \mathbf{B})^c$



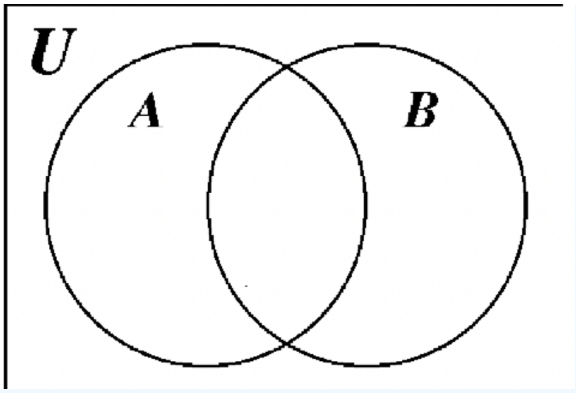
Exercise 45

$\mathbf{A}^c \cap \mathbf{B}^c$



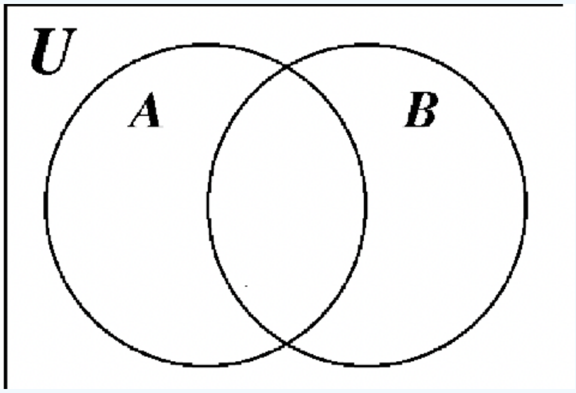
Exercise 46

$\mathbf{A}^c - \mathbf{B}^c$



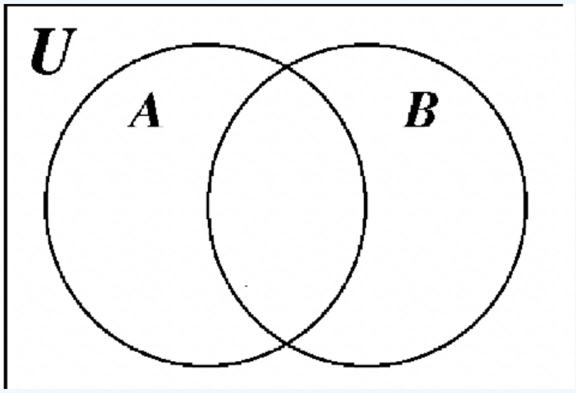
? Exercise 47

$(\mathbf{C}-\mathbf{B})^c$



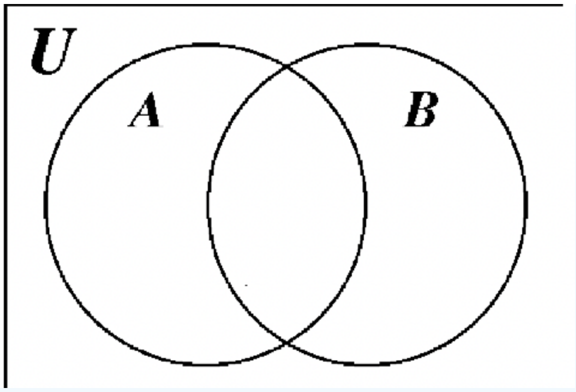
? Exercise 48

$(\mathbf{B}^c \cap \mathbf{C})-\mathbf{A}$



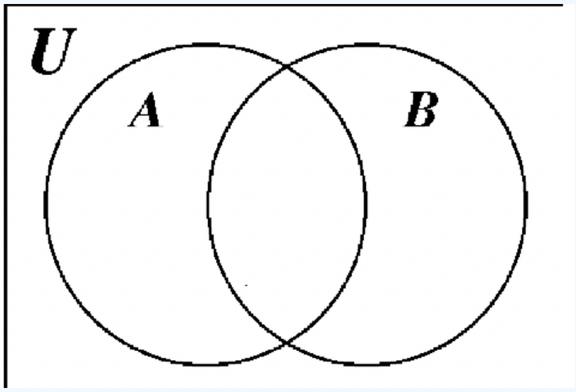
? Exercise 49

$(\mathbf{A}-(\mathbf{B}\cup\mathbf{C}))\cup(\mathbf{B}-(\mathbf{A}\cup\mathbf{C}))\cup(\mathbf{C}-(\mathbf{A}\cup\mathbf{B}))$



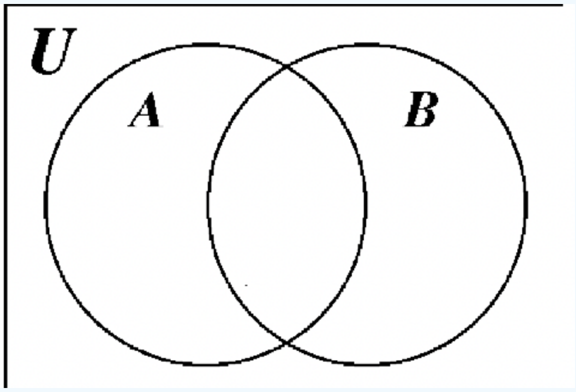
Exercise 50

$(\mathbf{A} \cap \mathbf{C})^c$



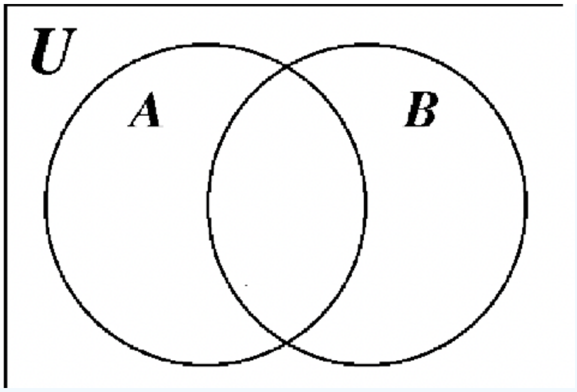
Exercise 51

$((\mathbf{A} \cap \mathbf{B}) - \mathbf{C}) \cap (\mathbf{C} - \mathbf{A})$



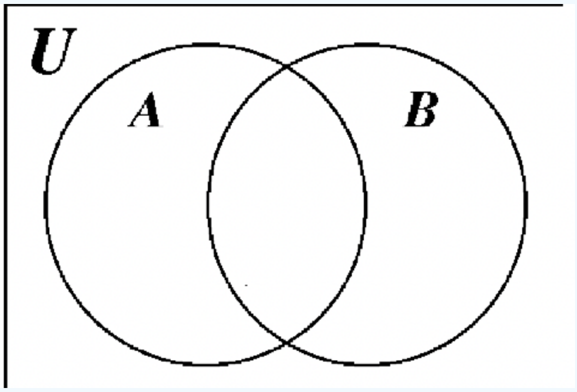
Exercise 52

$\mathbf{A}^c \cup \mathbf{C}^c$

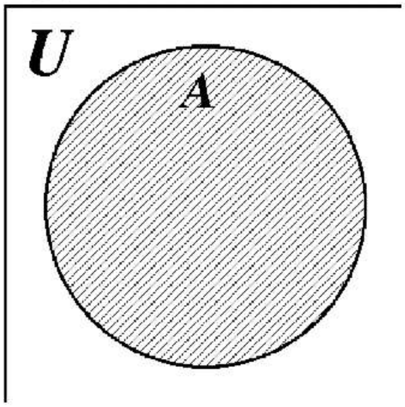


Exercise 53

B ∩ (**C** ∪ **A**^{c})



Here are the correct shadings to the exercises on the previous pages. After mastering these shadings, reverse the process by looking at the shadings on this page and try to describe them. It takes practice and patience and remember that there may be more than one way to describe some of these. In fact, many times you'll see there is a simpler way to describe them than was on the original exercise!!



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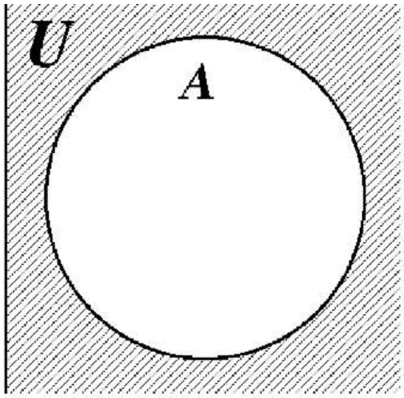

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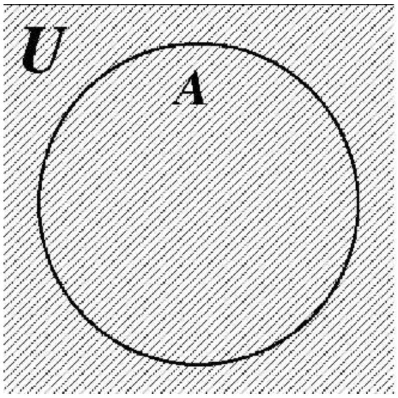

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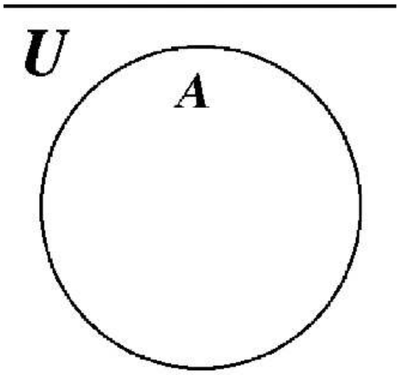

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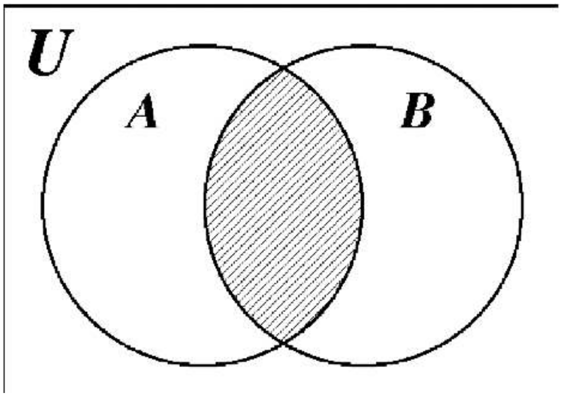

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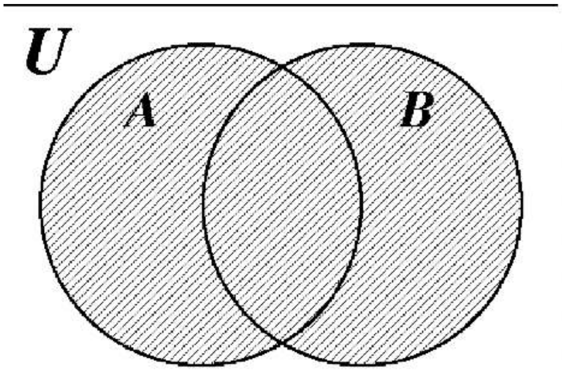

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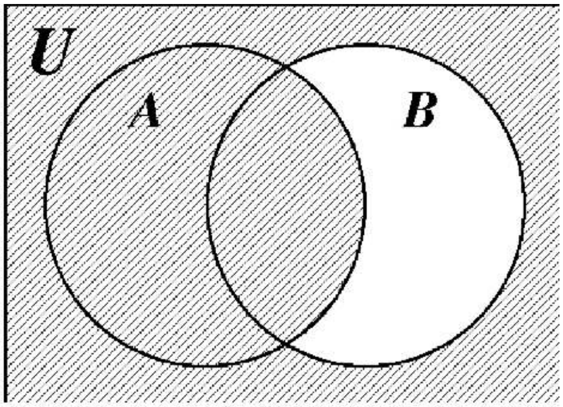

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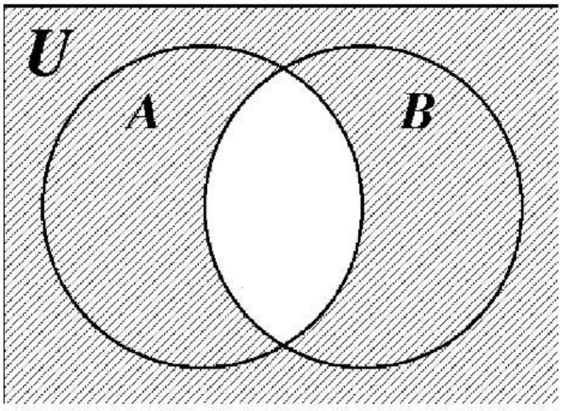

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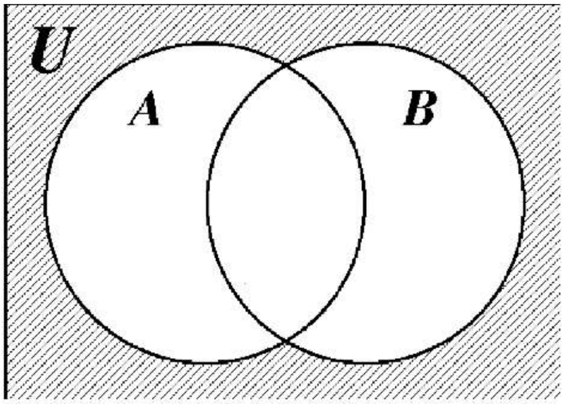

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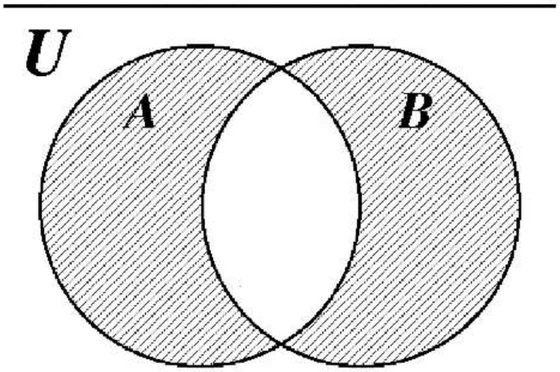

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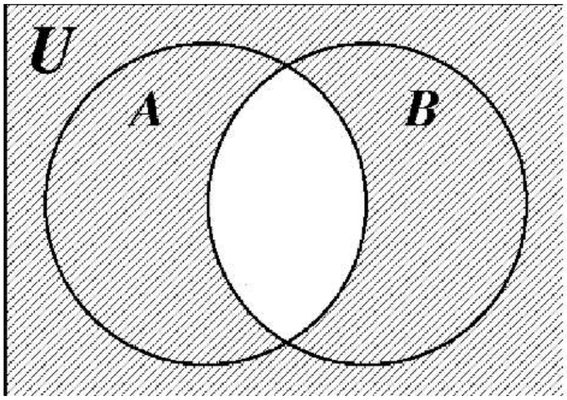

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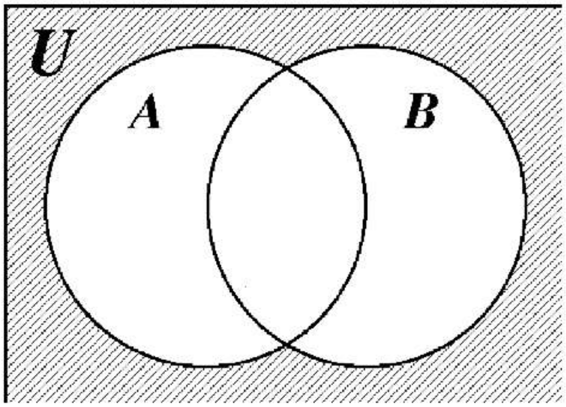

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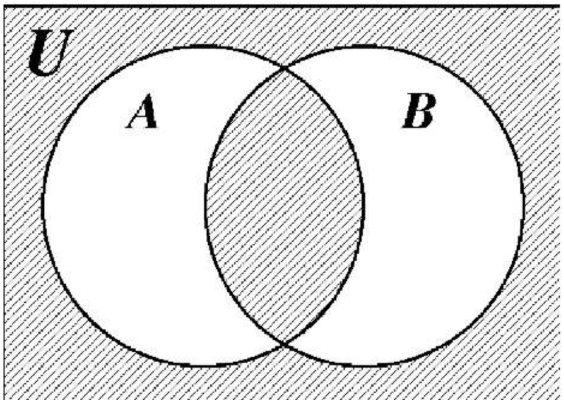

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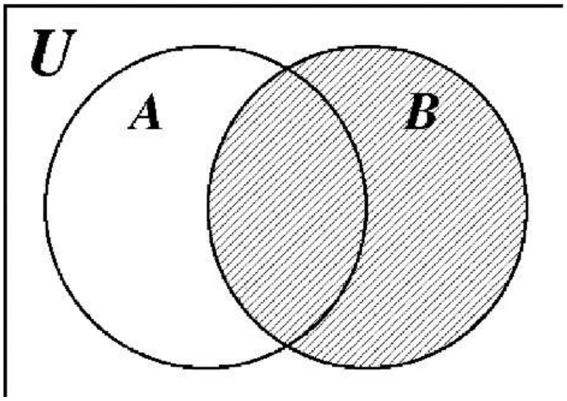

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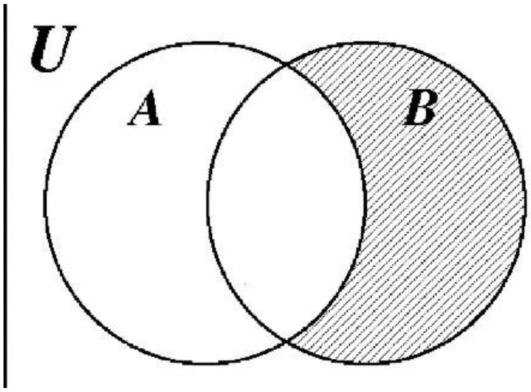

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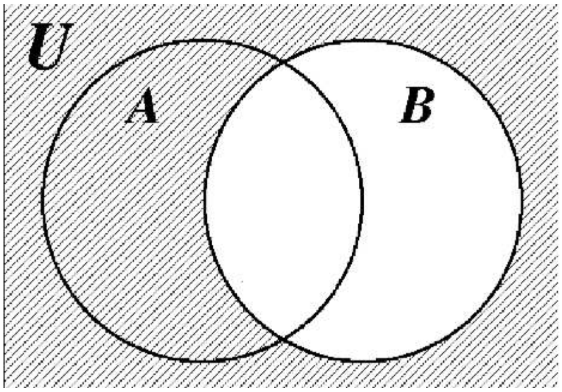

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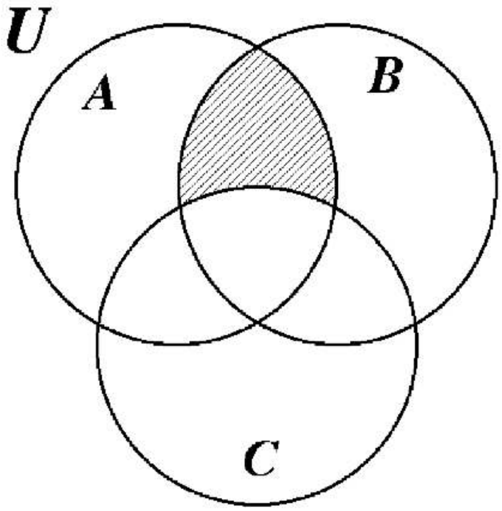

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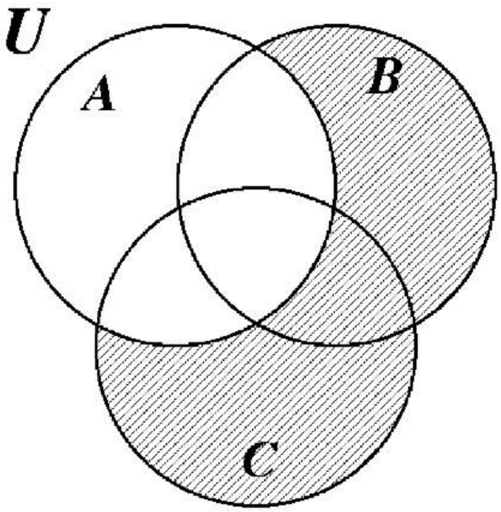

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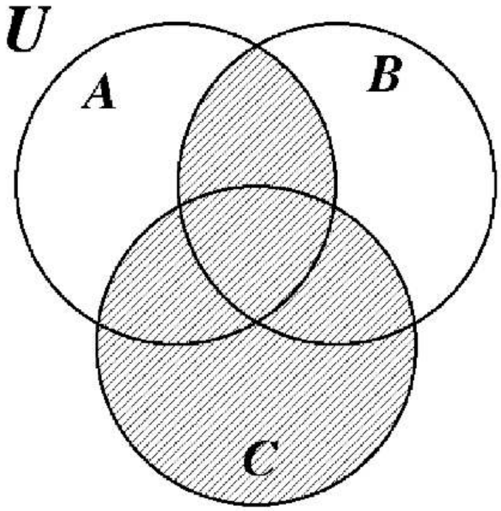

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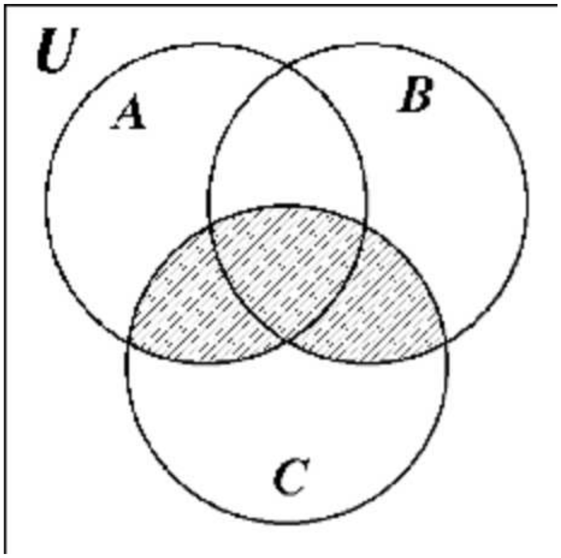

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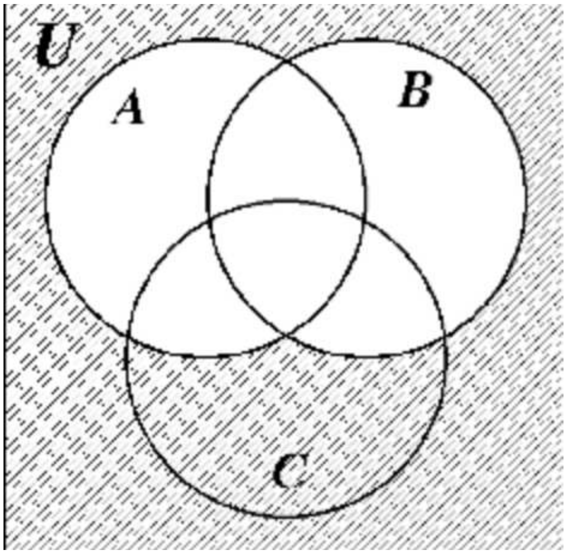
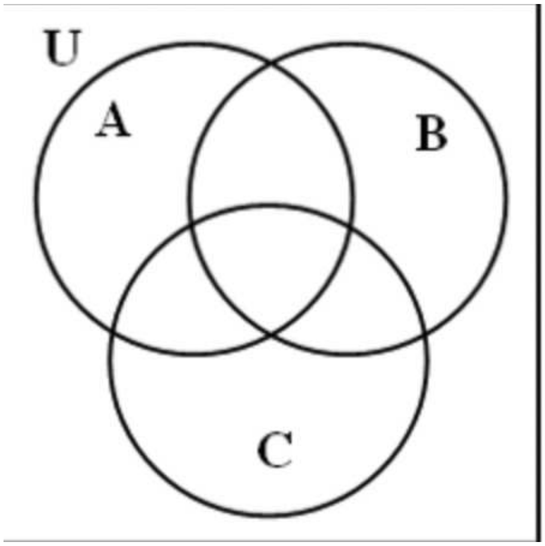
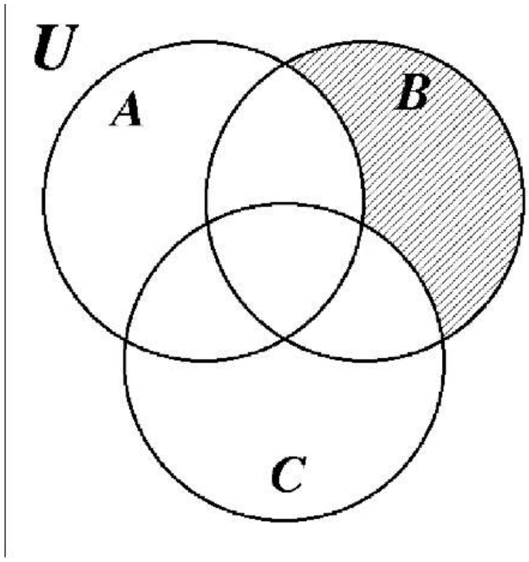


38.



39. Nothing is shaded.



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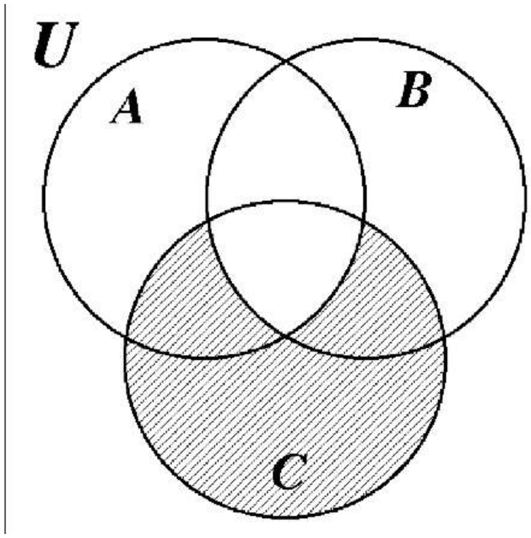

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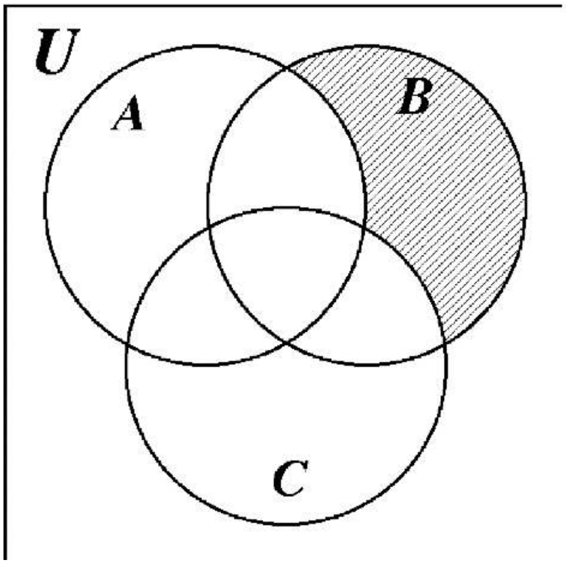

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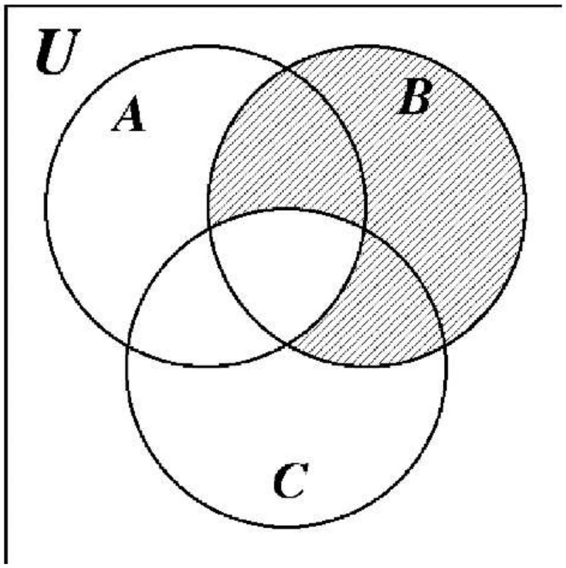

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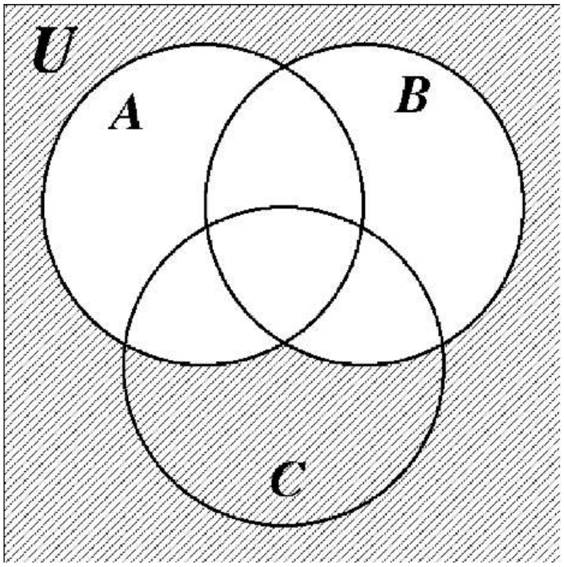

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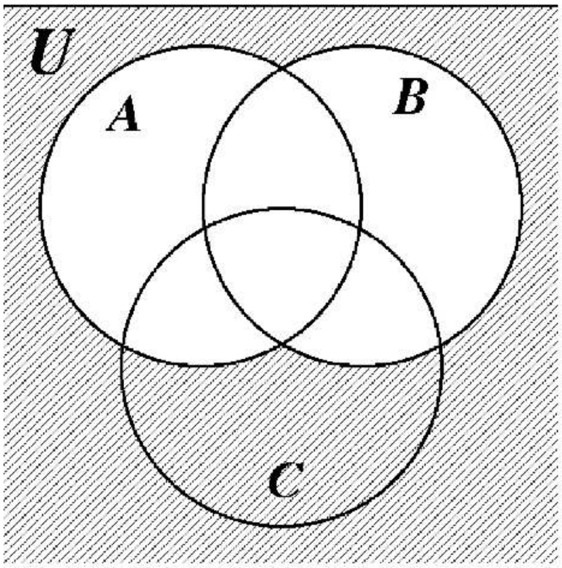

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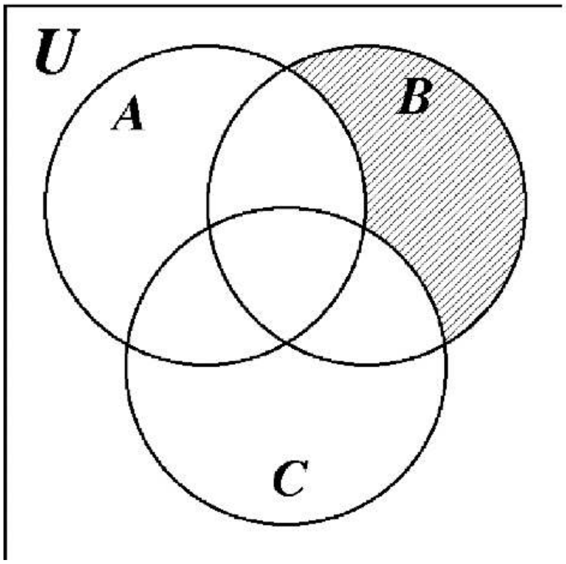

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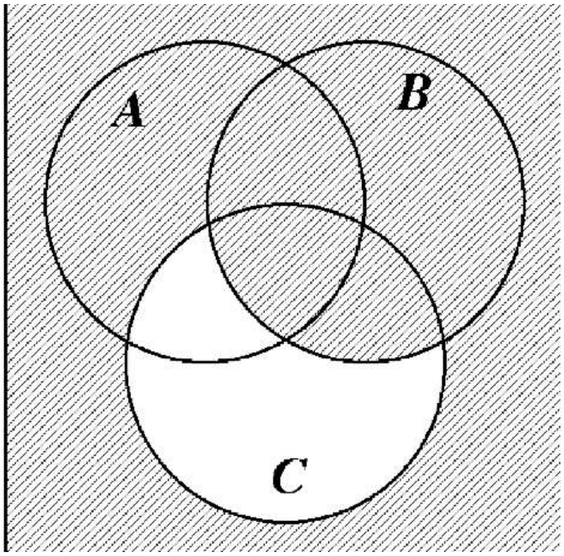

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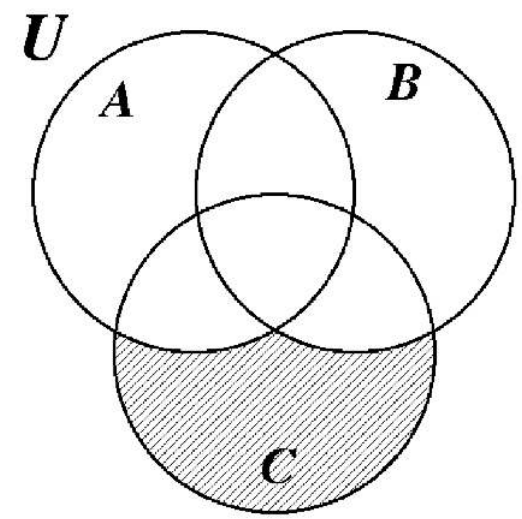

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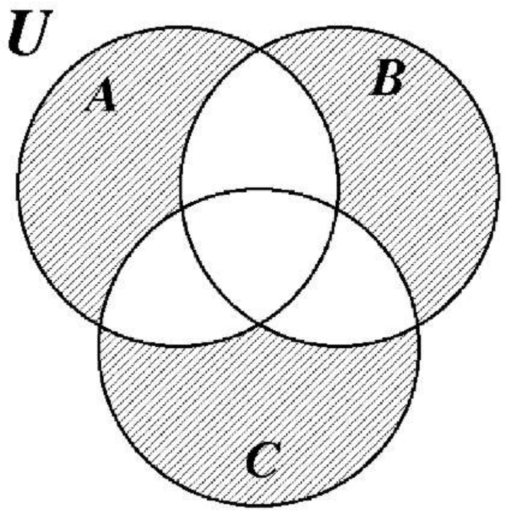
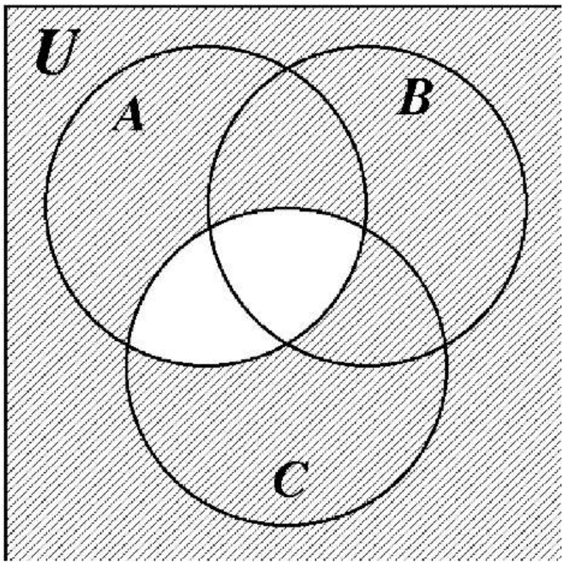
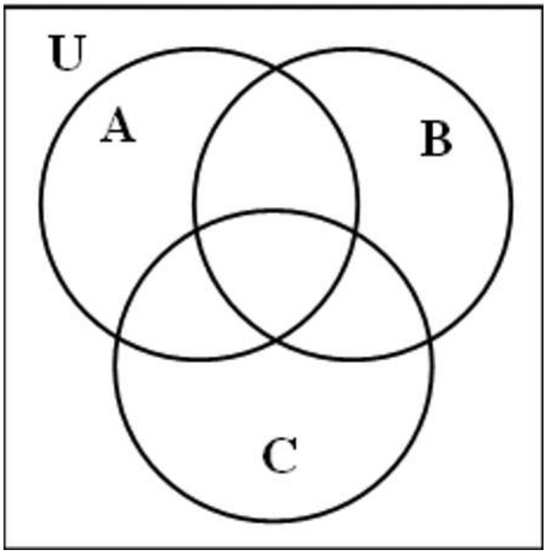


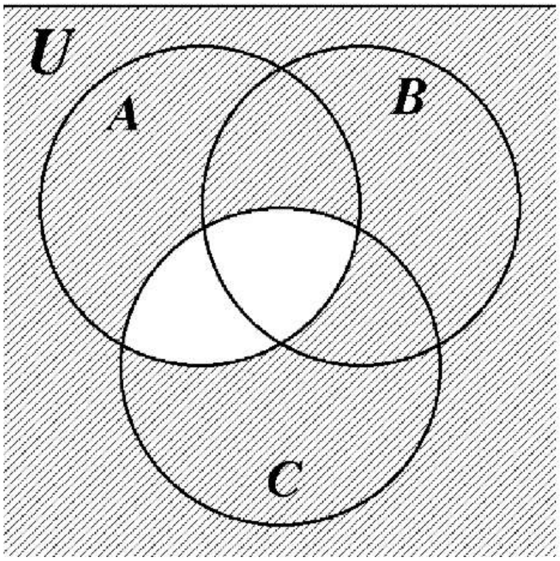
49.



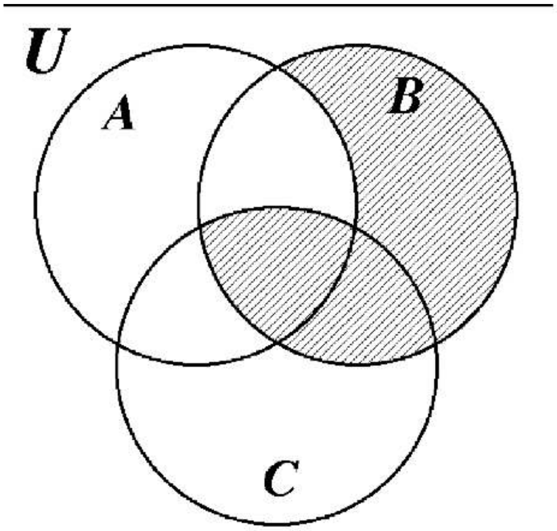
50.



51. Nothing is shaded.



52.



53.

In the Material Card section there are blank Venn diagram templates you can use for practice.

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1.3: More about Sets

You will need: Scissors, A-Blocks (Material Cards 2A-2E), a large Poster with a three set Venn diagram drawn on it. *If you don't have a poster, look at the diagram to the left. The dashes indicate where you can tape four sheets of paper or card stock (or two open file folders) together and then draw a big three set Venn diagram.*

Now that you've mastered shading Venn diagrams (don't laugh), it's time to get out your Set of A-blocks (which include the white value label cards). Each circle represents a set and we'll be labeling each circle with different value label cards (the white cards in each set of A-blocks) in the various exercises.

Exercise 1

Start by choosing a value (a particular color, shape or size). Put its value label card - **R**ed, **B**lue, **G**reen, **Y**ellow, **T**riangle, s**Q**uare, **C**ircle, **S**mall or **L**arge on one of the circles (representing a set) on the Venn diagram. Place all the A-blocks having that value (e.g., all squares) in that circle. Leave those pieces there. Now choose another value and put its label card on one of the other circles. Place all the A-blocks having this second value in this second circle. Notice that the circles overlap. Make sure a piece from one value isn't in the overlap unless it also belongs in the other circle. You can use abbreviations when answering these questions.

What was your first value? ____ What was your second value? ____

Did any pieces belong in both circles? ____ If so, which ones? ____

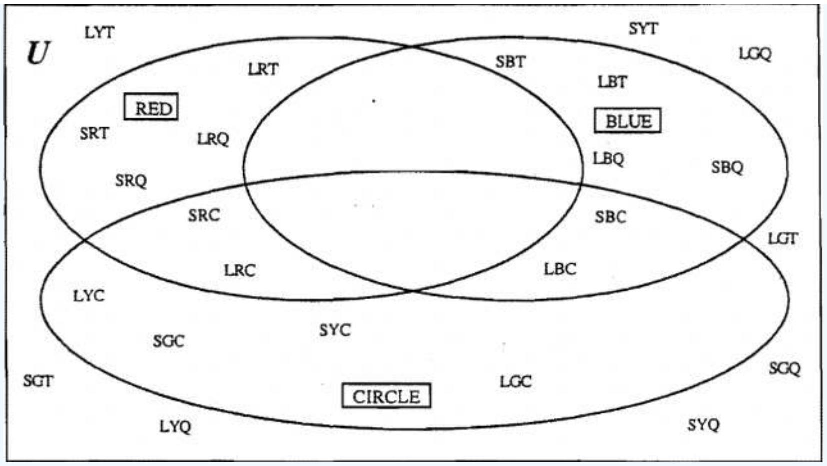
What do we call the overlap? ____

Now add a third value for the third circle on the Venn.

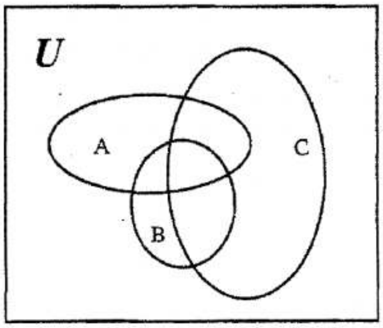
What was your third value? ____

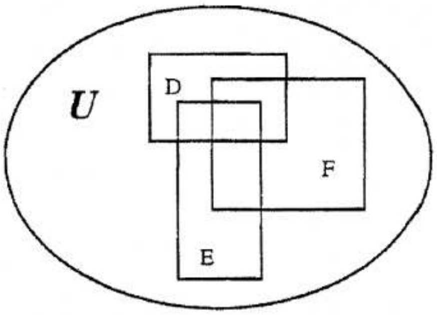
Arrange all 24 pieces on the poster so that each one is in the correct region. Note there are eight different regions on the Venn diagram. Some pieces might be outside all three circles. It all depends on what values you chose. To check yourself, look, carefully at each circle (which represents a Set), one at a time. Let's say squares was the value you chose for one circle. There are four regions in that one circle. All of the squares must be in one of those four spots. Now check each of the next two circles, one at a time. If one circle had a value of small, every single small piece must be in one of the four regions of that circle. Last, you'll check the third circle. Most people have trouble getting it exactly right the first time through. Just work until you are satisfied every element is in its proper place. Be sure that all 24 pieces are accounted for and that you check your work by looking at each set individually. Then, on the blank Venn diagram shown on the top of the next page, use abbreviations to designate where each piece was placed. Also, label each circle with the value you chose for that set. How many different regions are there in a three set Venn diagram? ____ (Hint: look at the beginning of this paragraph for the answer.)

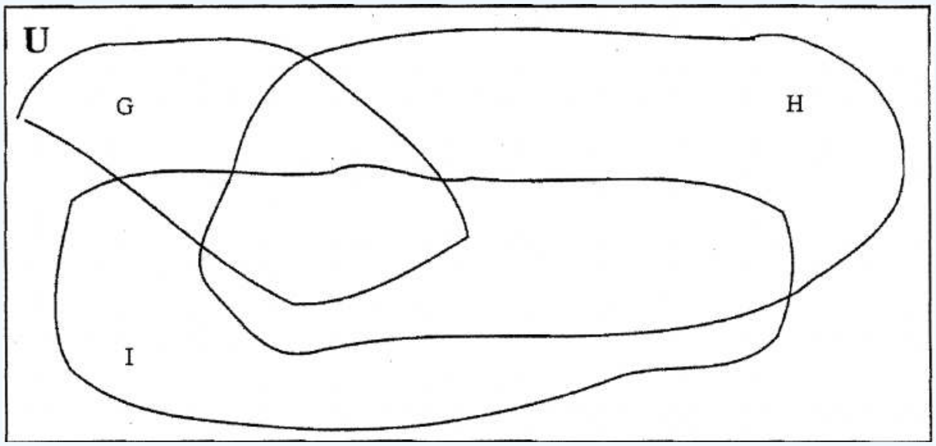
If you had trouble with this first exercise, you might want to study the Venn diagram below. The value labels chosen were RED (**R**), BLUE (**B**) and CIRCLE (**C**). The A–blocks are shown in their correct region. Notice there are two of the eight regions that are empty. That is because no element is both red and also blue! The abbreviations for the A–blocks are shown. To check, focus on one set (circle) at a time. All of the red pieces and no others must be in the set labeled RED. All of the blue pieces and no others must be in the set labeled BLUE and all of the circles and no others must be in the set labeled CIRCLE. The remaining eight pieces are outside of the three sets. All 24 pieces must be accounted for.



There is a point I want to make about the drawing of Venn diagrams. The sets you draw on a given Venn diagram do not have to be circles and they don't have to be the same size either. You just have to enclose some space. In the previous problem, you may have noticed that I stretched the sets out. I did that so it would be less crowded to fill in the placement of A–blocks. Below are three Venn diagrams each containing three sets. Although they are unconventional, they are legitimate and work fine. It is just usually convenient to make one template and stick with it for uniformity.





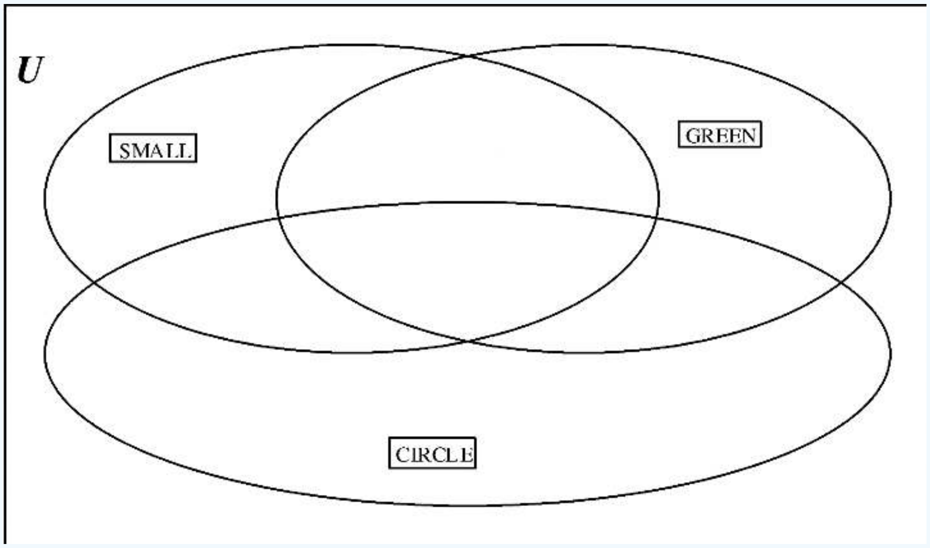


In the next few exercises, I've stretched the sets out for you so it will be less crowded to fill in the abbreviations for the placement of A–blocks.

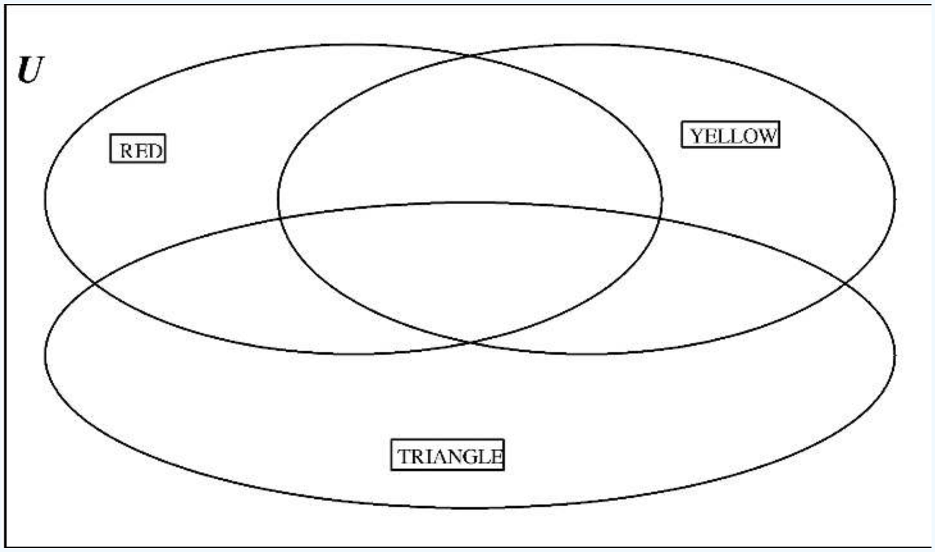
? Exercise 2

Repeat exercise #1 on each of the next few Venn diagrams but use the three value label cards shown for your three sets. First, use your A–blocks to place each element in the right region on the poster. Be sure you place all 24 pieces somewhere within the Venn diagram (there are eight different regions, and each piece belongs in exactly one of those eight regions.) Check your work by looking at each set individually. After you feel confident you have all pieces placed in the correct region, check your solution against the solution shown in the solution module. Then, on each Venn diagram shown on the page, write where you placed the pieces – use three letter abbreviations or a picture representation. You should feel confident that given any particular A–block, you could place it in the correct region.

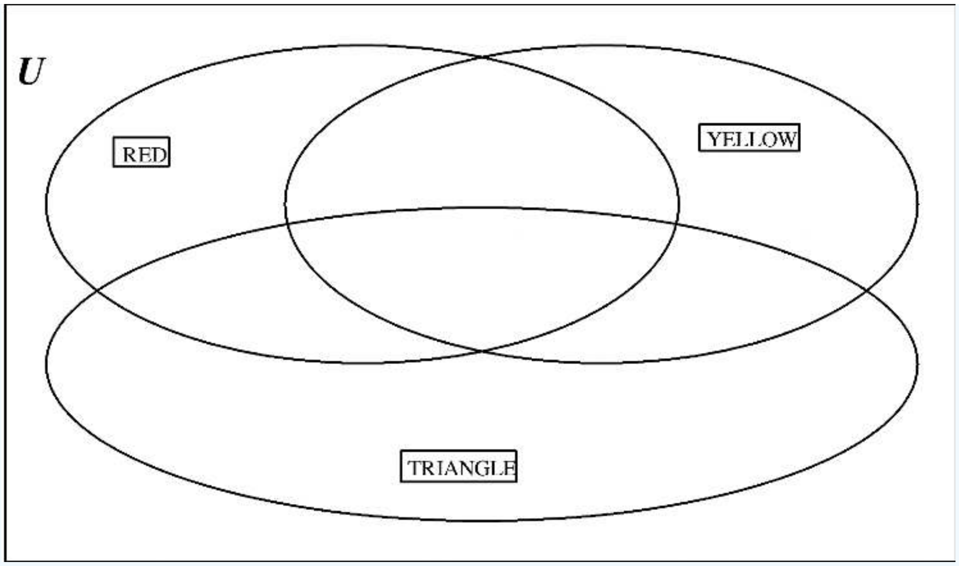
a. Values: SMALL (**S**), GREEN (**G**), CIRCLE (**C**)


U
SMALL
GREEN
CIRCLE

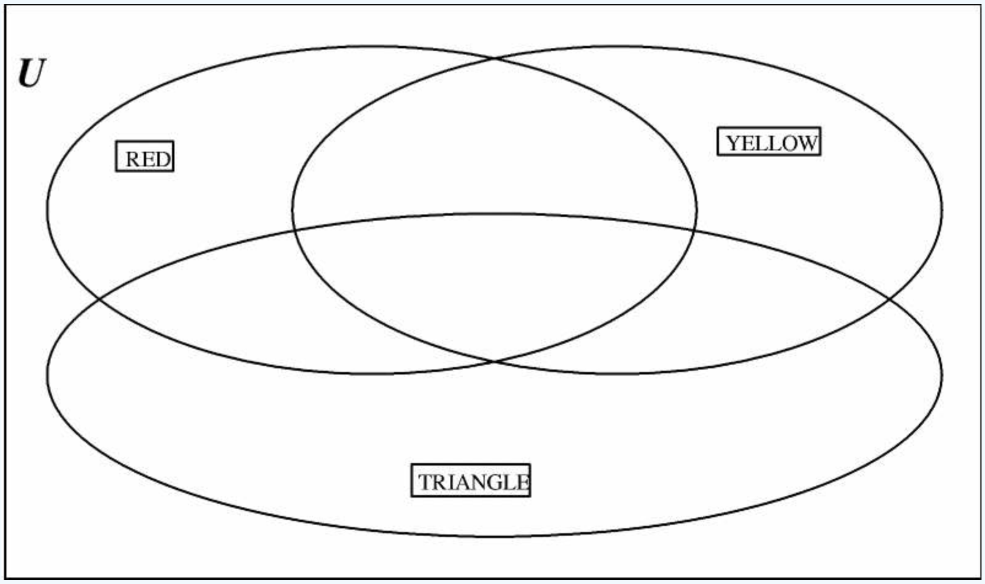
b. Values: RED (**R**), YELLOW (**Y**), TRIANGLE (**T**)


U
RED
YELLOW
TRIANGLE

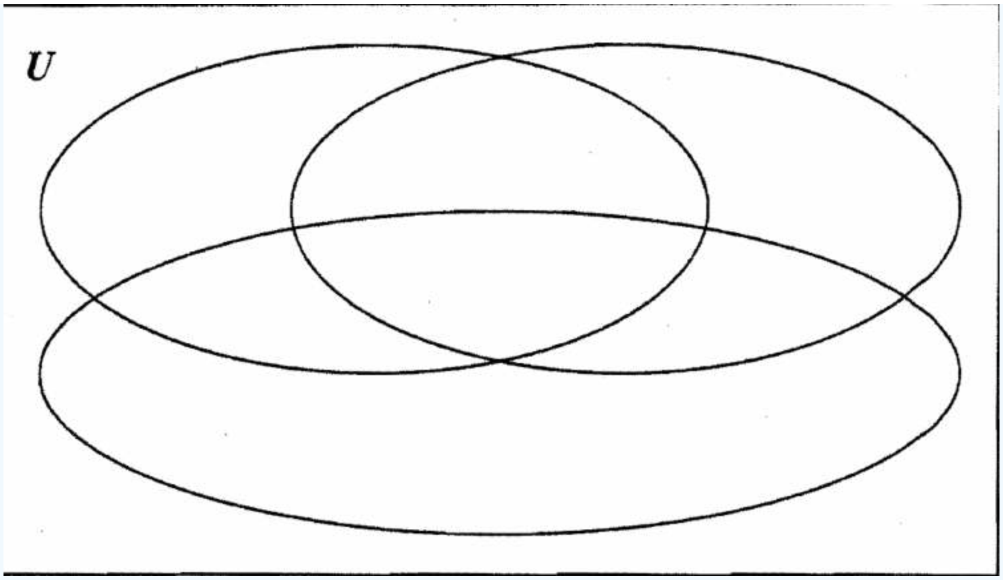
c. Values: SQUARE (**Q**), CIRCLE (**C**), TRIANGLE (**T**)


U
RED
YELLOW
TRIANGLE

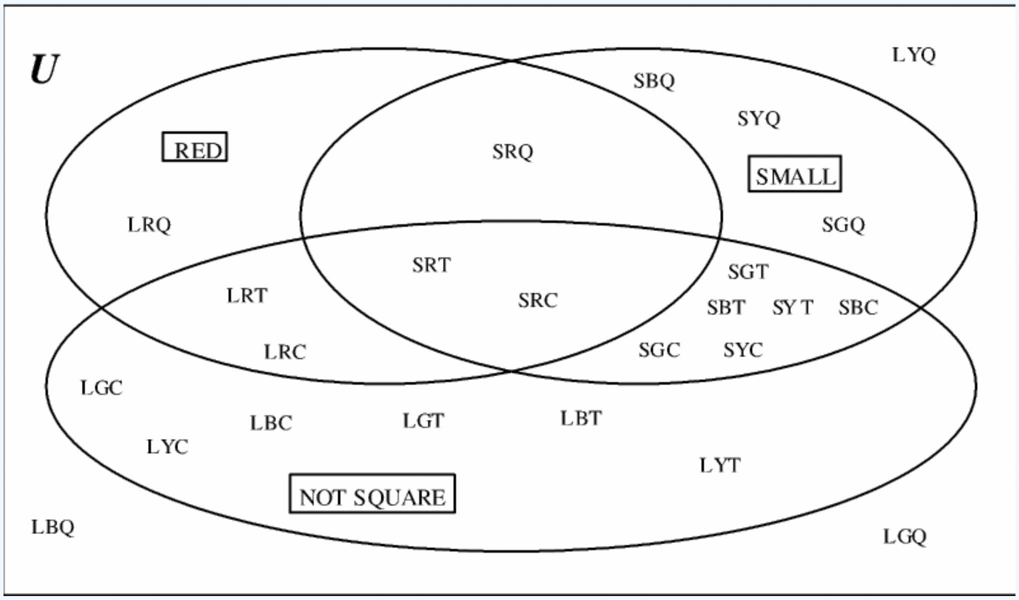
d. Values: BLUE (**B**), RED (**R**), GREEN (**G**)


U
RED
YELLOW
TRIANGLE

e. Make up a new problem of your own now, labeling each set with a value from your value label cards as in the previous exercises. Show the solution of where each block was placed on the Venn diagram below.

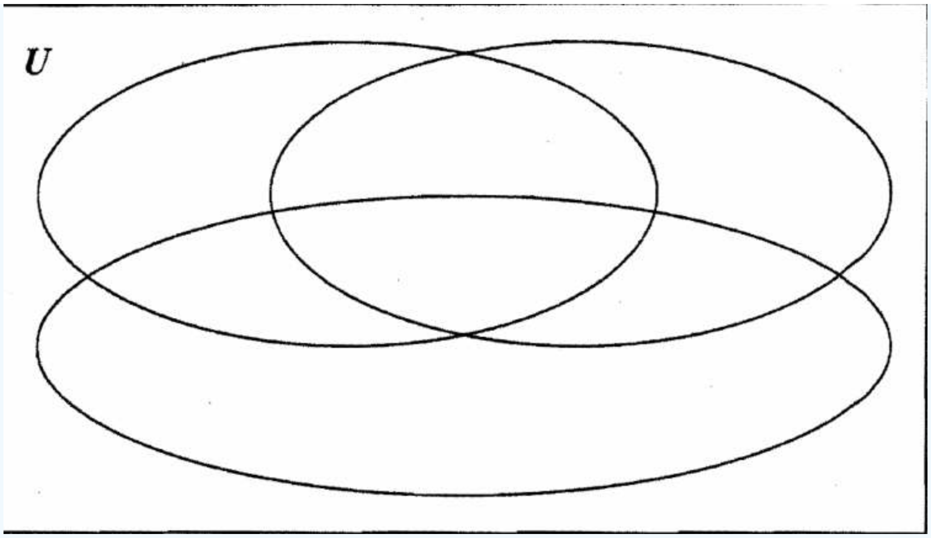


Below is a picture of a Venn diagram where the labels NOT SQUARE ($\mathbf{Q}^c$), RED (**R**) and SMALL (**S**) are chosen. It is easy to check because I make sure only red pieces and all red pieces are in the RED circle, all the small and no large pieces are in the SMALL circle and only the squares are outside the NOT SQUARE circle. All 24 pieces are accounted for.


U
LYQ
SBQ
SYQ
RED
SRQ
SMALL
LRQ
SGQ
SRT
SGT
LRT
SRC
SBT
SY T
SBC
LRC
SGC
SYC
LGC
LBC
LGT
LBT
LYC
LYT
NOT SQUARE
LBQ
LGQ

? Exercise 3

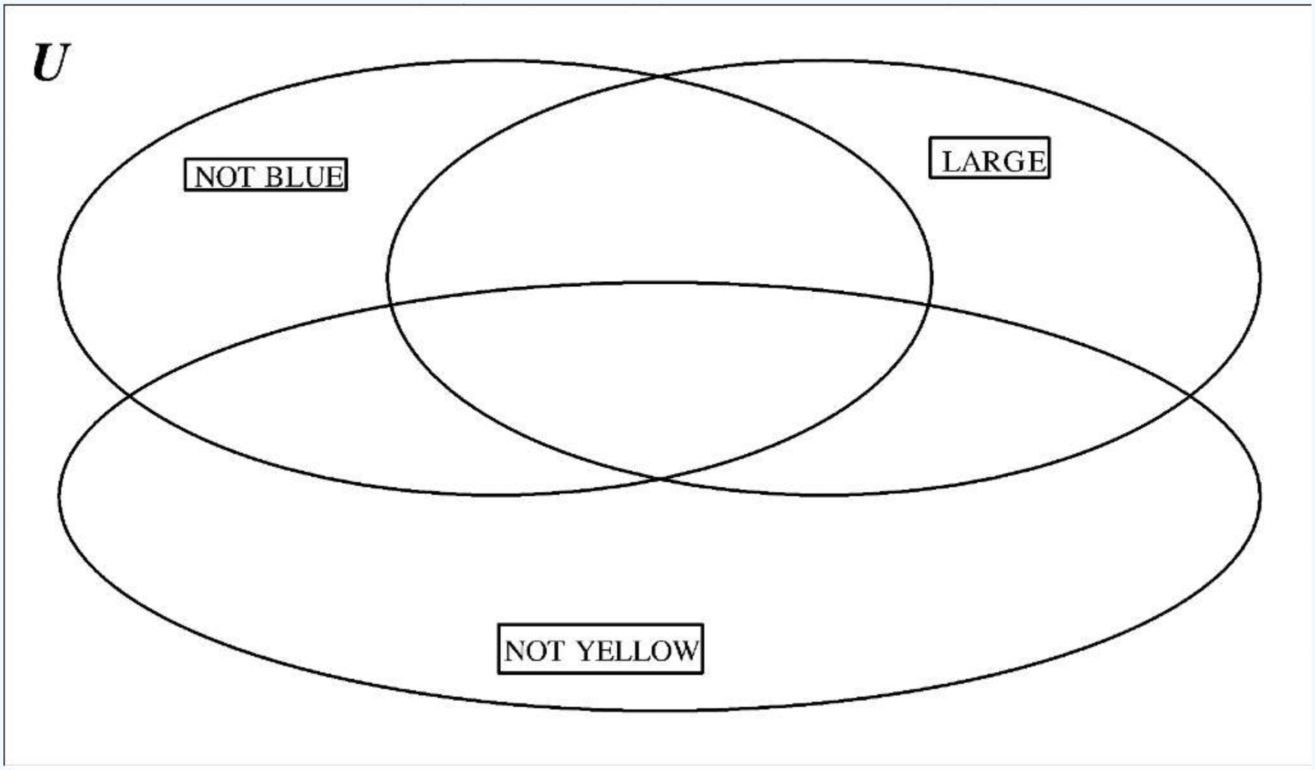
Use some negative (NOT) values now. SMALL and LARGE can be thought of as positive or negative values because SMALL is the same as NOT LARGE. Label each set with a negative value from your value label cards. After placing the pieces on the poster and checking, show the solution of where at least eight A–blocks were placed on the Venn diagram below.



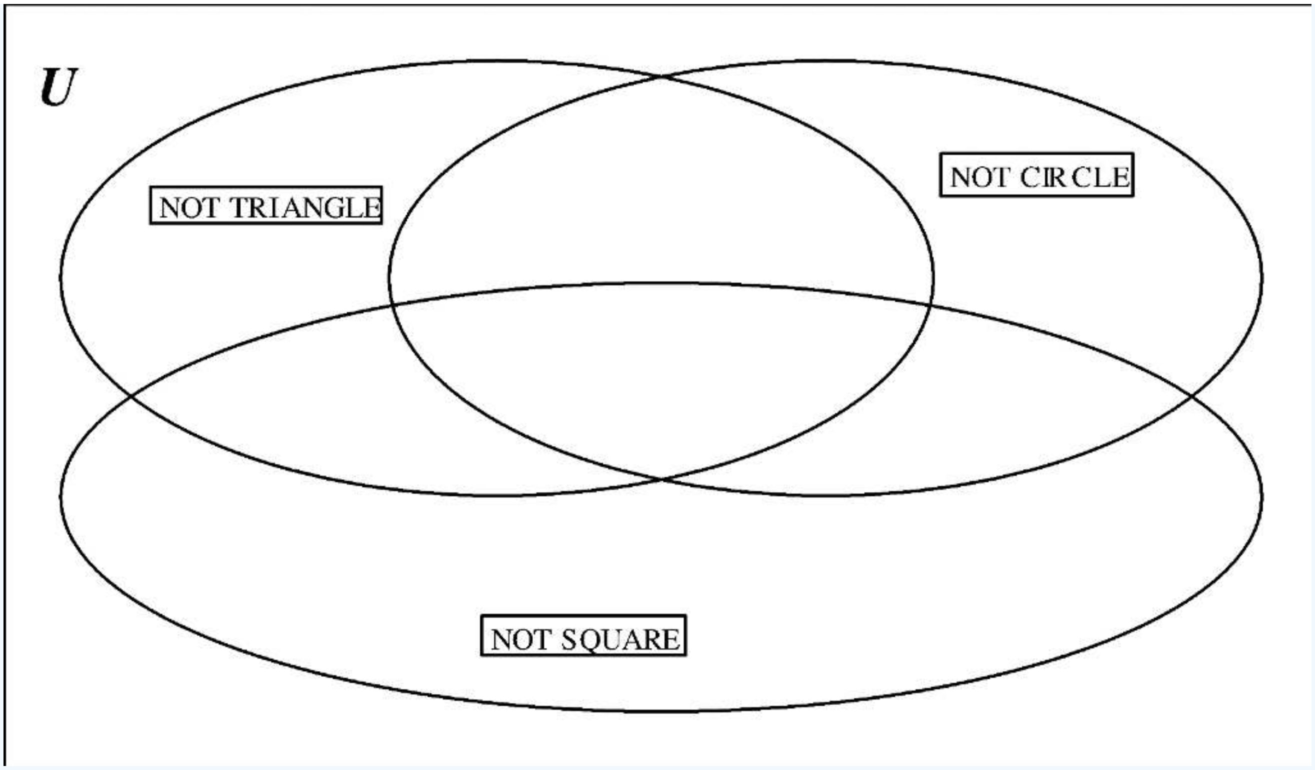
? Exercise 4

Some negative value label cards are shown on the next few Venn diagrams. Use your A–blocks to place each A–block in the correct region on the poster. Make sure you account for all 24 pieces and check your work by looking at each set individually. Then, on each Venn diagram shown, use abbreviations to designate where eight pieces were placed.

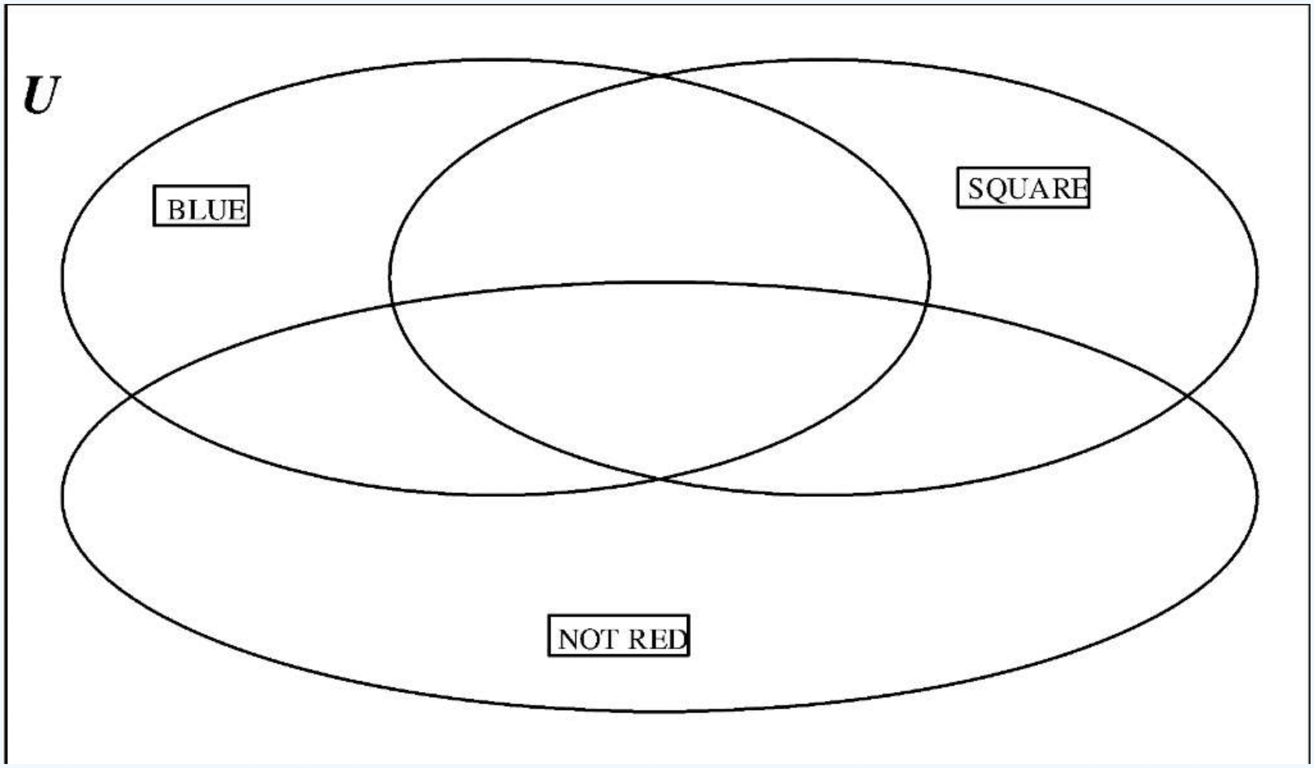
a. Values: NOT BLUE ($\mathbf{B}^c$), LARGE ($\mathbf{L}$), NOT YELLOW ($\mathbf{Y}^c$)



b. Values: NOT TRIANGLE ($\bar{\mathbf{T}}$), NOT CIRCLE ($\bar{\mathbf{C}}$), NOT SQUARE ($\bar{\mathbf{Q}}$)


U
NOT TRIANGLE
NOT CIRCLE
NOT SQUARE

c. Values: BLUE (**B**), SQUARE (**Q**), NOT RED ($\mathbf{R}^c$)


U
BLUE
SQUARE
NOT RED

Exercise 5

Name some combinations of value labels that leave no pieces outside the three sets.

Exercise 6

Name some combinations that leave exactly one piece outside the three sets. [Hint 1: Pick a piece you want to be left out. Hint 2: Use some NOT labels.]

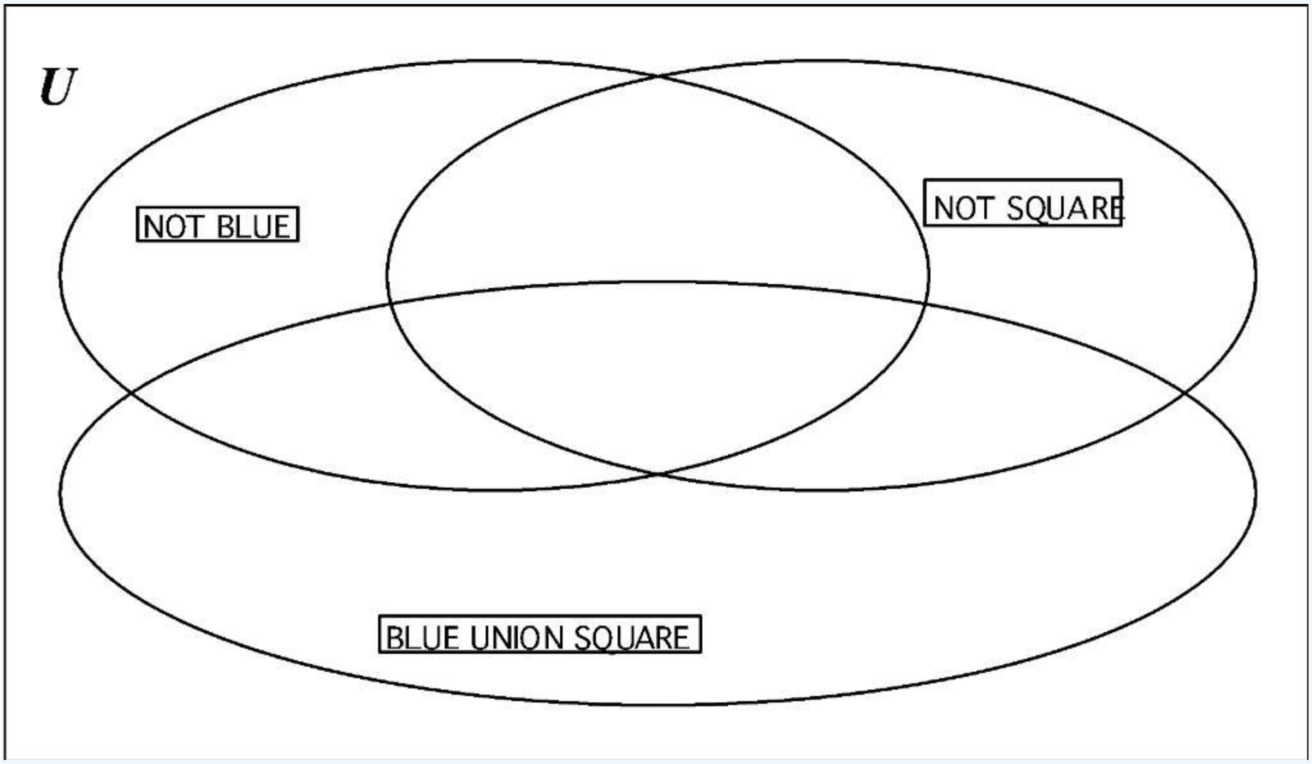
Exercise 7

Write down some combinations that leave the most number of pieces outside the sets. What do you think is the most number of pieces that can be left outside the three circles?

Exercise 8

Take one of your blank value label cards and on it, write $\mathbf{B} \cup \mathbf{Q}$ (for the UNION of BLUE and SQUARE). Take the label cards NOT BLUE ($\mathbf{B}^c$) and NOT SQUARE ($\mathbf{Q}^c$) from your set. On the poster, put each label on one of the sets. Place each A–block in the correct region.

a. On the Venn diagram below, show where you placed each A – Block.



b. Look at the set BLUE UNION SQUARE ($\mathbf{B} \cup \mathbf{Q}$). Look at all of the pieces *outside* the set $\mathbf{B} \cup \mathbf{Q}$ so you can list the elements in $(\mathbf{B} \cup \mathbf{Q})^c$: ____

c. Look at the intersection of NOT BLUE and NOT SQUARE $\mathbf{B} \cap \mathbf{Q}^c$, and list its elements : ____

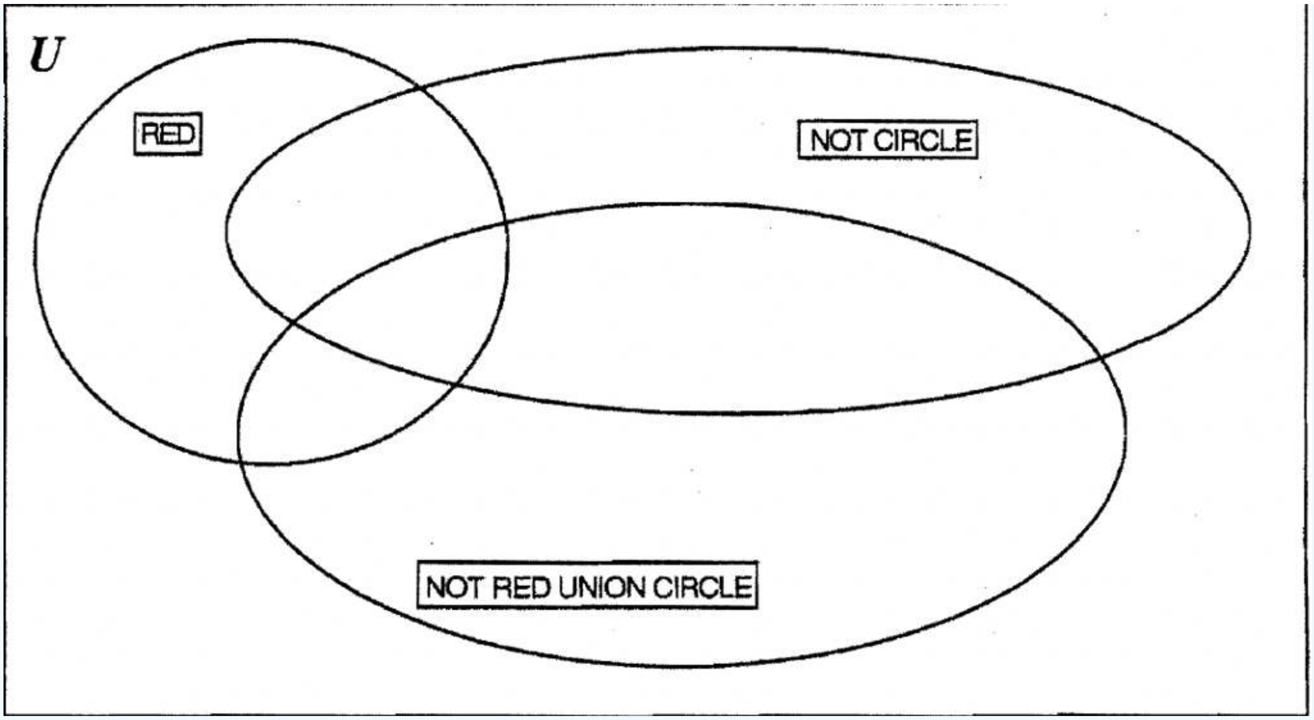
d. What do you notice about the elements listed in part b and part c above? ____

e. Write an equation from what you noticed. ____

Exercise 9

Make a label card NOT RED UNION CIRCLE ($\mathbf{R}^c \cup \mathbf{C}$) and take the labels RED ($\mathbf{R}$) and NOT CIRCLE ($\mathbf{C}^c$) from your set.

a. Following the directions of exercise 8 with these three new labels, show where you placed each A–block on the Venn diagram below.



b. Look at the set NOT RED UNION CIRCLE ($\mathbf{R}^c \cup \mathbf{C}$). Look at all of the pieces *outside* the set $\mathbf{R}^c \cup \mathbf{C}$ so you can list the elements in $(\mathbf{R}^c \cup \mathbf{C})^c$: ____

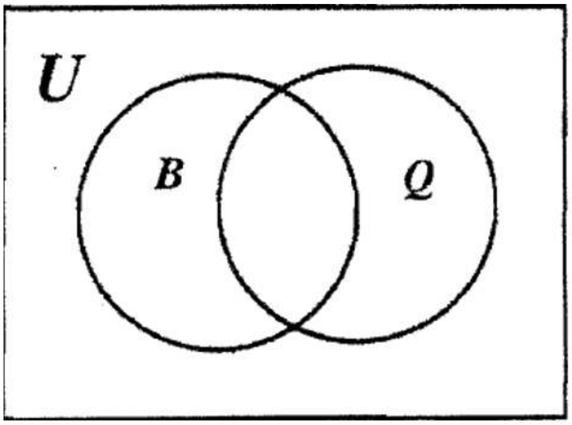
c. Look at the intersection of RED and NOT CIRCLE ($\mathbf{R} \cap \mathbf{C}^c$) and list its elements: ____

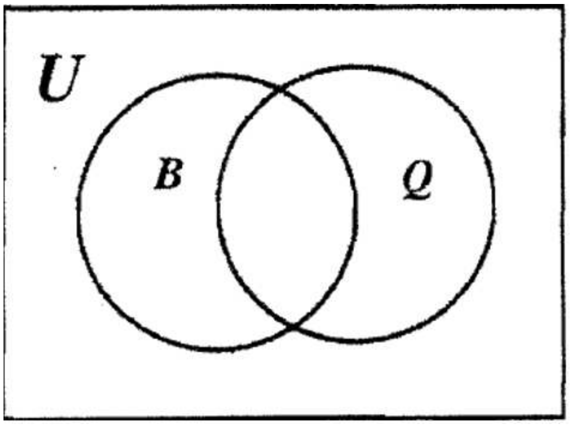
d. What do you notice about the elements listed in part b and part c above? ____

e. Write an equation from what you noticed. ____

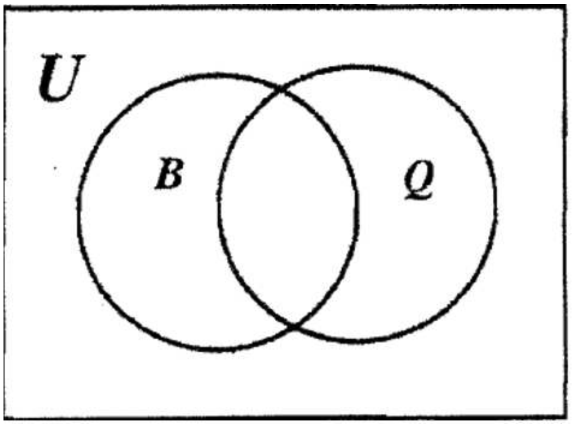
Exercise 10

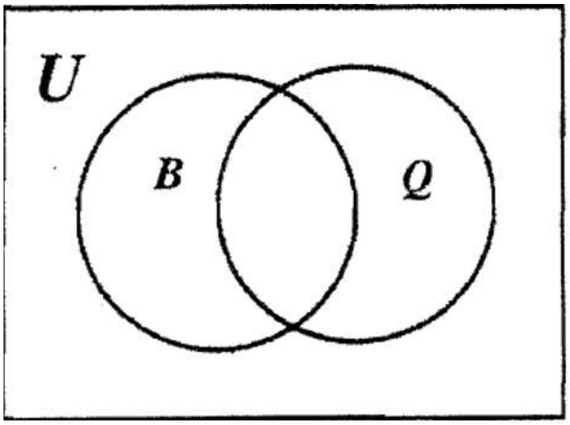
a. Below, on the Venn diagram to the left, shade with vertical lines and shade $\mathbf{B}^c$ with horizontal lines. Look at the overlap of both lines and shade this $\mathbf{Q}^c$ intersection on the Venn diagram to the right. Therefore, the Venn on the right represents ($\mathbf{B}^c \cap \mathbf{Q}^c$).





b. Below, on the Venn diagram to the left, shade $\mathbf{B} \cup \mathbf{Q}$. On the Venn diagram to the right, shade everything except $\mathbf{B} \cup \mathbf{Q}$. Then, the Venn on the right represents $(\mathbf{B} \cup \mathbf{Q})^c$.





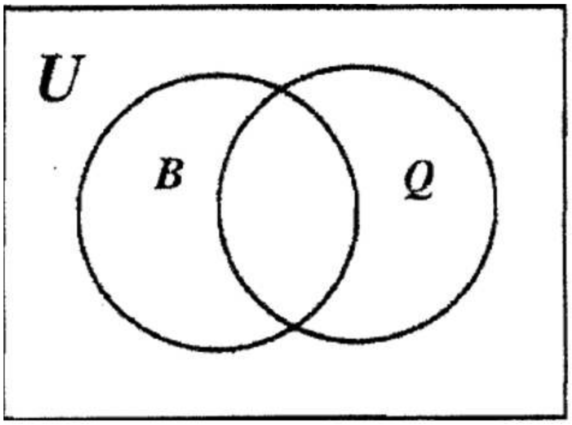
c. What do you notice about the shadings for $(\mathbf{B}^c \cap \mathbf{Q}^c)$ and $(\mathbf{B} \cup \mathbf{Q})^c$ (the Venns on the right for a and b)?

d. Write an equation from what you noticed ___

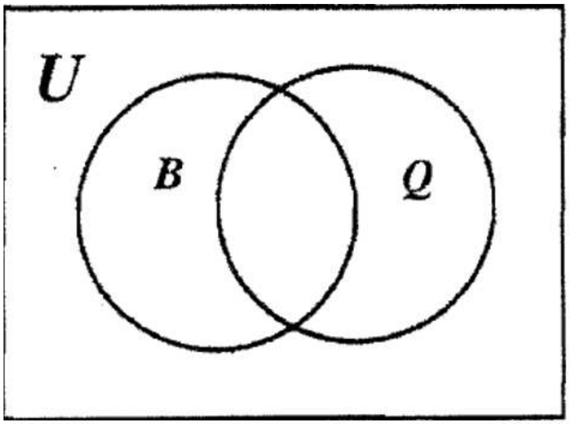
Exercise 11

Below, on the Venn diagram to the left, shade $(\mathbf{A} \cap \mathbf{B})^c$. On the Venn diagram to the right, shade $\mathbf{A}^c$ with vertical lines and $\mathbf{B}^c$with horizontal lines. Together, the *total shading* on the right represents $\mathbf{A}^c \cup \mathbf{B}^c$.

a. $(\mathbf{A} \cap \mathbf{B})^c$



b. $\mathbf{A}^c \cup \mathbf{B}^c$



c. What do you notice about the shadings for $(\mathbf{A} \cap \mathbf{B})^c$ and $\mathbf{A}^c \cup \mathbf{B}^c$?

d. Write an equation from what you noticed ____

Exercises 8, 9, 10 and 11 illustrate examples of deMorgan's Laws.

One law states that $(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c \cap \mathbf{B}^c$.

The other deMorgan's Law states that $(\mathbf{A} \cap \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$.

Remember that **A** and **B** are just dummy variables.

Basically, in English, the first of deMorgan's Laws ($(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c \cap \mathbf{B}^c$) says that the complement of the union of two sets is equal to the intersection of the complement of each set. Some examples of this law follow.

$(\mathbf{M} \cup \mathbf{N})^c = \mathbf{M}^c \cap \mathbf{N}^c$	$(\mathbf{L}^c \cup \mathbf{K})^c = \mathbf{L} \cap \mathbf{K}^c$	$(\mathbf{X}^c \cup \mathbf{Y}^c)^c = \mathbf{X} \cap \mathbf{Y}$

I like to think of it as distributing the complement over the parentheses. In order to remove the parentheses, take the complement of each of the two sets in parentheses and change the union to intersection.

The second of deMorgan's Laws ($(\mathbf{A} \cap \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$) says the complement of the intersection of two sets is equal to the union of the complement of each set. Again, it's like distributing the complement over the parentheses. But this time, the intersection changes to union when the parentheses are removed. Here are a few examples of this law:

$(\mathbf{M} \cap \mathbf{N})^c = \mathbf{M}^c \cup \mathbf{N}^c$	$(\mathbf{L}^c \cap \mathbf{K})^c = \mathbf{L} \cup \mathbf{K}^c$	$(\mathbf{X}^c \cap \mathbf{Y}^c)^c = \mathbf{X} \cup \mathbf{Y}$

If you are given the union or intersection of any two sets, you can use deMorgan's Law to write it as the complement of a quantity. For instance, if someone asked you to write an equivalent statement for $\mathbf{R} \cup \mathbf{S}^c$, you use deMorgan's Laws to write it as $(\mathbf{R}^c \cap \mathbf{S})^c$. Note how this is done: To get parentheses with a complement outside it when there is none to begin with, as in $\mathbf{R} \cup \mathbf{S}^c$, begin by putting parentheses with a complement and room to put something inside like this: $\mathbf{R} \cup \mathbf{S}^c$ = $(\ )^c$. To figure out what goes inside, change the three original parts to their "opposites." From R to R^c, from $\cup$ to $\cap$, and from S^c to S. This gives use $\mathbf{R} \cup \mathbf{S}^c$ = $(\mathbf{R}^c \cap \mathbf{S})^c$. You'll have to use this technique for 12b, d and f.

? Exercise 12

Use deMorgan's Laws to write an equivalent expression for each statement below.

$(\mathbf{R}^c \cap \mathbf{S}^c)^c$	$\mathbf{M} \cap \mathbf{N}$
$(\mathbf{F} \cup \mathbf{G}^c)^c$	$\mathbf{H}^c \cup \mathbf{I}$
$(\mathbf{P}^c \cap \mathbf{Q})^c$	$\mathbf{S} \cap \mathbf{T}^c$

? Exercise 13

Make up two of your own equivalent statements (like in exercise 12) to demonstrate deMorgan's Law.

a.	b.

It's time to demonstrate deMorgan's Laws in yet another way. Back in Exercise Set 1, exercise 13, you worked on problems where we let **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9}, and we defined these subsets of **U**: **A** = {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}. We'll use these defined sets to compute the following and therefore verify deMorgan's Laws:

Example

Show that = $(\mathbf{A} \cap \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$.

Solution

To show the two sides are equal, *first simplify the left side* by computing the intersection of A and B, (by listing the elements of each and then listing the elements in the intersection). Then take the complement of that set.

$(\mathbf{A} \cap \mathbf{B})^c = (\{\mathbf{1,2,3,4,5}\} \cap \{\mathbf{2,4,6,8}\})^c = (\{\mathbf{2,4}\})^c = \{\mathbf{1,3,5,6,7,8,9}\}$

Second, *simplify the right side* by computing the complement of A and also the complement of B, and then take the union of those two sets.

$\mathbf{A}^c \cup \mathbf{B}^c = (\{\mathbf{1,2,3,4,5}\})^c \cup (\{\mathbf{2,4,6,8}\})^c = \{\mathbf{6,7,8,9}\} \cup \{\mathbf{1,3,5,7,9}\} = \{\mathbf{1,3,5,6,7,8,9}\}$

Since $(\mathbf{A} \cap \mathbf{B})^c$ and $\mathbf{A}^c \cup \mathbf{B}^c$ have the exact same elements, they are equal.

Therefore, $(\mathbf{A} \cap \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$

? Exercise 14

Let **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9} and these defined subsets:

A = {1, 2, 3, 4, 5}	**B** = {2, 4, 6, 8}	**C** = {3, 5, 7}

Verify the following. **Show *all the steps* as done in the above example.** State the final conclusion in words.

a. Show that $(\mathbf{A}\cup\mathbf{B})^{c} = \mathbf{A}^{c}\cap\mathbf{B}^{c}$

b. Show that $(\mathbf{A}\cap\mathbf{C})^{c} = \mathbf{A}^{c}\cup\mathbf{C}^{c}$

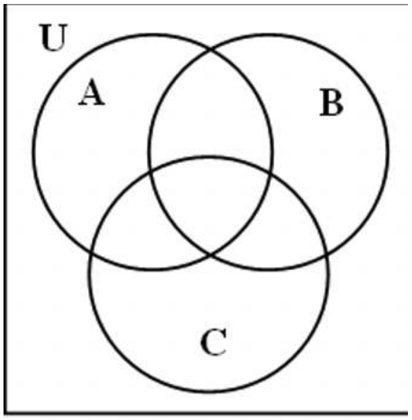
c. Show that $(\mathbf{B}^{c}\cup\mathbf{C})^{c} = \mathbf{B}\cap\mathbf{C}^{c}$

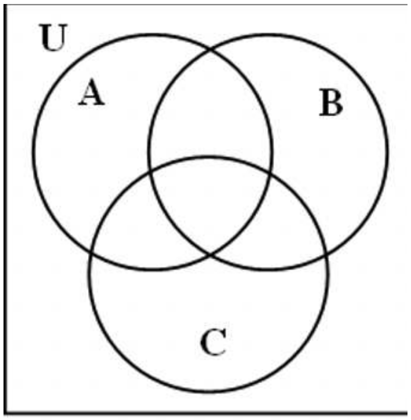
You have seen deMorgan's Laws demonstrated in many ways. Do you think you'll remember them? There were several exercises you did in Exercise Set 2 demonstrating these laws. Now is a good time to look back at these pairs of Venn diagram exercises that you shaded in Exercise Set 2 #'s 25 & 28, 26 & 29, 44 & 45 and 50 & 52.

It's time to demonstrate two more interesting properties about sets.

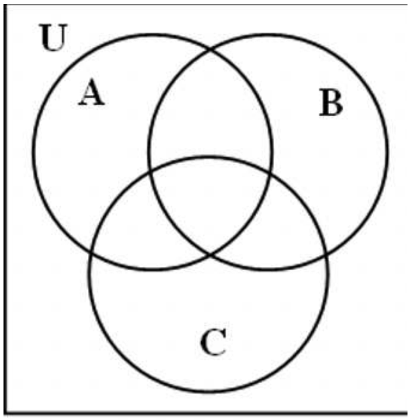
Exercise 15

a. We will shade Venn diagrams representing $\mathbf{A}\cap(\mathbf{B}\cup\mathbf{C})$ and $(\mathbf{A}\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{C})$. On the left Venn diagram, shade $\mathbf{A}$ with horizontal lines $\mathbf{B}\cup\mathbf{C}$ and with vertical lines. Then, the overlap is $\mathbf{A}\cap(\mathbf{B}\cup\mathbf{C})$. Shade only the overlap on the middle Venn diagram. Since $(\mathbf{A}\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{C})$ is a union, shade both $\mathbf{A}\cap\mathbf{B}$ and $\mathbf{A}\cap\mathbf{C}$ on the right Venn diagram.





$\mathbf{A}\cap(\mathbf{B}\cup\mathbf{C})$



$(\mathbf{A}\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{C})$

b. What do you notice about the final shadings of the Venn diagrams for $\mathbf{A}\cap(\mathbf{B}\cup\mathbf{C})$ and $(\mathbf{A}\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{C})$?

As in exercise 14, let **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9}, and define these subsets of **U**:

A = {1, 2, 3, 4, 5}	**B** = {2, 4, 6, 8}	**C** = {3, 5, 7}

Compute the following:

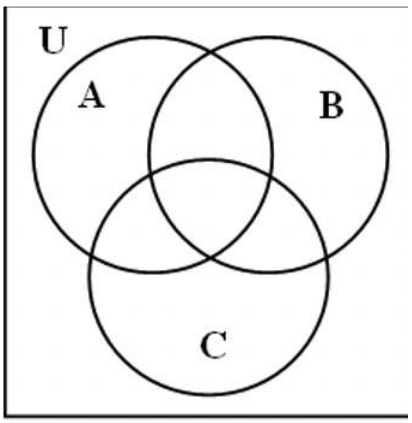
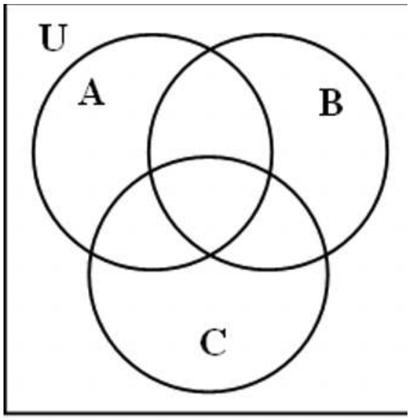
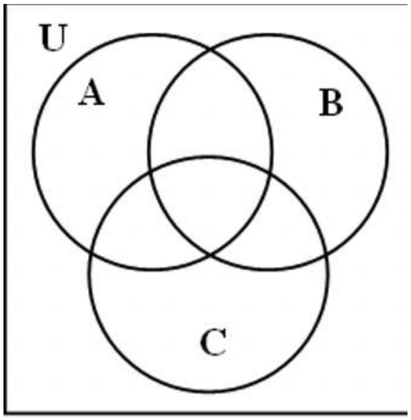
c. $\mathbf{A}\cap(\mathbf{B}\cup\mathbf{C})$ = ____

d. $(\mathbf{A}\cap\mathbf{B})\cup(\mathbf{A}\cap\mathbf{C})$ = ____

e. What do you notice about the answers for c and d? ____

Exercise 16

a. We will shade Venn diagrams representing $(\mathbf{A}\cup\mathbf{B})\cap(\mathbf{A}\cup\mathbf{C})$ and $\mathbf{A}\cup(\mathbf{B}\cap\mathbf{C})$. On the left Venn diagram, shade $\mathbf{A}\cup\mathbf{B}$ with horizontal lines and $\mathbf{A}\cup\mathbf{C}$ with vertical lines. Then, the overlap is $(\mathbf{A}\cup\mathbf{B})\cap(\mathbf{A}\cup\mathbf{C})$. Shade only the overlap on the middle Venn diagram. Since $\mathbf{A}\cup(\mathbf{B}\cap\mathbf{C})$ is a union, shade both **A** *and* $\mathbf{B}\cap\mathbf{C}$ on the right Venn diagram.

	 $(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$	 $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})$

b. What do you notice about the final shadings of the Venn diagrams for $(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$ and $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})$?

As in exercise 14, let **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9}, and define these subsets of **U**:

A = {1, 2, 3, 4, 5}	**B** = {2, 4, 6, 8}	**C** = {3, 5, 7}

Compute the following:

c. $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})$ = ____

d. $(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$ = ____

e. What do you notice about the answers for c and d? ____

Exercises 15 and 16 demonstrate two distributive properties of sets.

The first distributes intersection over union:

For three sets, **A**, **B** and **C**, $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$.

The second distributes union over intersection:

For three sets, **A**, **B** and **C**, $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$.

Below are some examples of these properties.

✓ Example 1

$\mathbf{A}^c \cup (\mathbf{M}^c \cap \mathbf{N}) = (\mathbf{A}^c \cup \mathbf{M}^c) \cap (\mathbf{A}^c \cup \mathbf{N})$

✓ Example 2

$(\mathbf{B} \cap \mathbf{T}^c) \cup (\mathbf{B} \cap \mathbf{S}) = \mathbf{B} \cap (\mathbf{T}^c \cup \mathbf{S})$

? Exercise 17

Use the distributive properties of sets to rewrite each of the following (don't be thrown off by complements):

a. $\mathbf{X} \cap (\mathbf{Y} \cup \mathbf{Z})$ = ____

b. $\mathbf{P} \cup (\mathbf{Q}^c \cap \mathbf{R})$ = _____

c. $(\mathbf{K}^c \cap \mathbf{L}) \cup (\mathbf{K}^c \cap \mathbf{M})$ = ______

d. $(\mathbf{D} \cup \mathbf{E}^c) \cap (\mathbf{D} \cup \mathbf{F})$ = _____

Exercises 53 and 54 in Exercise Set 2 demonstrated one of the distributive laws. Now is a good time to look back at these.

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1.4: Survey Problems

In this exercise set, you won't be working with manipulatives. We'll be constructing and using Venn diagrams in a different way to answer specific questions given a variety of information.

We're going to start off by learning how to construct a two set Venn diagram where

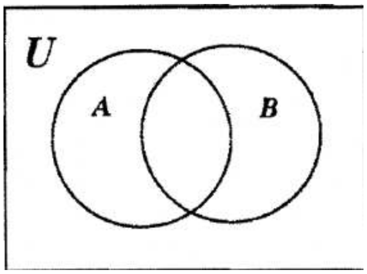
U = {1, 2, 3, 4, 5, 6, 7, 8, 9},

A = {1, 2, 3, 4, 5} and

B = {2, 4, 6, 8}.

The goal is to construct a Venn diagram showing the placement of all the elements in the universe.

Notice that a blank two set Venn diagram has four distinct regions.

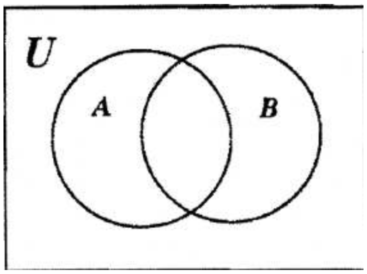


Method 1

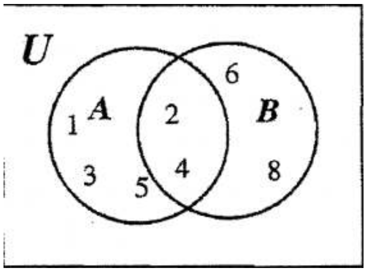
One way to obtain the solution is to decide where to place each number, one at a time. Starting with 1, note it is only in set A, and place it in that region (which notationally is A – B). You can then figure out where to place each of the other eight elements one at a time. It doesn't matter which element you place first. But, in the end, each of the nine elements must be placed in one of the four regions. **Try using this method. Place each of the nine elements in the Venn diagram shown above.**

Method 2

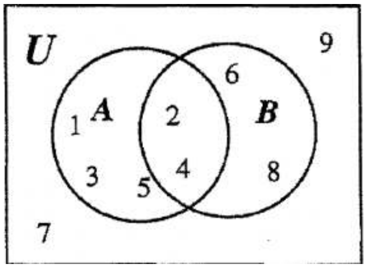
Here is another way to obtain the solution. Since we can compute that **A** ∩ B = {2, 4}, we can place those two elements in the intersection at once, and we know that no other numbers go there.



Next, we note that **A** also contains the elements 1, 3 and 5, so we can now fill these elements in the *other* region of **A**. Similarly, we note B also contains 6 and 8, so we can fill those elements in the other region of **B**. This is shown below.



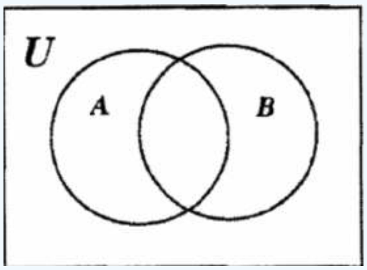
The only elements in the universe not accounted for are 7 and 9. These go in the part of the universe not containing A or B. After putting those in, the entire diagram can be completed as shown below. To check, see if the original information given above matches the answer.



Were you able to obtain the correct answer as shown above by using the first method proposed?

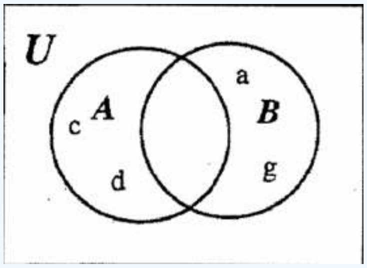
? Exercise 1

Put each element of the universe in its correct region on the Venn diagram to the right if **U** = {1 ,2, 3, 4, 5, 6, 7, 8, 9}, **A** = {2, 4, 5,7, 9} and **B** = {1, 2, 5, 6}

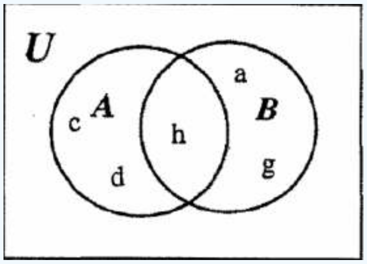


In this next example, information is provided in a different form. Method 1 is very difficult to use when the sets A and B are not explicitly given. We should still be able to fill in the Venn diagram. Again, we will work with two sets **A** and **B**. In the end, we will answer these questions: What elements are in **A, B** and **U**? We are given the following information: **A – B** = {c, d}, **B**^{c} = {b, c, d, e, f}, **B – A** = {a, g} and **A** ∪ **B** = {a, c, d, g, h}

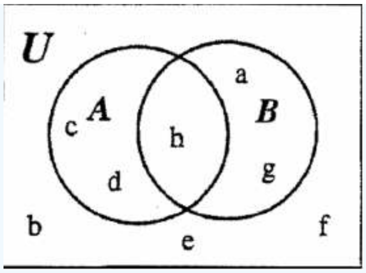
STEP 1: Since **A – B** and **B – A** are both isolated regions, we could start by filling in that information on a blank Venn diagram as shown below:



STEP 2: Next, we can fill in the rest of **A** ∪ **B**, putting the element h in the intersection, the region not yet accounted for.



STEP 3: Finally, we can fill in the rest of **B**^{c}. Note that so far, we see the elements c and d in one of the regions of **B**^{c}, so the remaining elements b, e and f must go in the only region not yet accounted for as shown:



STEP 4: Verify that the information given at the start of the problem is still true when looking at the finished Venn diagram as shown above.

STEP 5: Answer the questions asked:

U = {a, b, c, d, e, f, g, h}

A = {c, d, h}

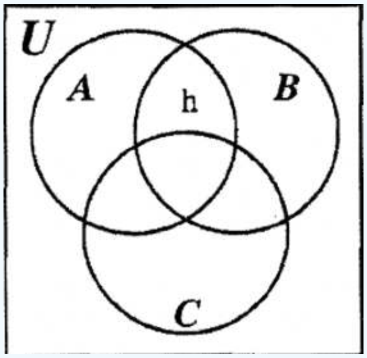
B = {a, g, h}

Note that someone else could have done the above problem using different steps. There's more than one way to proceed and still arrive at the correct solution. You should try it again on your own. The outcome should be the same no matter which region you filled in first.

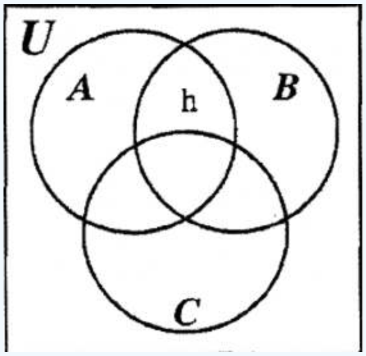
We will do one more example of this type, but this time we will work with a three set Venn diagram. Since there are eight regions, it will take a little longer to construct. The given information follows: $(\mathbf{A} \cap \mathbf{B}) - \mathbf{C}$ = {h}, **B – C** = {h, n}, $(\mathbf{A} \cup \mathbf{B}) - \mathbf{C}$ = {l, m, h, n}, **A – B** ={l, m, f}, $\mathbf{A} \cap \mathbf{B}$ ={g, h, i}, **B**={h, n, j, g, i}, **C**={f, g, i, j, k} and $(\mathbf{C} \cup \mathbf{B})^c$ = {l, m, d, e}

Below is just one way to take steps that lead to the correct answer.

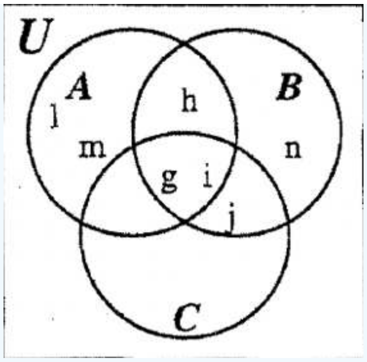
STEP 1: First, look at a blank three set Venn diagram and find an isolated region you can fill in first from the given information. $(\mathbf{A} \cap \mathbf{B}) - \mathbf{C}$ satisfies that criterion and thus that region can be filled in first as shown below.



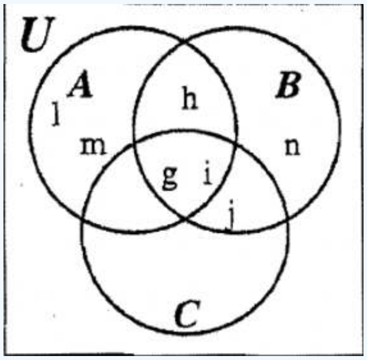
STEP 2: Next, **B – C** can be completed by putting the element n into the region not already filled in as shown below.



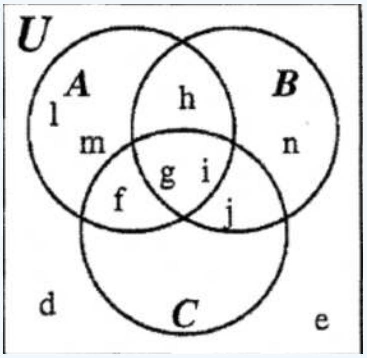
STEP 3: $\mathbf{A} \cap \mathbf{B}$ can be completed now too with g and i going into the region not yet accounted for:



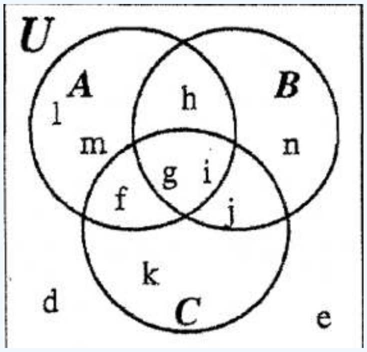
STEP 4: Next, we can fill in both $(\mathbf{A} \cup \mathbf{B}) - \mathbf{C}$ and **B** because they also are down to isolated regions.



STEP 5: We're getting close. There are only three regions left to fill in. $(\mathbf{C} \cup \mathbf{B})^c$ and the remaining region of **A – B** can be filled in as shown below:



STEP 6: We can now fill in the remaining region of set **C** to complete the diagram.



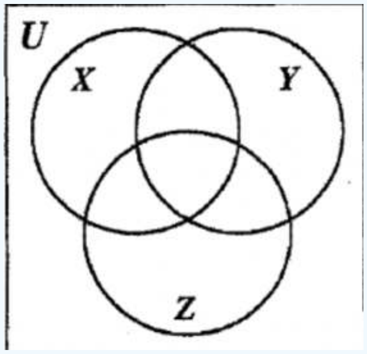
STEP 7: Check to see that all of the given information still holds true when looking at this finished diagram. We can now state the elements in sets **A,B,C** and **U**.

U = {d, e, f, g, h, i, j, k, l, m, n}	**A** = {f, g, h, i, l, m}

B = {g, h, i, j, n}	**C** = {f, g, i, j, k}

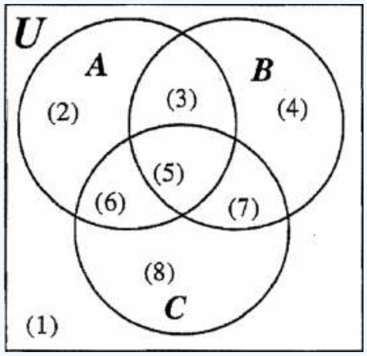
Exercise 2

Given that **U** = {a, b, c, d, e, f, g, h, i, j, k}, **X** = {a, b, c, d, e}, **Y**={c, d, f, g, h} and **Z** = {b, c, h, i, j}, fill in each of the elements of the universe in its proper region on the Venn diagram to the right.



Exercise 3

Now is a good time for you to practice identifying the eight regions of a three set Venn diagram by describing each region using set notation. Describe each region as indicated in parentheses on the Venn diagram. There may be more than one way to describe a region.



a. Region 1: _____

b. Region 2: _____

c. Region 3: ______

d. Region 4: ______

e. Region 5: ______

f. Region 6: ______

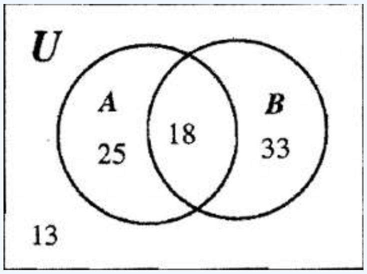
g. Region 7: ______

h. Region 8: _______

Before going on to survey type problems, we need to practice looking at a Venn diagram where the *cardinality* (number of elements) of each region is given and then be able to answer questions about the cardinality of other combinations of regions.

Exercise 4

Look at the Venn diagram below and answer the questions about the cardinality of the regions indicated.



a. $n(\mathbf{A})$ = ____	b. $n(\mathbf{B})$ = ____
c. $n(\mathbf{A}^c)$ = ____	d. $n(\mathbf{B}^c)$ = ____
e. $n(\mathbf{A} \cap \mathbf{B})$ = ____	f. $n(\mathbf{A} - \mathbf{B})$ = ____
g. $n((\mathbf{A} \cup \mathbf{B})^c)$ = ____	h. $n(\mathbf{U})$ = ____

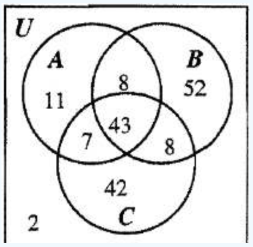
Imagine that the above problem was really a model for what kind of fruit a group of people packed in their lunches. **A** stood for those who packed apples and **B** stood for those who packed bananas. Assume apples and bananas are the only kind of fruit available. Then the questions you answered were these:

a. How many packed apples?
b. How many packed bananas?
c. How many didn't pack apples?
d. How many didn't pack bananas?
e. How many packed both kinds of fruit?
f. How many packed apples only?
g. How many didn't pack either fruit?
h. How many people were surveyed?

Look to see if you would have looked in the same regions if the questions were asked in this way.

Exercise 5

Let's see how well you do working with a three set Venn diagram. The Venn diagram below shows the cardinality of each region. A survey of college students showed the grades they received on their last report card. **A** stands for getting an A, **B** stands for getting a B and **C** stands for getting a C. Some people took only one course whereas others took more. Look at the Venn diagram and answer the following questions. *Extra Credit: Instead of just writing the number, write down what the question is asking* ***symbolically****. In other words, if the question is "How many received both A's and B's?" you would write* $n(\mathrm{A} \cap \mathrm{B}) = 51$.



a. How many got all A's? ____
b. How many got A's? ____
c. How many got B's and C's? ____

d. How many got only A's or B's? _____

e. How many didn't get a C? _____

f. How many got B's or C's? _____

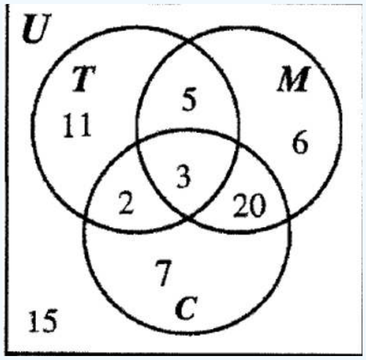
g. How many got B's, but not C's? ____

h. How many got at least two different grades of A, B or C? ____

i. How many got exactly two different grades of A, B or C? ____

Exercise 6

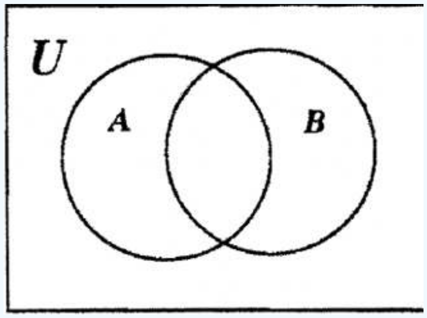
In another survey, questions were asked to see how many people were in their teens, how many were married and how many had children. **T** represents teens, **M** represents those who were married and **C** represents those having children. Look at the Venn diagram which shows the cardinality of each region to answer the following questions:



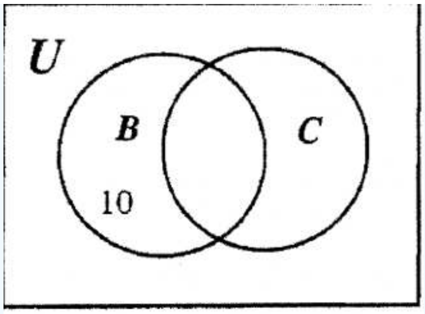
a. How many people were married? ____
b. How many teens had children? ____
c. How many people were in the survey? ____
d. How many people did not have children? ____
e. How many teens weren't married? ____
f. How many teens were married with children? ____
g. How many teens were in the survey? ____
h. How many people were unmarried with children? ____
i. How many people were married with children? ____

At the beginning of this exercise set, you learned how to place elements in their proper regions given information about the elements in some sets. The same process can be used when only considering the *cardinality* of a set or region. We will do an example involving two sets, **C** and **B**, where $n(\mathbf{U}) = 35$, $n(\mathbf{C}) = 20$, $n(\mathbf{B}) = 15$ and $n(\mathbf{B} - \mathbf{C}) = 10$.

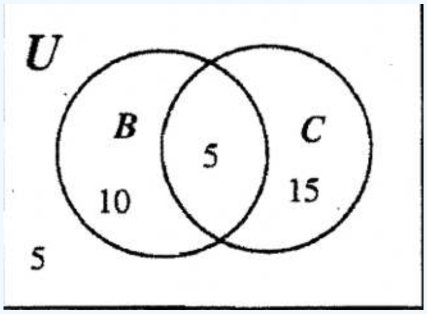
STEP 1: Again, we first have to isolate a region. **B – C** is an isolated region and so we can fill 10 into that spot as shown in the Venn below:



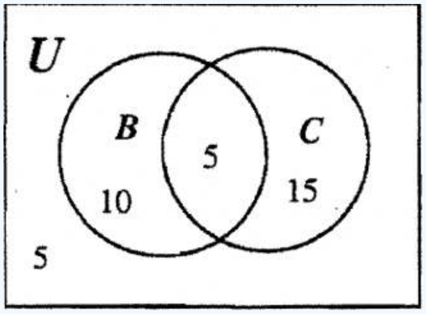
STEP 2: We can fill 5 into the other region of **B** since **B** has a total of 15 elements (10 + 5 = 15) .



STEP 3: Since **C** has a total of 20 elements, we can fill in the other region of **C** with the number 15.



STEP 4: There is only one region left to fill and since the total number of elements is 35, the final region to fill in must be 5.



Usually, the above problem would have been a word problem worded something like this:

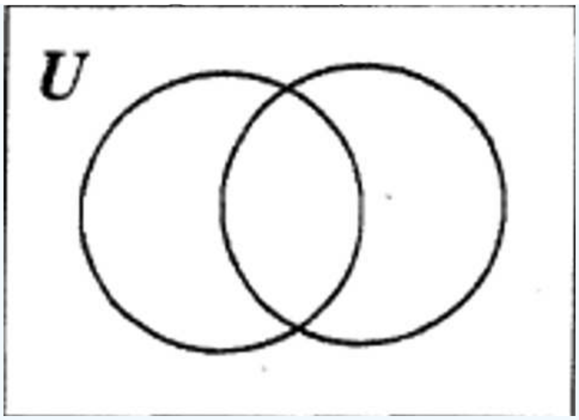
A class of 35 students was surveyed and it was determined that 20 students owned a car, 15 owned a bike and 10 of those owning a bike did not have a car. How many students owned both a bike and a car? How many students owned only a car? How many students owned neither a bike or car?

In order to answer these questions, one might consider drawing a Venn diagram containing two sets, **C** and **B**, where **C** would represent the number of students owning cars and **B** would represent the number of students owning bikes. Notice the first step would be in defining your variables! Next, after reading through the information given, one could conclude that the information could be written down symbolically like this: n(**U**) = 35, n(**C**) = 20, n(**B**) = 15 and n(**B – C**) = 10. Does this look familiar? From here, the steps would be exactly the same as we just did in the above example. The person could then look at the Venn diagram and answer the questions asked.

- How many students owned both a bike and a car? (Look at $\mathbf{B} \cap \mathbf{C}$): 5
- How many students owned only a car? (Look at **C – B**)): 15
- How many students owned neither a bike or car? (Look at $(\mathbf{B} \cup \mathbf{C})^c$): 5

? Exercise 7

Twenty preschoolers were surveyed about whether or not they had dogs and/or cats as pets in their home. Ten children owned both animals. Four had no pets at home and 14 owned dogs. Construct a Venn diagram representing how many had each kind of animal and answer the questions asked based on your diagram. Use meaningful letters to represent your sets!!



a. How many children owned cats?
b. How many children owned cats only?
c. How many kids didn't own a dog?

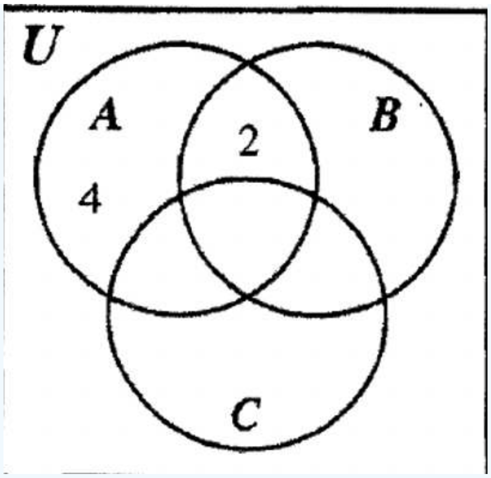
It's now time to practice constructing a three set Venn diagram given the following information provided about three sets, **A**, **B** and **C:**

$n((\mathbf{A} \cap \mathbf{B}) - \mathbf{C}) = 2$	$n(\mathbf{B} - \mathbf{C}) = 8$
$n(\mathbf{A} - \mathbf{B}) = 7$	$n(\mathbf{A}) = 13$
$n(\mathbf{A} - (\mathbf{B} \cup \mathbf{C})) = 4$	$n(\mathbf{B}) = 20$
$n(\mathbf{C} - \mathbf{B}) = 13$	$n(\mathbf{U}) = 42$

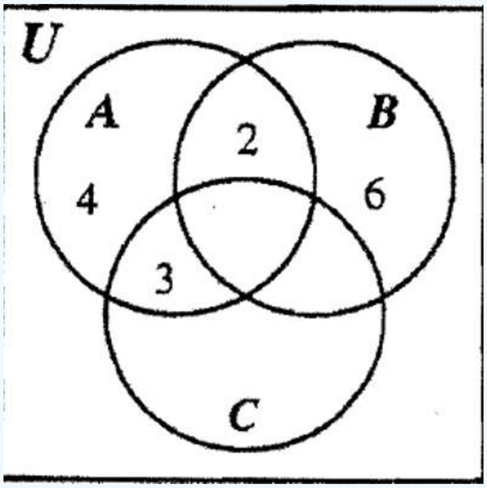
We will construct a Venn diagram showing the cardinality of each region and then find:

a. $n((\mathbf{B} \cap \mathbf{C}))$
c. $n(\mathbf{B}^c)$
e. $n((\mathbf{A} \cup \mathbf{C}) - \mathbf{B})$
f. $n((\mathbf{C} - (\mathbf{A} \cup \mathbf{B})$
g. $n(\mathbf{A})$
h. $n((\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})^c)$
b. $n(\mathbf{A} - \mathbf{C})$
d. $n(\mathbf{C})$

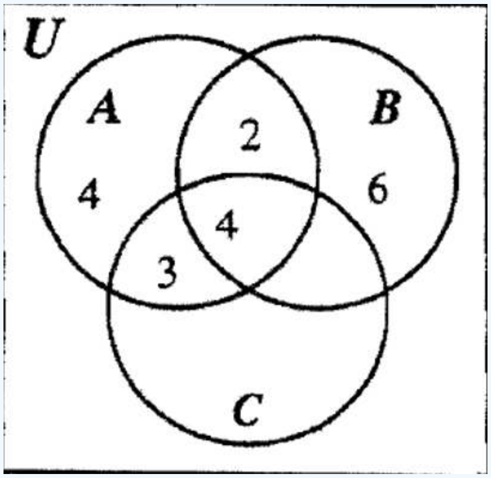
STEP 1: Draw a three set Venn diagram and see if any of the information given is specific enough to cover just one region. $(\mathbf{A} \cap \mathbf{B}) - \mathbf{C}$ and $\mathbf{A} - (\mathbf{B} \cup \mathbf{C})$ each cover one region so fill in those regions first.



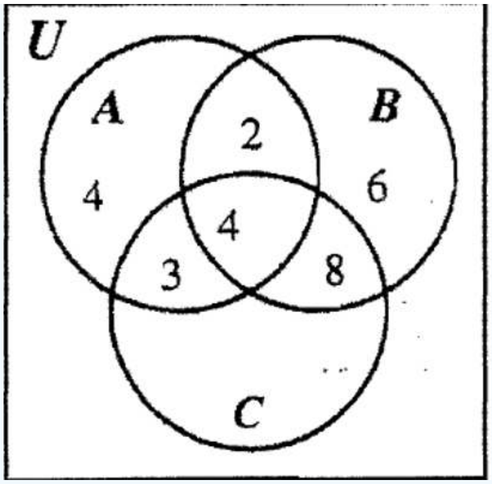
STEP 2: **B – C** and **A – B** each have two regions with one region of each already filled in. The two regions of **B – C** must add up to 8 so that leaves 6 for the other part. Similarly, the two regions of **A – B** must add up to 7 leaving 3 for the other region.



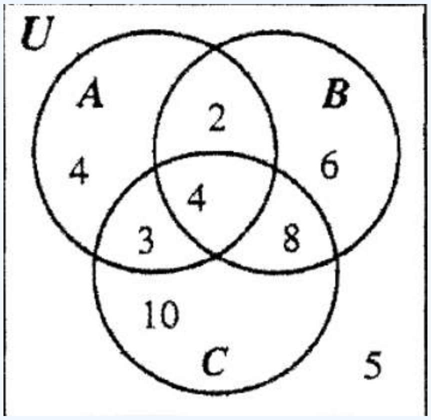
STEP 3: **A** has three of its four regions filled in now, so we can fill 4 into the last region of **A** since $n(\mathbf{A}) = 13$, which means all four regions of **A** must add up to 13.



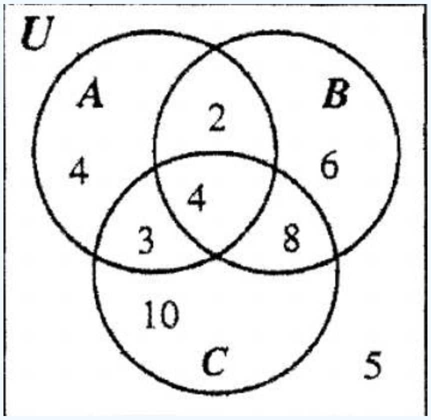
STEP 4: Note that **B** has all but one of its regions filled in now, so we can fill in that last region with 8 since $n(\mathbf{B}) = 20$.



STEP 5: Since $n(\mathbf{C} - \mathbf{B}) = 13$ and one of the two regions of **C – B** is 3, that leaves 10 to be filled into the other region.



STEP 6: There is only one region left to fill in and since the universe contains 42 elements, there are 5 elements left to be accounted for. This completes this part of the process.



STEP 7: This step is ***very important!*** Make sure you go back and check your work very carefully. Eight pieces of information were given at the beginning of the problem. Go back and check each one individually. For instance, make sure that when you look at your completed diagram, it is still true that $n(\mathbf{B} - \mathbf{C}) = 8$, $n(\mathbf{A} - \mathbf{B}) = 7$, $n(\mathbf{A}) = 13$, $n(\mathbf{A} - (\mathbf{B} \cup \mathbf{C})) = 4$, $n(\mathbf{C} - \mathbf{B}) = 13$, etc.

STEP 8: Look at the completed Venn diagram and now answer the questions posed:

a. $n(\mathbf{B} \cap \mathbf{C}) = 12$

b. $n(\mathbf{A} - \mathbf{C}) = 6$

c. $n(\mathrm{B}^c) = 22$

d. $n(\mathbf{C}) = 25$

e. $n((\mathbf{A} \cup \mathbf{C}) - \mathbf{B}) = 17$

f. $n((\mathbf{C} - (\mathbf{A} \cup \mathbf{B})) = 10$

g. $n(\mathbf{A}) = 13$

h. $n((\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})^c) = 5$

More often, the above problem would have been presented as a survey–type problem where answers could be answered using set theory. Consider the next example.

A survey of 42 men was taken asking about their likes of certain female names: Amy, Beth and Cindy. It was discovered that 13 men liked the name Amy, 20 of them liked Beth, 8 liked Beth but not Cindy, 13 liked Cindy but not Beth, 7 liked Amy but not Beth, 4 liked only the name Amy and 2 liked Amy and Beth but not the name Cindy. Find the following:

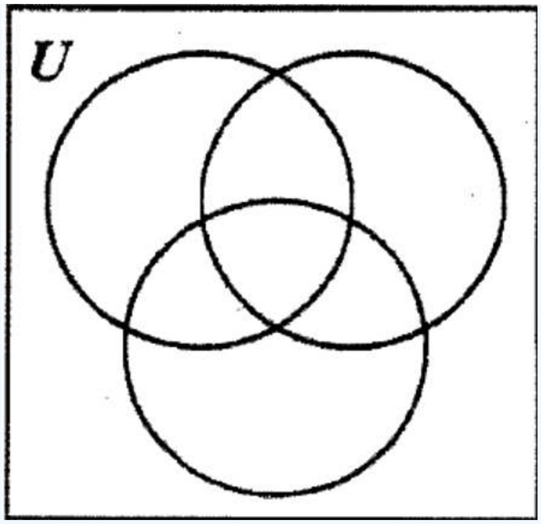
a. How many men liked both the names Beth and Cindy?
b. How many men liked the name Amy but not the name Cindy?
c. How many men didn't like the name Beth?
d. How many men liked the name Cindy?
e. How many men didn't like the name Beth, but did like at least one of the other names?
f. How many men liked only the name Cindy?
g. How many men liked the name Amy?
h. How many men didn't like any of the names?

This problem could be solved by first letting the letters **A**, **B** and **C** represent the sets of men who liked the names Amy, Beth and Cindy, respectively. Next, you could write out the eight clues given using set notation, *which will give you the exact same eight clues given at the beginning of the previous example!* Or, you might go straight to a Venn diagram, deciding which part of the Venn diagram each clue is linked to and start filling in the numbers. The questions asked are also exactly the same as in the previous example. Try this problem in this form and then check your answers by looking back at the previous example. Make sure you can do this one successfully on your own before trying the next two exercises.

? Exercise 8

A survey of 60 college students was taken asking about which meals they regularly ate at school. Thirty students usually ate breakfast at school, 26 usually ate lunch there

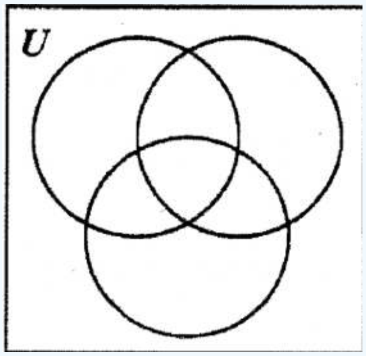
and 23 usually ate dinner at school. Six students said they ate all three meals at school. Eleven ate both breakfast and lunch there whereas 15 only ate breakfast at school and 12 only ate lunch there. Label the sets and fill in the Venn diagram below representing this information and answer the following questions:



a. How many students ate both lunch and dinner at school?
b. How many didn't eat any meals at school?
c. How many only ate dinner at school?
d. How many ate breakfast and/or dinner at school?
e. How many ate exactly one meal at school?
f. How many ate at least two meals at school?
g. How many ate both lunch and dinner at school, but not breakfast?

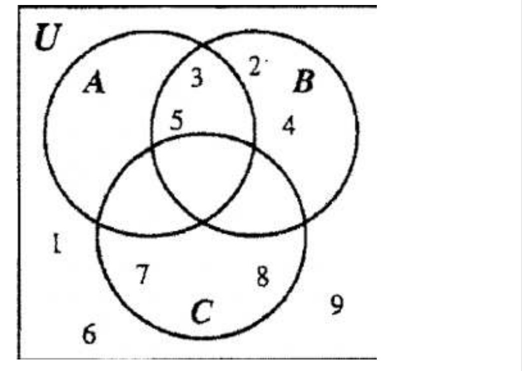
? Exercise 9

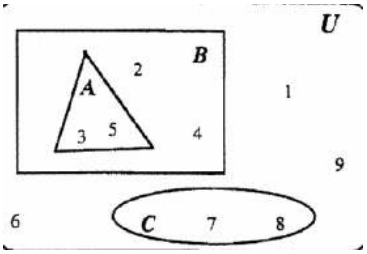
Another group of students was asked which old reruns of Dick Van Dyke, Gilligan's Island and The Honeymooners they liked to watch on late night T.V. 34 liked both Dick Van Dyke and Gilligan's Island while 7 liked only Dick Van Dyke and 11 liked only Gilligan's Island. 15 liked Dick Van Dyke but not Gilligan's Island whereas 21 liked Gilligan's Island but not Dick Van Dyke. 27 didn't like The Honeymooner's but liked at least one of the other programs. 31 didn't like The Honeymooners and 20 liked The Honeymooners but not Gilligan's Island. Label the sets and fill in the Venn diagram below representing this information. Then, answer the following questions:

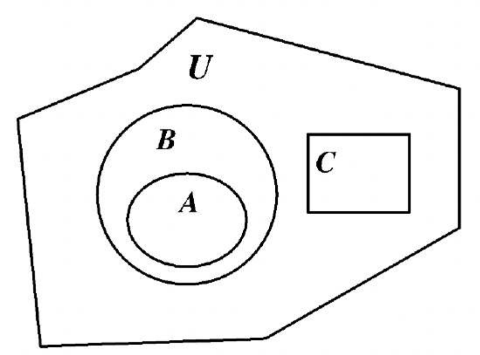


In working with Venn diagrams so far, we've basically restricted the picture to three sets and shown the same template over and over again where the sets overlap each other. For a particular problem, a Venn diagram might be represented very differently. For instance, consider **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9} with subsets **A**, **B** and **C** defined as follows:

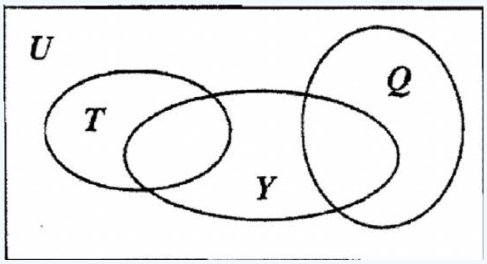
A={3, 5}, **B**={2, 3, 4, 5} and **C**={7, 8}. We could show the elements of the universe in the two ways shown below. The Venn on the left shows the conventional way like we've been doing for this exercise set whereas the one on the right is actually a little more specific because it is clear from the picture that **A** ⊂ **B** and that **C** has no elements in common with **A** or **B.**







It is especially useful to be able to draw a Venn diagram like the second Venn diagram shown above when you are asked to do something like this: Draw a Venn diagram containing sets **A**, **B** and **C** such that **A** ⊂ **B** with **C** having no elements in common with **A** or **B**. Notice you are not given the specific elements of **A** and **B**. The diagram on the right shows one way of representing this information.



Consider these subsets of the A–blocks: Yellow, Triangle and Square. One way of showing how they might be arranged on a Venn diagram is shown on the Venn to the right. Triangles and Squares are disjoint with one another whereas both Triangles and Squares intersect with Yellow.

Exercise 10

In the space below, arrange these subsets of the A–blocks, Red (R), Green (G) and Circle (C) in a Venn diagram that clearly shows their relationship to each other.

Exercise 11

In the space below, arrange these subsets of the A–blocks, Green (G), Yellow (Y), Blue (B), Red(R), Small (S), in a Venn diagram that clearly shows their relationship to each other.

Exercise 12

Let **U**={1, 2, 3, 4, 5, 6, 7, 8, 9} with these subsets: **A**={6, 7, 8}, **B**={2, 3}, **C**={3, 4}, **D**={5}, **E**={1, 2, 3, 4, 5, 6}. In the space below, arrange these sets in a Venn diagram that clearly shows their relationship to each other. It's a bit tricky because there are five subsets, but you can do it!

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1.5: Homework

- **Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)**
- **Start a new module on the front side of a new page and write the module number on the top center of the page.**
- **Answers without supporting work will receive no credit.**
- **Some solutions are given in the solutions manual.**
- **You may work with classmates but do your own work.**

Hw #1

Let **U** = {a, c, e, m n, r, u, v, w, x, z} with subsets **A**, **B**, **C** and **D** defined below: **A** = {m, n, r, u, x} **B** = { a, c , r, u, x} **C** = {e, v, w, x, z } **D** = {a, c, z} Using correct notation, find the following, show all work

a. $\mathbf{B} \cap \mathbf{D}$	f. $\mathbf{A} - \mathbf{C}^c$	k. $(\mathbf{A} \cap \mathbf{D}) - \mathbf{B}^c$
b. $\mathbf{A} \cap \mathbf{D}$	g. $\mathbf{B} - (\mathbf{A} \cap \mathbf{C})$	l. $n(\mathbf{A} \cup \mathbf{B})$
c. **C** - **B**	h. $\mathbf{D}^c - (\mathbf{B} \cup \mathbf{C})$	m. n(A) + n(B)
d. $(\mathbf{A} \cup \mathbf{C})^c$	i. $(\mathbf{A} \cap \mathbf{B}) \cap (\mathbf{C} \cup \mathbf{D})$	n. n(**D** - **C**)
e. $\mathbf{D}^c \cap \mathbf{B}$	j. $(\mathbf{B}^c \cup \mathbf{D})^c$	o. $n(\mathbf{B} \cap \mathbf{C})$

HW #2

Let **U** = {a, c, e, m n, r, u, v, w, x, z} with subsets **A**, **B**, **C** defined as follows:

A = {m, n, r, u, v}	**B** = {r, u, w, x}	**C** = {n, r, x, c}

Draw a Venn Diagram and place each element of the universe in the correct region

HW #3

Use deMorgan's Laws to rewrite each of the following:

a. $\mathbf{N} \cup \mathbf{P}^c$	b. $\mathbf{R}^c \cap \mathbf{S}$

HW #4

Use the distributive properties of sets to rewrite each of the following:

a. $(\mathbf{A}^c \cup \mathbf{E}) \cap (\mathbf{A}^c \cup \mathbf{F})$	b. $\mathbf{B} \cap (\mathbf{A} \cup \mathbf{C})$

HW #5

A survey was given to determine which of the three beverages (tea, milk and/or coffee) people drank each day. The results were as follows:

7 only drank coffee	6 drank all three	11 drank tea and coffee
21 drank coffee	4 drank none of the three	9 drank neither coffee or tea
21 drank tea	1 drank only tea and milk	

a. Draw a Venn diagram indicating how many people would belong in each region. Label the three sets with meaningful letters.

b. How many people were surveyed?

c. How many drank milk?

d. How many drank only coffee and milk?

e. How many drank only milk?

f. How many drank tea or coffee but not milk?

g. How many drank exactly two kinds of beverages?

h. How many didn’t drink milk or tea?

? HW #6

Take out your A–blocks and arrange them into subsets so that each subset only contains elements that have the same size and color.

a. How many subsets are there?

b. How many pieces are in each subset?

? HW #7

Let **A**, **B** and **C** represent any sets. Answer True or False for the following statements. In order for a statement to be true, it must always be true. For each False statement, give an example of why it is False.

a. **B** is always a subset of $\mathbf{A} \cup \mathbf{B}$

b. $(\mathbf{A}-\mathbf{B})^c = \mathbf{A}^c - \mathbf{B}^c$

c. **B** is always a subset of $\mathbf{A} \cap \mathbf{B}$

d. $n(\mathbf{A} \cup \mathbf{B}) = n(\mathbf{A}) + n(\mathbf{B})$

e. $n(\mathbf{A} \cup \mathbf{B}) = n(\mathbf{A}) + n(\mathbf{B}) - n(\mathbf{A} \cap \mathbf{B})$

f. If $n(\mathbf{A} \cup \mathbf{B}) = n(\mathbf{A}) + n(\mathbf{B})$, then **A** and **B** are disjoint.

? HW #8

Draw a Venn diagram and shade in the region that represents the following

a. $(\mathbf{C} \cup \mathbf{A}) - \mathbf{B}$

b. $(\mathbf{C} \cap \mathbf{B}) \cup \mathbf{A}$

c. $(\mathbf{C} \cap \mathbf{B}) - \mathbf{A}$

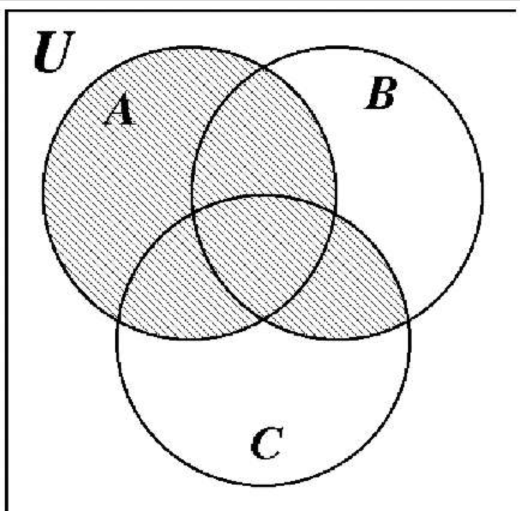
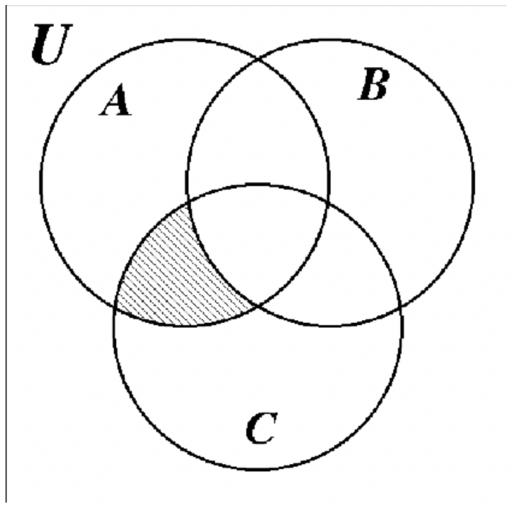
d. $(\mathbf{A} \cup \mathbf{C}) \cap \mathbf{B}$

e. $\bar{\mathbf{A}} - (\mathbf{B} \cap \mathbf{C})$

f. $(\mathbf{B} - \mathbf{A}) \cap (\mathbf{B} - \mathbf{C})$

? HW #9

Identify the shaded area of each Venn diagram by set notation.

 a.	 b.

? HW #10

List all possible subsets for each set given.

a. { }	b. {a}	c. {a, b}	d. {a, b, c}

? HW #11

Let A = {1, 2, 4}, B = {(a, c), 5} and C = {x}. Find the following:

a. $A \times A$	a. $A \times B$	c. $B \times C$
d. $C \times A$	a. $C \times C$	

? HW #12

Use your A–blocks to do this problem. Let **X** represent the set of large circles and **Y** represent the set of red circles. Using set notation and abbreviations, find the following:

a. **X** - **Y**	b. **X** ∩ **Y**	c. **X** ∪ **Y**	d. **Y** - **X**

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CHAPTER OVERVIEW

2: Counting and Numerals

One, two,

buckle my shoe,

three, four,

open the door

to the world

of counting

I often think of my grandpa, Benjamin Cohn, who tirelessly and happily read and recited nursery rhyme after nursery rhyme to me as he rocked me in his lap...I sat on his lap more gently once I was in college. He rocked many children through the years. Although he had only an eighth grade education, he was smart, well-read, highly respected and he laid down the foundations for life-long learning. He was a major influence in my life and also the kindest man I have ever met. Grandpa was loved by all and his memory is alive in the hearts of all who knew him.

- 2.1: Numbers and Numerals
- 2.2: Numeration Systems
- 2.3: Trading and Place Value
- 2.4: Other Base Systems
- 2.5: Supplementary Topics
- 2.6: Homework

Thumbnail: The abacus is a calculating tool that has been in use since ancient times and is still in use today. The abacus consists of a number of rows of movable beads or other objects, which represent digits. One of two numbers is set up, and the beads are manipulated to implement an operation involving a second number (e.g., addition), or rarely a square or cubic root. (Unsplash Lisense; **Crissy Jarvis via Unspash**)

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2.1: Numbers and Numerals

You will need: ***A-blocks*** *(Material Cards 2A, 2B, 2C, 2D, 2E)*

? Exercise 1

Imagine you came across a person from a remote English-speaking colony where everyone had three fingers on each hand. Although the people of this colony were intelligent, they had never heard of counting and did not know what it was, what it meant or how to do it. They knew nothing about numbers, what they were, what they were called or what they looked like. Think about how you might teach this person about counting and numbers so they could go back and share this wonderful knowledge with their friends. What tools would you use to teach? What are some of the concepts you would have to get across? What difficulties might arise? Do you think this would be an easy or hard task? Write down some of your ideas, strategies and conclusions.

? Exercise 2

What do you remember about learning about numbers or how to count? Have you ever helped to teach a young child how to count? If so, what was it like? How long do you think it takes for someone to learn how to count for the first time?

Most of us in the modern world pretty much take the concept of numbers and counting for granted. In fact, we take language for granted –but that topic is too involved to go into right now. Imagine what a breakthrough it was when people first thought to give names to objects. Eventually they used words, which we will call number names, to describe sets of objects (remember sets?!) in terms of their size. This was a considerable mathematical advancement. Although there are still some cultures that have not assigned names for numbers larger than three, most cultures have developed this concept. Some tribes even have different number names for different types of objects.

One difficulty with the concept of numbers as we use them today is that *they are concepts* and not necessarily used only for counting objects. The number four, for instance, is used in many ways –four years old, room 4, four feet high, 4 children, four hours, four minutes after eight, mail station 4, 4 fingers, a grade of 4 on a quiz, count to 4, four times bigger than something, 4th place, four wishes, fourth in line, call 444-4444, code 44, etc.

So, what is a number?

Well, a **number** is an abstract idea that represents a quantity. You can't see or touch a number. The symbols that we see and touch *that are used to represent numbers* are actually *numerals*. A **numeral** is made up of one or more symbols. In the Hindu-Arabic system we use, the numerals are made up of one or more of these ten symbols –0,1,2,3,4,5,6,7,8 and 9. For instance, in the Hindu-Arabic system, the number three is represented by writing the numeral three which we write like this: 3 and the number two million is represented by writing the numeral two million which we write like this: 2,000,000. Notice, it took one symbol to write the number for three and seven symbols to write the number for two million (we don't count the commas as symbols because they aren't essential to writing the numeral). Although it took seven symbols to write the numeral two million, note that there are *only two different symbols* used to write that numeral –a two (2) and a zero (0).

? Exercise 3

In your own words, describe the word **number**.

? Exercise 4

In your own words, describe the word **numeral**.

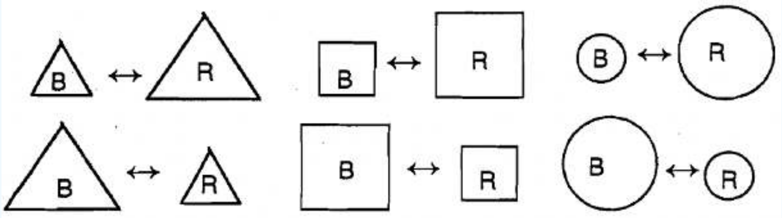
There are three common uses of numbers. Describing how many elements are in a finite set is the most common use of the counting numbers. When used this way, the number is referred to as a **cardinal number.** Another way numbers are used concerns order. For instance, someone might be the fifth place winner or you might be eighth in line. When used in this way, numbers are called **ordinal numbers**. There are also numbers used for identification such as your phone number, zip code, social security number or credit card number. These are called **identification numbers** and it is the actual order of the symbols that is important – not the value of the number

? Exercise 5

When people recognized that a collection of four pebbles and a collection of four bananas had a common property, they were learning the concept of **matching**. Take out your A-blocks and divide them into subsets according to color. Take the subset of Red A-Blocks and the subset of Blue A-blocks. We will match each element in the Blue A-blocks with an element in the Red A-blocks. There are several ways to do this. I will show two ways to match them. Show a different way to match them under Matching #3 below.

MATCHING #1	MATCHING #2	MATCHING #3
SBT ↔ LRT	SBT ↔ SRQ	
LBT ↔ SRT	LBT ↔ LRQ	
SBQ ↔ LRQ	SBQ ↔ SRC	
LBQ ↔ SRQ	LBQ ↔ LRC	
SBC ↔ LRC	SBC ↔ SRT	
LBC ↔ SRC	LBC ↔ LRT	

Pictures may be used to represent a matching. Below is another way to show Matching #1:



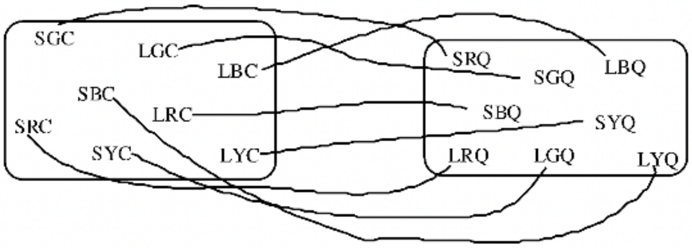
We say that the set of Blue A-blocks **match** the set of Red A-blocks because we can **pair** each element of the Blue A-blocks with exactly one element of the Red A-blocks.

? Exercise 6

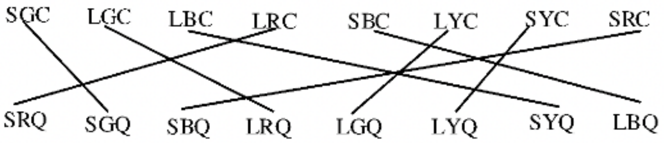
Using pictures, show a matching between the large triangles and the small circles. There is more than one possible way to match them up. Make sure each large triangle is "dancing" with one of the small circles.

NOTE: Two sets that match DO NOT have to match in a natural or pleasing way the way we might match the colors of our clothes together. It might be easier to think of the word **pairing** instead. Think of showing a matching between two sets up by taking one element in the first set and pairing it up to dance with one element in the second set. Now that they are "dancing" together, take another element from the first set and pick someone in the second set for it to dance with. Continue doing this until all elements are paired up and dancing! This is a matching. If your first set had 5 girls and your second set had 4 boys, they wouldn't match because one girl would be left without a partner. That is why it is necessary to have the same number of elements in each set in order to represent a matching.

Let's match up the subset of Circles with the subset of Squares now. Two different matchings are shown below. Note I've used different formats than in exercise 5.



MATCHING #1



MATCHING #2

? Exercise 7

Show two different ways to match the set of small triangles with the set of large squares.

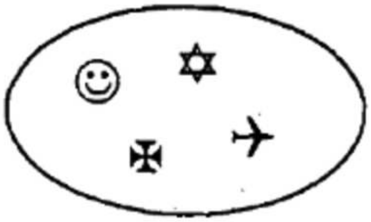
Each of the pairings we have matched so far illustrates a **one-to-one correspondence (1-1)** between two sets. We say that two sets **match** if there is a **one-to-one correspondence** between their elements.

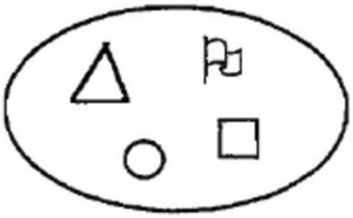
? Exercise 8

If a 1-1 correspondence between two sets exists, what does that imply about the cardinality of the two sets?

? Exercise 9

Draw a one-to-one correspondence between the elements of the sets below.





To show **all** of the 1-1 correspondences, or matchings between two sets, I think of all the ways two sets might partner up to do a single dance on the dance floor. Everyone from each set gets exactly one partner from the other set to dance with, and everyone gets to dance. For instance, think of a group of 4 boys–A, B, C and D and 4 girls–W, X, Y and Z. Each boy needs to be paired up with one girl for a dance, so there will be four couples dancing for one dance. One matching would be to show how they pair up for that one dance: maybe A with X, B with Z, C with W and D with Y –that is Matching 11, as shown below. *That's only one matching.* Note that you do not pair up an element of one set with an element of the same set. For this example, that means a girl only dances with a boy. To show ALL possibilities for this example, you'd have to list all the different ways they might end up as couples on the dance floor for one dance. Each matching shows the four couples, as they are paired up for one dance. To be a different matching, at least two couples would have changed partners. There are actually 24 possible 1-1 correspondences (or different matchings) for this example of four boys and four girls. Below is one way to list all of these possibilities:

Matching 1 A ↔ W B ↔ X C ↔ Y D ↔ Z	*Matching 2* A ↔ W B ↔ X C ↔ X D ↔ Y	*Matching 3* A ↔ W B ↔ Y C ↔ X D ↔ Z	*Matching 4* A ↔ W B ↔ Y C ↔ Z D ↔ X	*Matching 5* A ↔ W B ↔ Z C ↔ X D ↔ Y	*Matching 6* A ↔ W B ↔ Z C ↔ Y D ↔ X
Matching 7 A ↔ X B ↔ W C ↔ Y D ↔ Z	*Matching 8* A ↔ X B ↔ W C ↔ Z D ↔ Y	*Matching 9* A ↔ X B ↔ Y C ↔ W D ↔ Z	*Matching 10* A ↔ X B ↔ Y C ↔ W D ↔ Z	*Matching 11* A ↔ X B ↔ Z C ↔ W D ↔ Y	*Matching 12* A ↔ X B ↔ Z C ↔ Y D ↔ W
Matching 13 A ↔ Y B ↔ W C ↔ X D ↔ Z	*Matching 14* A ↔ Y B ↔ W C ↔ Z D ↔ X	*Matching 15* A ↔ Y B ↔ X C ↔ W D ↔ Z	*Matching 16* A ↔ Y B ↔ X C ↔ Z D ↔ W	*Matching 17* A ↔ Y B ↔ Z C ↔ W D ↔ X	*Matching 18* A ↔ Y B ↔ Z C ↔ X D ↔ W
Matching 19 A ↔ Z B ↔ W C ↔ Y D ↔ X	*Matching 20* A ↔ Z B ↔ W C ↔ X D ↔ Y	*Matching 21* A ↔ Z B ↔ X C ↔ W D ↔ Y	*Matching 22* A ↔ Z B ↔ X C ↔ Y D ↔ W	*Matching 23* A ↔ Z B ↔ Y C ↔ W D ↔ X	*Matching 24* A ↔ Z B ↔ Y C ↔ X D ↔ W

If you were asked to list one matching between two sets {A, B, C, D} and {W, X, Y, Z}, any of the 24 matchings shown above would be fine. However, if you are asked to list ALL possibilities, you must be very clear what constitutes a matching, as I've illustrated above. In each matching, note the individual pairings (who is dancing with whom) must be shown. If the sets have one element each, then *each* matching has one pairing. If the sets have two elements each, then *each* matching has two pairings. If the sets have three elements each, then *each* matching has three pairings. And, if the sets have four elements each (as in the example above), then *each* matching has four pairings.

See if you notice any patterns above before going on. It is much easier to show the matchings when there are fewer elements in the sets. If there is one element in each set, then it's like being on a desert island, one man and one woman. There isn't much choice, is there?

? Exercise 10

Show every possible 1-1 correspondence (or matching) between the sets.

a. {1} and {A}	b. {2,3} and {B,C}	c. {4,5,6} and {D,E,F}

It's important to realize the two sets that are being matched may contain one, some or all of the same elements. Study the following examples, where each set contains the exact same elements.

Examples: Show every possible 1-1 correspondence (or matching) between the sets listed.

✓ Example 1

{3} and {3}

Solution

Matching 1

3 ↔ 3

✓ Example 2

{6, 7} and {6, 7}

Solution

Matching 1	*Matching 2*
6 ↔ 6	6 ↔ 7
7 ↔ 7	7 ↔ 6

✓ Example 3

{a, b, c} and {a, b, c}

Solution

Matching 1	*Matching 2*	*Matching 3*	*Matching 4*	*Matching 5*	*Matching 6*
a ↔ a	a ↔ a	a ↔ b	a ↔ b	a ↔ c	a ↔ c
b ↔ b	b ↔ c	b ↔ a	b ↔ c	b ↔ b	b ↔ a
c ↔ c	c ↔ b	c ↔ c	c ↔ a	c ↔ a	c ↔ b

? Exercise 11

Show every possible 1-1 correspondence (or matching) between the sets listed. This exercise continues on the next page. Label the matchings, so it is clear how many matchings there are, and what pairs are in each matching.

a. {M} and {M}	b. {x,y} and {x,z}	c. {1, 2} and {1, 2}	d. {1, 2, 3} and {1, 2, 3}	e. {1, 2, 3} and {3, 4, 5}

? Exercise 12

If you must list all the possible 1-1 correspondences between 2 sets, each having the cardinality listed below, how many possible matchings are there?

a. 1 element each: ___	b. 2 elements each: ___	c. 3 elements each: ___

The formula for figuring out the number of 1-1 correspondences between sets having n elements is n! (this is read "n factorial"). To figure out what number n! is, multiply all the counting number together up to n. For instance, 1! = 1, 2! = 1 · 2 = 2, 3! = 1 · 2 · 3 = 6, and 4! = 1 · 2 · 3 · 4 = 24. In the very first example where I listed all the possible matchings for two sets, each having 4 elements, there were 24 possibilities. Did you get the right number of possibilities for all the ones you did?

It's time to point out some straightforward, yet important properties of matching.

Property 1

A set always matches itself.

Two sets that match do not have to be different from each other. For instance, you could match {X,Y,Z} with {X,Y,Z} by showing this natural one-to-one correspondence: X « X, Y « Y and Z « Z. (Or you can match them some other way. Here are two other ways to show a matching: X « Z, Y « X and Z « Y OR X « X, Y « Z and Z « Y.)

Property 2

For any two sets A and B, if A matches B, then B matches A.

This property emphasizes that the order in which the sets are mentioned is irrelevant.

Property 3

For any three sets, if A matches B and B matches C, then A matches C.

Exercise 13

Illustrate Property 3 by letting A be the set of small circles, B be the set of small triangles and C be the set of small squares. List the abbreviations (SRC, etc.) or draw the picture for each of the individual elements in each set! *Three separate matchings, each having 4 pairings,* should be shown. The first matching should show one matching between A and B, then there should be another matching shown between B and C. Last, there should be a matching shown between A and C.

Matching between A and B:	Matching between B and C:	Matching between A and C:

Today, most but not all civilizations around the world use the Hindu-Arabic system of counting and numerals 1,2,3,4,5,6,7,8,9,10,11,12,... There are many advantages to this number system. While you might think this common, widely used system of counting the obvious choice, you might consider both the advantages and disadvantages of Hindu-Arabic as we learn about other numeration systems.

Exercise 14

Consider the STROKE system where there was only one symbol, a simple vertical stroke, to express a number. A vertical stroke like this, |, expressed *one*. To express the number *seven*, you would write seven strokes: | | | | | | |

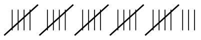
a. Show how you would write the numeral *eleven* in STROKE:

b. *Describe in words* how to write the numerals *five hundred twelve* ; and *two million* in STROKE. Don't attempt to write the actual strokes!! We only have a semester!

Exercise 15

Name some advantages and disadvantages of the STROKE system, compared with the Hindu-Arabic system.

The tally system improved the STROKE system by introducing the concept of **grouping**. You are probably familiar with this system where the fifth stroke was placed across the previous four so it would be easier to read since you can count by fives. STROKE is a single symbol system, but Tally can be thought of as a two symbol system, although the second symbol is really just made up of five strokes. Note the difference in writing the numeral twenty-eight in the two systems.

<table>
<tr><td>STROKE system:
||||||||||||||||||||||||||||</td><td>Tally system:
</td></tr>
</table>

Exercise 16

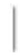
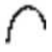
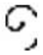
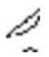
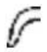
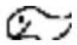
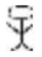
a. Write the numeral for 17 in STROKE.

b. Write the numeral for 17 in Tally.

Exercise 17

If you think of a single stroke (|) as one symbol and the 5-stroke tally (𝍸) as a separate symbol, then it takes eight symbols to express the numeral for 28 as shown above Exercise 13. In STROKE, it took 28 symbols to write the same number. How many symbols does it take to express the numeral for 172 in TALLY?

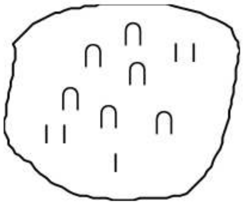
The Old Egyptian numeral system which was developed around 3400 B.C., employs an additive system of counting, where the symbols of the numeral do not have to be in any special order or even on a line going from left to right. The seven symbols in this ancient system are listed below, along with their Hindu-Arabic equivalents.

						
Staff	Heel bone	Scroll	Lotus flower	Pointing finger	Polliwog	Astonished man
One	Ten	Hundred	Thousand	Ten thousand	Hundred thousand	Million
1	10	100	1,000	10,000	100,000	1,000,000

Here are three different ways to write the numeral *sixty five* in the Old Egyptian system:

∩∩∩∩∩∩|||||	or	∩∩||| ∩	∩∩|∩

or as shown below with all the symbols enclosed in a visual set



You simply add up the values of each of the symbols to get the answer.

Definition: additive principle

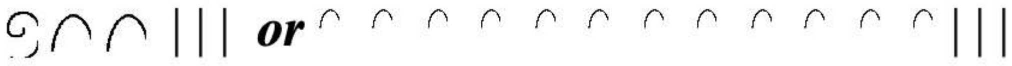
The **additive principle** states that the value of a set of symbols is the sum of the value of the symbols

In this additive system, the Egyptian numeral represents

1,000,000 + 10,000 + 1,000 + 1,000 + 1,000 + 100 + 100 + 1 + 1 + 1 + 1 = 1,013,204.

Remember: The order in which the symbols are written is irrelevant in an additive system.

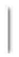
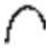
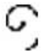
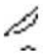
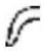
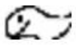
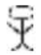
In addition to being able to write the symbols of the numeral in any order, you may also use a *different combination* of symbols used to represent the same number. Here are two different ways to write the number for *one hundred twenty three*:



Exercise 18

Explain in words two more ways you could write the numeral *one hundred twenty three* by using a different combination of symbols than shown above.

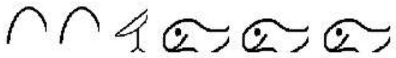
Below is a reminder of what each symbol stands for in order to do the next few exercises.

						
Staff	Heel bone	Scroll	Lotus flower	Pointing finger	Polliwog	Astonished man
One	Ten	Hundred	Thousand	Ten thousand	Hundred thousand	Million
1	10	100	1,000	10,000	100,000	1,000,000

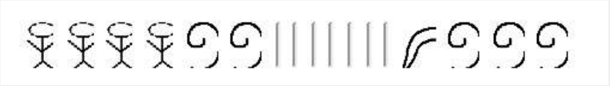
Exercise 19

Express each Egyptian numeral below as a Hindu-Arabic numeral. Show work.

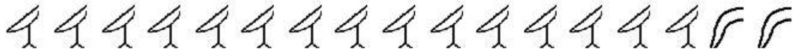
a.



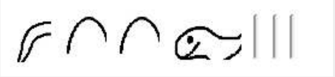
b.



c.



d.



Exercise 20

Express each Hindu-Arabic numeral below as an Egyptian numeral using the least number of symbols possible. Show work.

a. 407

b. 3,051,040

c. 232,501

d. 4,000,000

e. 1,111,111

The Egyptian system employs the technique of grouping, where a certain number of the same symbols are grouped together and replaced with a new symbol. This is necessary in order to symbolize very large numbers efficiently. Systems that use this grouping principle are said to be of a certain base depending on *how many* symbols it takes to exchange to a new symbol. For instance, a system that groups six symbols together and then replaces them with a new symbol is said to be a Base Six system.

? Exercise 21

In what base is the Egyptian system? What is the limitation of this system as far as grouping is concerned? (Think about how to write four trillion!)

? Exercise 22

Name some advantages and disadvantages of the Old Egyptian system compared with the Hindu-Arabic sytem, the STROKE system and the Tally system.

? Exercise 23

Here are examples of some Roman numerals: XVI, MCMLX, XIX, xii, iv, iii. Name some places you still see Roman numerals today.

Extra Credit: Post a picture on the Forum of Roman Numerals in the real world. It must be your own photo not one copied from the internet or another source. Make sure you are in the picture and include the value as a Hindu-Arabic numeral.

The Roman numeration system was in use by 300 B.C. The basic Roman numerals and their corresponding values are listed below. We'll be writing with upper case letters.

I: 1	V: 5	X: 10	L: 50	C: 100	D: 500	M: 1000

Originally, the Roman Numeration System was a simple additive system –for example, DCCCLXXXVII = 500 + 100 + 100 + 100 + 50 + 10 + 10 + 10 + 5 + 1 + 1 = 887. It is actually a Base Ten system with additional symbols for 5, 50 and 500. These extra symbols significantly reduce the amount of writing one must do. For instance, writing the Roman numeral *eight hundred eighty seven* without the use of the additional symbols would look like this: CCCCCCCCXXXXXXXXIIIIIII. Writing this same numeral using the additional symbols would look like this: DCCCLXXXVII. Without the V, L and D, one would need to write out more than twice as many symbols for this particular numeral.

? Exercise 24

Write each Hindu-Arabic numeral as a Roman numeral using the least amount of symbols:

a. 32:	b. 561:
c. 708:	d. 2053:

? Exercise 25

Write each Roman numeral as a Hindu-Arabic numeral and show work:

a. MMDCLXXXV11: ____	b. MCCXXXII:_____

The Roman numeration system developed and incorporated changes over time. One limitation of a simple additive system is that you eventually run out of symbols for very large numbers. Did you already figure that out when answering exercise 19? The Romans devised a clever way to take care of this problem. If a bar was placed over a symbol or set of symbols, it indicated that number was to be multiplied by a thousand. More than one bar could be used on any given symbol or set of numerals. For example, two bars indicated the number was to be multiplied by 1000 1000 (or one million). Simply put, symbols with one bar over them are in the thousands, those with two bars over them are in the millions, those with three bars over them are in the billions and so on. Thus, the Roman numeration system is *multiplicative* as well as *additive*. Study this feature in the following examples:

✓ Example 1

CCL̄VICCX = (100 + 100 + 50 + 5 + 1) 1000 + 100 + 100 + 10 = 256,210

? Exercise 2

V̄L̄XDX = 5 1000000 + (50 + 10) 1000 + 500 + 10 = 5,060,510

? Exercise 26

Write each Hindu-Arabic numeral as a Roman numeral using the least amount of symbols possible.

a. 330,802 ____	b. 70,001,651 ____

? Exercise 27

Write each Roman numeral as a Hindu-Arabic numeral: Show work.

a. IĪICCXII	b. LX̄DCL

Over time, the Roman numeration system developed a *subtractive principle*. If a symbol representing a smaller number is to the left of a symbol representing a larger number, then the total value of those two symbols together represented the value of the larger symbol minus the value of the smaller symbol. There were specific conditions. Only symbols representing powers of ten (I,X,C,M) could be subtracted and each of these four could only be paired with the next two larger symbols. So, the following are the only possibilities for using the subtractive principle:

IV: 4	IX: 9	XL: 40	XC: 90	CD: 400	CM: 900

As you read a Roman numeral from left to right, use the additive principle. If a symbol denoting a smaller value precedes a symbol denoting a larger value, use the subtractive principle when reading those two symbols. Study the following examples:

✓ Example 1

CMXXXIX = (1000-100) + 10 + 10 + 10 + (10 - 1) = 939

✓ Example 2

XC̄IVCCXCII = (100-10 + 5-1) 1000 + 100 + 100 + (100-10) + 1 + 1 = 94,292

? Exercise 28

Rewrite each Roman numeral *without* using the subtractive principle:

a. CMXLIV: ____

b. CDXCIX: ____

? Exercise 29

Rewrite each Roman numeral using the least number of symbols possible and the subtractive principle where applicable:

a. CCCCCCCXXXXIIIIIIII ____

b. MMMMMMMMMMMMMCCCCXXXXXXXX ____
c. CCCCCCCCCXXXXXXIIIIIIIII ____
d. CCCCXXXXII ____

? Exercise 30

Write each Hindu-Arabic numeral as a Roman numeral using the least number of symbols possible and the subtractive principle where applicable:

a. 19,453 ____
b. 2,849 ____
c. 1,996 ____

From now on, because this is how they are used today, write all Roman numerals using the least number of symbols possible and the subtractive principle where applicable unless you are otherwise instructed.

My birthdate is July 15, 1958. If I were to write this date as a six digit numeral (the first two for the month, the second two for the day and the last two for the year), I would write 071558. This is how to write 71,558 as a Roman numeral: LX̄XIDLVIII.

? Exercise 31

Write your birthdate as a six digit numeral like I did for mine above. Then, write it in Roman numerals. If you prefer, you can lie or use the birthdate of someone you love!

? Exercise 32

For numbers under a thousand, indicate the most times each symbol below can be repeated in a row when written as part of a Roman numeral. Assume the subtractive principle is used.

In this exercise set, you have learned some basic concepts of counting and have also learned about some numeration systems. In the next set, you'll learn about two more numeration systems. We'll end these exercises by comparing the systems we have worked with so far. First, we will consider the number of symbols one has to memorize in order to understand each individual system. We will then determine how many symbols one has to write down in order to represent various numerals in the different systems.

? Exercise 33

For each of the numeration systems we've learned about so far, state how many different symbols a person has to memorize to understand the system. For Roman, do not consider symbols with bars over them as different symbols.

a. Hindu-Arabic: _____	b. STROKE: ____	c. Tally: ____	d. Egyptian: ____	e. Roman: ____

Let's write the numeral 67 in the different systems and note how many total symbols (not necessarily different symbols) it takes to express the number 67 in each system.

Hindu-Arabic:	We write: 67	2 symbols
STROKE:	We write 67 strokes, shown below ǀǀǀ	67 symbols
Tally:	WE write 13 tally groups and 2 strokes, shown below 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ǀǀ	15 symbols

Egyptian:	We write:	13 symbols
Roman:	We write: LXVII	5 symbols

Exercise 34

Write the numeral for 900 in each of the five systems below and state how many symbols it takes to write out the full numeral. Rather than writing the numerals in STROKE and Tally, *explain in words* how you would write them, still indicating the total number of symbols that would have to be written.

a. Hindu-Arabic: _____	b. STROKE: ____	c. Tally: ____	d. Egyptian: ____	e. Roman: ____

Exercise 35

For each Hindu-Arabic number across the top row, state the least number of symbols it takes to write the numeral in each of the given numeration systems on the left. Some answers are filled in for you. You don't need to actually write the numeral –just determine how many symbols you would have to write down if you *were* going to write the numeral. Write your answer in Hindu-Arabic.

	143	400	1,000,000	30,009	2,124
Hindu-Arabic	3		7		
Stroke				30,009	
Tally	31		200,000		
Egyptian					9
Roman		2			

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2.2: Numeration Systems

In this exercise set, you will learn about two ancient numeration systems –those of the Chinese and Mayans.

Chinese Numerals

Chinese numerals are still used today. Symbols for some Chinese numerals are shown below.

1	2	3	4	5	6	7	8	9	10	100	1,000
一	二	三	四	五	六	七	八	九	十	百	千

Chinese numerals are formed by writing the symbols vertically and using the multiplicative principle, which simplifies the recording of numerals by eliminating the repetition of symbols. For instance, the Chinese write the numeral for 3,058 by thinking $3 \times 1000 + 5 \times 10 + 8$ and write down the symbols 3, 1000, 5, 10 and 8 in that order to represent that number. Even though you can think of the 8 as 8×1, the 1 is not written down. This Chinese numeral (3,058) is shown to the left. The Chinese numeral for 872 is shown to the right.

三千五十八

八百七十二

Below are some more Chinese numerals. Make sure you understand how to read all of them before trying the exercises.

6,400	87	9,531	2,605	4,011	7,000
六千四百	八十七	九千五百三十一	二千六百五	四千一十一	七千

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? Exercise 1

Write each Hindu-Arabic numeral as a Chinese numeral.

a. 5,093
b. 610
c. 427
d. 8,008

So you won't have to keep turning back the page to remember these symbols, here again are the Chinese numerals you probably need to look at to do the following exercise.

1	2	3	4	5	6	7	8	9	10	100	1000
一	二	三	四	五	六	七	八	九	十	百	千

? Exercise 2

Rewrite each Chinese numeral in Hindu-Arabic numeral .

a. ____	b. ____	c. ____	d. ____	e. ____
一 百 九	六 千 六 十 三	四 十	二 千 八 百 一 十 五	七 千 八 百

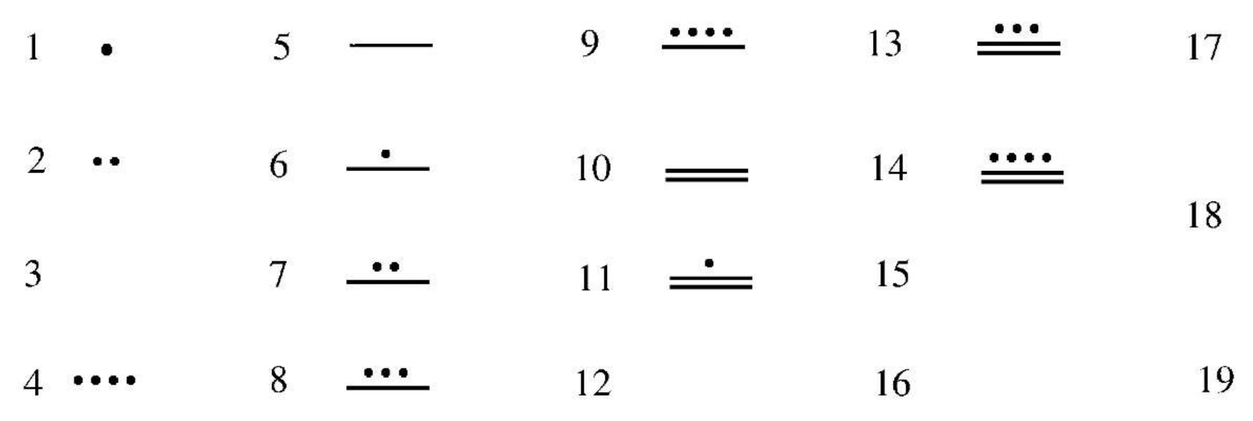
Notice in the Chinese system that numbers over nine have symbols written in pairs. To write 800, you must write the symbol for 8 above the symbol for 100. IMPORTANT NOTE: This is true even if there is only a "1" in the place value, see 2d above. You MUST remember to write the symbol for "one" above the symbol for "ten"! Although the multiplicative principle allows you to write down fewer symbols than a simple additive system (for most numerals, at least), a further simplification would allow us to skip writing the second numeral of each pair. This would work if we used the *position* of the symbol to indicate the size of that group (10, 100, 1000, etc.) This type of system is called a **positional numeration system.** In order to keep track of a position where no digit is used, a symbol for zero is necessary. Although the Chinese system doesn't need a symbol for zero, a circle was introduced to represent zero in the 1200's.

Mayan Numerals

The final system we'll learn about in this Exercise Set uses a positional system and is similar to the Chinese system in that the symbols for the numerals are written from top to bottom. Mayan Numerals were developed by the Mayan priests of southern Mexico and Central America around 300B.C. It is believed to be the earliest positional numeration system incorporating a zero and using it for a placeholder.

? Exercise 3

Some Mayan Numerals are shown below. Try to figure out the pattern and then fill in the missing numerals.



? Exercise 4

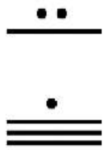
Explain what symbols are in this system, what they stand for and how the system works for at least the numerals one to nineteen.

From what you have seen of this system so far, it might look like a simple additive system. One might guess that the numeral for 20 would be four line segments and that the numeral for 103 would be twenty line segments and three dots. At first glance, it is very similar to the tally system. However, the Mayans used a vertical *positional* system. The bottom level represented how many units (or ones), the second level up represented how many 20's, the third level up represented how many 360's (20 18), the fourth level up represented how many 7200's (20 18 20), the fifth level up represented how many 144,000's (20 18 20 20), etc. Except between the second and third level, each place value increased by a multiple of 20. It is almost a Base Twenty system except for that strange third level. Why the third level is 18 times the second level is explained later. Looking at the chart to the left, which shows the first four place values, may help you understand the system.

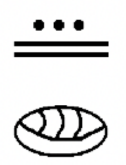
7200's
360's
20's
1's

To try and make sense of all of this, we will look at some Mayan numerals that have more than one position now. The numerals from one to nineteen only utilize the bottom level so it isn't apparent that the Mayan system is positional until you count past nineteen.

To the left is a two-level Mayan numeral. There is a 16 in the bottom level, representing 16 ones, or 16 (16×1), plus there is a 7 in the second level up, representing 7 groups of twenty, or 140 (7×20). We add the values of each level, 16 + 140, so the numeral you see represents 156.



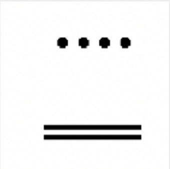
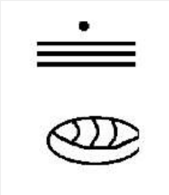
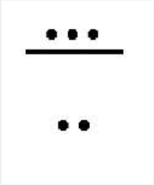
You have seen two basic symbols in this system so far –a dot ( ), which represents the number one and a line segment ( ), which represents the number five. As previously mentioned, there needs to be a symbol for zero to incorporate the use of place value. For that, the Mayans used a shell that looked something like this:



To the left is a two level Mayan numeral with a zero in the bottom level, representing zero ones, or 0, plus a 13 in the second level up representing 13 groups of twenty, or 260 (13×20). So after adding the values together (0 + 260), the numeral you see represents the number 260.

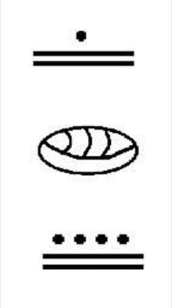
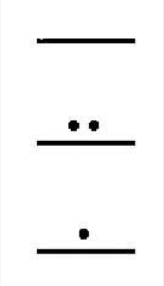
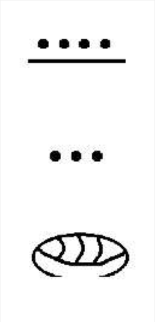
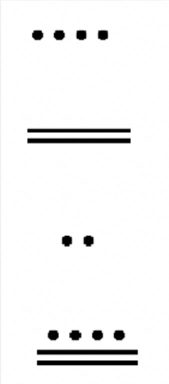
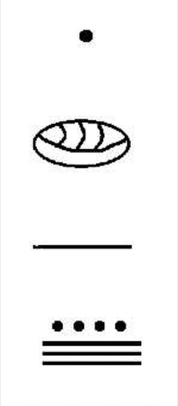
Exercise 5

State the Hindu-Arabic equivalent of each Mayan numeral. Show how you obtained your answers. Note that proper space should be left between each place value. Otherwise, someone might incorrectly conclude the number shown for 5a represents 14, which would be the answer if there was no space.

a. _____	b. _____	c. _____
		

Now, we'll go on to some three and four level Mayan numeral. Remember that the place value for the third level up is 360 and the place value for the fourth level up is 7200. See if you can figure these out on your own first.

Exercise 6

a. _____	b. _____	c. _____	d. _____	e. _____
				

How did you do? Make sure that if you still have any trouble understanding any of these that you go back and work through them again.

It's a little trickier to start with a Hindu-Arabic numeral and convert it to Mayan, but with a little patience and practice, you'll be doing it quickly and accurately! Before showing a method for doing this, try the following exercise. Hint: It should be easy, not hard –no calculator is required. Think about the place values of the various levels in the Mayan system.

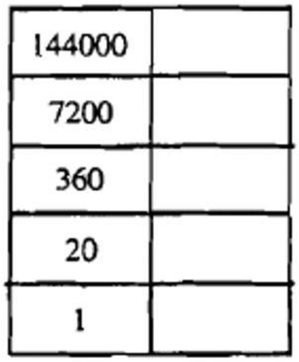
? Exercise 7

Write the Mayan numeral equivalents for each of the following numbers:

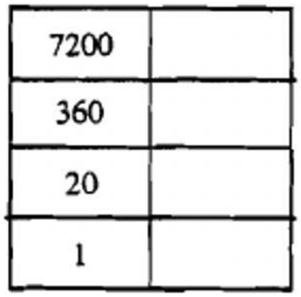
a. 1	b. 20	c. 360	d. 7200	e. 144000

To convert a number to a Mayan numeral, the first thing you'll have to determine is how many levels the numeral will have. Remember the levels: 1, 20, 360, 7200, 144000, 2880000, etc. So any numeral less than 20 has one level, a numeral between 20 and 359 has two levels, a numeral between 360 and 7199 has three levels, a numeral between 7200 and 143999 has four levels, and so on.

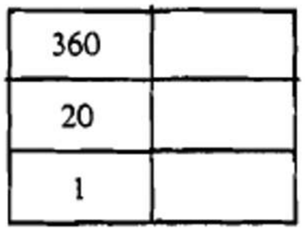
It might help if you set up a table with the correct number of levels set up already with a space to fill in what symbol you'll be using at each level. For instance, look below at the four most common charts you'll be using.



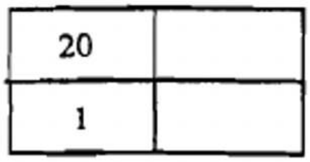
144000	
7200	
360	
20	
1	



7200	
360	
20	
1	

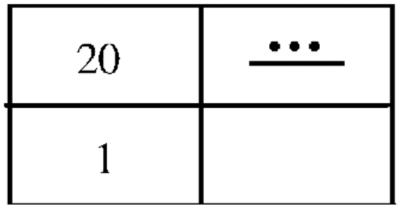


360	
20	
1	

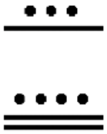


20	
1	

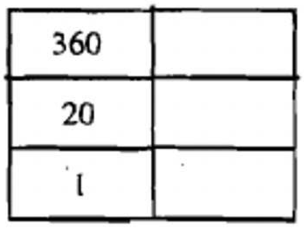
Let's start with the number 174. Convince yourself that this will be a two level numeral. We'll start at the top, which is the 20's place value. We have to ask ourselves how many 20's are in 174? This is a division question: $174 \div 20 = 8$, remainder 14. We can begin to construct the Mayan numeral by starting with the two-level chart and filling an 8 in second level up as shown below.

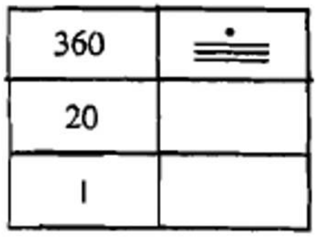


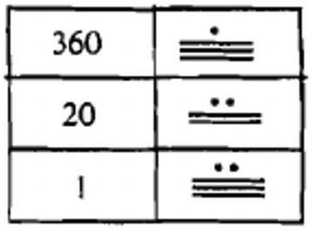
So far, we have eight groups of twenty, or 160 filled in, which leaves 14 more (the remainder) to accommodate. When you get down to the units place, the remainder is filled in there. So the next step is to fill 14 in the units place. Do that in the vacant place in the chart shown. Before feeling satisfied that everything is correct, check your answer by computing the Mayan numeral you have just constructed and see if it is indeed 174. Then, get rid of the chart and write the answer as a Mayan numeral as shown to the right.

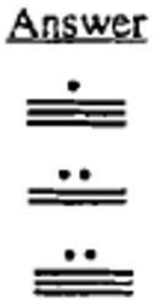


Let's try another number. We'll convert 6017 to a Mayan numeral. This will be a three level numeral, so we'll start off with a three level chart and figure out how many 360's are in 6017. We do this division problem: $6017 \div 360 = 16$, remainder 257. This tells us to put the symbol for 16 in the third level up (360's place). Now we have to take the remainder, 257, and find out how many 20's there are in it to find out what to put in the second level. We do this division problem: $257 \div 20 = 12$, remainder 17. Since only the units place value is left (the bottom level), the remainder of 17 goes in that level. The sequence of filling in the charts is shown below. Make sure you go back and check your work by converting the numeral back to Hindu-Arabic $(1 \times 17 + 12 \times 20 + 16 \times 360)$ and seeing if it really equals 6017.









Before going on, I'm going to show you a quick and easy way to find out the quotient and remainder by using a simple calculator when doing these division problems. If you can already do it easily or your calculator figures it out for you, skip on down to Exercise 8. Let's say you were going to change the number 5263 to Mayan. The first division problem you would have to compute would be $5263 \div 360$. When you do this on your calculator, it shows up something like 14.619444. This indicates that there are 14 360's in 5263, but the remainder isn't evident. At least, you know to put 14 in the third level up. To find the remainder on your calculator, key in 14 360 - 5263 and the number showing is the remainder if you ignore the negative sign! In this case, the remainder is 223. Remember that the remainder must be less than what you originally divided by –less than 360 in this case. If you wanted to find what to put in the 20's place, you would repeat this process by taking 223 (the remainder) and dividing it by 20. You might not need a calculator to continue. Let's try to find the quotient and remainder when dividing 24567 by 7200. On your calculator, $24567 \div 7200 = 3.4120833\ldots$ So the quotient is 3. To get the remainder, key in $3 \times 7200 - 24567$, which gives a remainder of 2967 (which is less than 7200!). So the answer is 3, remainder 2967. To check, 7200 3 + 2967 must equal 24567. Think about why this process works and try it on the next few problems.

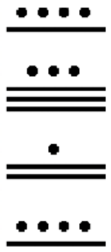
? Exercise 8

Use a calculator to find the quotient and remainder for these division problems.

a. $9876 \div 360 =$ _____
b. $71509 \div 7200 =$ _____
c. $333 \div 20 =$ _____
d. $430040 \div 144000 =$ _____

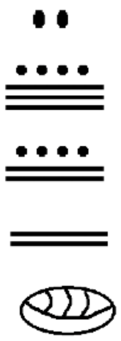
7200	
360	
20	
1	

Let's convert 71509 to Mayan. Fill in the Mayan symbols on the chart to the left as we work through the problem. This is a four level Mayan numeral and Exercise 8b should help us get started! Did you get 9, remainder 6709? So 9 is in the fourth level up. Now, we compute: $6709 \div 360 = 18$, remainder 229. So 18 is in the third level up. Next computation: $229 \div 20 = 11$, remainder 9, so 11 is the second level up and that leaves 9 in the units place. The final answer is shown to the right.

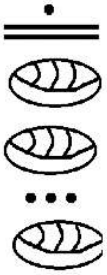


144000	
7200	
360	
20	
1	

Well, which number should we convert now? How about 430040 so we can use the help of Exercise 8d? Fill in the Mayan symbols on the chart to the left as we work through the problem. We first note that this is a five level Mayan numeral and so we do the division in Exercise 8d which gives a 2 at the fifth level up with a remainder of 142040. This is a big remainder but less than 144000. Now, we divide 142040 by 7200 to get 19 at the fourth level up and a remainder of 5240. Next, divide 5240 by 360 to get 14 at the third level up and a remainder of 200. There are exactly ten 20's in 200 and no remainder for the units, so we need to have a zero at the bottom level. Finally, the numeral we seek is shown on the right. **Make sure you check it** by computing $2 \times 144000 + 197200 + 14 \times 360 + 10 \times 20 + 0$! Does it really equal 430040? If so, it's right.



Let's convert 1584060 to Mayan. This is another five level Mayan numeral, so our first division of $1584060 \div 144000$ gives us 11 at the fifth level up with a remainder of 60. There are no 7200's or 360's in 60, so there must be zeroes at those two levels. There are 3 20's in 60 with no remainder, so the units level must also have a zero. The numeral is shown on the right. Please check it. Is it really 1584060?



? Exercise 9

Think about the highest numeral value that can be on any given level. For instance, the bottom level could go up to 19, or . But what about the second level up? What number would be represented by a two level Mayan numeral if there was a 19 at the second level up and a zero at the bottom level? Write the Mayan numeral for 380.

? Exercise 10

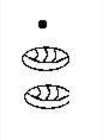
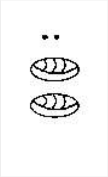
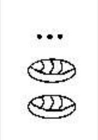
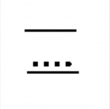
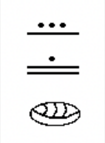
What is the highest Mayan numeral value that should be on the second level up? What is the highest Mayan symbol that could be on any level except the second level up?

? Exercise 11

Convert each of the following to Mayan numerals. (show work) CHECK Answers!

a. 1549
b. 4750
c. 53981
d. 145804
e. 1000000

You may be wondering why the Mayans chose 360 for the third level up instead of 400, which seems more natural. Their counting system was based on their calendar, which consisted of 18 months of 20 days each, hence 360. The extra five days were considered "useless" and they didn't worry about them. Their system made it easy to count time. They didn't count the "useless days." For instance, consider the following:

a. 1 year	b. 2 years	c. 3 years	c. 5 months, 9 days	d. 8 years, 11 months
				

? Exercise 12

Write the Mayan numeral for the number of (non-useless) days in:

a. 7 months, 15 days
b. 13 Mayan years

c. 20 Mayan years

The final exercise of this set will be to compare the Hindu-Arabic, Chinese and Mayan numerals.

? Exercise 13

State how many different symbols a person has to memorize to understand each system:

a. Hindu-Arabic: _____
b. Chinese: _____
c. Mayan: _____

? Exercise 14

Write each number as a numeral in Chinese and Mayan. Note how many symbols it takes to write the given number in Hindu-Arabic, Chinese and Mayan. Show work.

a. 15
b. 100
c. 100
d. 9999

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2.3: Trading and Place Value

*You will need:**Base Two Models** (Material Card 3), **Calculator***

You will be exploring more about place value in this exercise set. We'll start off by working with the Base Two Models on Material Card 3, which has a model set of cups (C), pints (P), quarts (Q), half-gallons (H), gallons (G) and double-gallons (D). Cut them out and use them to do exercises 1 - 4. Note that we'll be using triangles for our models except for a large square which will represent the double-gallon. By working with these models, it should be clear how many cups are in a pint, how many pints are in a quart, etc.

Exercise 1

Fill in the blanks with the correct numeral.

a. There are _____ cups in a pint.
b. There are _____ pints in a quart.
c. There are _____ quarts in a half-gallon.
d. There are _____ half-gallons in a gallon.
e. There are _____ gallons in a double-gallon.

Exercise 2

Suppose you were sick for nine days and a friend came over three times a day, each time leaving you a cup of soup. Suppose you never ate any of the soup, but instead kept putting each cup in the refrigerator in your garage. After those nine days were up, your refrigerator was stocked with all these individual cups of soup. You decide to consolidate these into as few containers as possible. You have five different kinds of containers –those holding cups, pints, quarts, half-gallons and gallons. Fill in each blank with the correct numeral.

a. Using your model cups, set aside three cups a day for nine days into a pile. There are _____ cups in this pile.
b. Trade in the cups for as many pints as you can. There are now _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.
c. Trade in the pints for as many quarts as you can. There are now _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.
d. Trade in the quarts for as many half-gallons as you can. There are now _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.
e. Trade in the half-gallons for as many gallons as you can. There are now _____ gallon(s), _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup
f. Write down the information from part e in a consolidated fashion by filling in each space in the chart with the correct numeral.

G	H	Q	P	C

g. Originally, from part (a), there were _____ cups of soup. By consolidating (see part e and f), the number of containers was reduced to only _____ containers of soup!

Exercise 3

Repeat exercise 2, but change the circumstances so that the friend only brings over two cups of soup a day for nine days.

a. Using your model cups, set aside two cups a day for nine days into a pile. There are _____ cups in this pile
b. Trade in the cups for as many pints as you can. There are now _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

c. Trade in the pints for as many quarts as you can. There are now _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.

d. Trade in the quarts for as many half-gallons as you can. There are now _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

e. Trade in the half-gallons for as many gallons as you can. There are now _____ gallon(s), _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

f. Write down the information from part e by filling in each space in the chart with the correct number

G	H	Q	P	C

Exercise 4

Repeat exercise 2 one more time, but change the circumstances so that the friend brings over five cups of soup a day for nine days. Also, let's assume you also own a 2 gallon container, which we'll refer to as a double-gallon.

a. Using your model cups, set aside five cups a day for nine days into a pile. There are _____ cups in this pile

b. Trade in the cups for as many pints as you can. There are now _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

c. Trade in the pints for as many quarts as you can. There are now _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

d. Trade in the quarts for as many half-gallons as you can. There are now _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

e. Trade in the half-gallons for as many gallons as you can. There are now _____ gallon(s), _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

f. Trade in the gallons for as many double-gallons as you can. There are now _____ double-gallon(s), _____ gallons, _____ half-gallon(s), _____ quart(s), _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup

g. Write down the information from part f by filling in each space in the chart with the correct number.

G	H	Q	P	C

In the previous exercises, exchanges or trades were being made any time you had two of one container. You were basically employing the grouping technique we discussed back in Exercise Set 1 (look back at the bottom of page 8 of that exercise set). When employing the grouping technique, we are working in a particular base, depending on how many it takes to form a group. In the exercises you just completed, you were working in Base Two. The final answers that you obtained in Exercises 2, 3 and 4 are the way to express the original number of cups as a Base Two numeral. In exercise 4, five cups of soup for nine days translates to 45 in our Base Ten system (four groups of ten and five units). In Base Two, the final answer of 1 double gallon, 0 gallons, 1 half-gallon, 1 quart, 0 pints and 1 cup is written as a Base Two numeral like this: 101101_{two}. In other words, $27_{\text{ten}} = 11011_{\text{two}}$. On the next pages, I'll explain in detail what this numeral means and how the place value system works in various bases. But first, there are some important points you need to know about how to write, read and say numerals in different bases.

To write a numeral in a given base, notice that the base is written out in words to the right and a little below the numeral, as in 11011_{two}. The only time you do not have to write the base is when it is a Base Ten numeral! A numeral without the base explicitly written out is assumed to be a Base Ten numeral.

- In exercise 3, the Base Ten numeral 18 (two cups for nine days) is written as 10010_{two}.
- In exercise 4, the Base Ten numeral 45 (five cups for nine days) is written as 101101_{two}.

It is imperative that you read and say these numerals in different bases correctly. It's much easier to make mistakes if you say the numeral wrong and much easier to avoid mistakes if you say the numeral correctly. The following paragraph explains the proper way to read or say the numerals.

In Base Ten, we have abbreviated ways of saying numerals out loud. For instance, "13" is read "thirteen". But, "13_{five}" is read "One, three, base five" and "246201_{eight}" is read "two, four, six, two, zero, one, base eight." *It's extremely important that you learn to read and say* "13_{five}" as "one, three, base five" AND NOT AS "thirteen, base five"! THERE IS NO THIRTEEN IN BASE FIVE!!! When you *say* thirteen, it refers to the base ten numeral meaning one group of ten and three ones. Although I haven't yet explained what we mean when we say and write "13_{five}", or "one, three, base five", it's important that you practice saying these numerals correctly from the start. Even though when we read it, we say the word "base", *do not write the word "base"*.

The following are incorrect: $^{13}5$.......$^{13}\text{Base } 5$........$^{13}\text{Base five}$

Exercise 5

Write out in words how to say each of the following numerals.

a. 302_{six} ______
b. 1011_{two} ______
c. 435_{seven} ______

Exercise 6

Write the given numerals

a. "Five, zero, one, six, base eight" ______
b. "one, zero, one, zero, zero, one, base two" ______

When you see a Base Two numeral like 11011_{two} from exercise 2, you need to understand what each place value stands for. *In all bases*, the rightmost place value is the units or 1's place. Next, the number name "two" to the right of the numeral tells you this is a Base Two numeral, which means each place value as you move left increases by a multiple of 2. To the right is a chart showing the first eight place values in Base Two. They are written in Base Ten.

128	64	32	16	8	4	2	1

To check that the Base Two numeral 11011_{two} really is 27_{ten}, put it into a chart with just five place values showing (since it's a five-digit numeral) and check the total value of the numeral. Look at the numeral written to the right. The place value for each digit is written below each digit of the numeral. The numeral is written in bold so it does not get confused with the place values. You check this exactly the way you did for the Mayan numerals:

$$\begin{aligned}(1 \text{ group of } 16) + (1 \text{ group of } 8) + (0 \text{ groups of } 4)\\ + (1 \text{ group of } 2) + (1 \text{ group of } 1) &= (\mathbf{1} \times 16) + (\mathbf{1} \times 8) + (\mathbf{0} \times 4)\\ &\quad + (\mathbf{1} \times 2) + (\mathbf{1} \times 1)\\ &= 16 + 8 + 2 + 1\\ &= \mathbf{27}.\end{aligned}$$

All right! It really works! In fact, it's really easy to check the arithmetic for a Base Two numeral because for each place value containing a 1, that place value gets added to the total, whereas those place values containing a zero do not get added to the total. There's no multiplication you have to do –you can skip directly to the addition: 16 + 8 + 2 + 1 = **27**.

$$\boxed{\frac{\mathbf{1}}{16}\,\frac{\mathbf{1}}{8}\,\frac{\mathbf{0}}{4}\,\frac{\mathbf{1}}{2}\,\frac{\mathbf{0}}{1}\ \mathbf{two}}$$

Let's convert $\mathbf{10\ 011\ 010}_{\text{two}}$ to Base Ten. Again, we'll write it with the place values shown under each digit and add up the place values containing a 1. This numeral converts to 128 + 16 + 8 + 2 = **154**.

$$\boxed{\frac{\mathbf{1}}{128}\,\frac{\mathbf{0}}{64}\,\frac{\mathbf{0}}{32}\,\frac{\mathbf{1}}{16}\,\frac{\mathbf{1}}{8}\,\frac{\mathbf{0}}{4}\,\frac{\mathbf{1}}{2}\,\frac{\mathbf{0}}{1}\ \mathbf{two}}$$

Exercise 7

Fill in the chart to show what values are missing in this Base Two place value chart.

				128	64	32	16	8	4	2	1

Exercise 8

Convert each Base Two numeral to Base Ten.

a. $10\ 011_{\text{two}}$	b. $1\ 000\ 001_{\text{two}}$	c. $111\ 111_{\text{two}}$

Let's convert some numerals in other bases. Consider the Base Five numeral, 24_{five}. First, we need to establish the place values in Base Five just like we did for Base Two.

Exercise 9

Fill in the chart to show what values are missing in this Base Five place value chart.

			25		1

In Base Five, $\mathbf{24}_{\text{five}}$ is a two digit numeral. Note there are **2** groups of 5 and **4** units or $2\times 5+4\times 1=10+4$ = **14**.

$$\boxed{\frac{\mathbf{2}}{5}\,\frac{\mathbf{4}}{1}\ \textbf{five}}$$

Exercise 10

Convert $\mathbf{13}_{\text{five}}$ to a Base Ten numeral. *Remember not to read this as "thirteen"!!!*

Here is how to convert $\mathbf{31204}_{\text{five}}$ to Base Ten. Look below to understand this computation:

$$\begin{aligned}\mathbf{3}\times 625+\mathbf{1}\times 125+\mathbf{2}\times 25+\mathbf{0}\times 5+\mathbf{4}\times 1 &= 1875+125+50+4\\ &=\mathbf{2054}.\end{aligned}$$

$$\boxed{\frac{\mathbf{3}}{625}\,\frac{\mathbf{1}}{125}\,\frac{\mathbf{2}}{25}\,\frac{\mathbf{0}}{5}\,\frac{\mathbf{4}}{1}\ \textbf{five}}$$

Let's take the Base Three numerals $\mathbf{1\ 002\ 021}_{\text{three}}$ and $\mathbf{22122}_{\text{three}}$ and convert to Base Ten. First, we should make a place value chart for Base Three.

Exercise 11

Fill in the missing place values for Base Three.

		243			9		

To convert $\mathbf{1\ 002\ 021}_{\text{three}}$, look at the above chart to understand this computation:

$$\mathbf{1}\times 729+\mathbf{2}\times 27+\mathbf{2}\times 3+\mathbf{1}=729+54+6+1=\mathbf{790}.$$

To convert $\mathbf{22122}_{\text{three}}$, look at the above chart to understand this computation:

$$\mathbf{2}\times 81+\mathbf{2}\times 27+\mathbf{1}\times 9+\mathbf{2}\times 3+\mathbf{2}=162+54+9+6+2=\mathbf{233}.$$

Exercise 12

Convert the following Base Three numerals to Base Ten.

a. 200 112_{three} = _____
b. 12 002 110_{three} = _____
c. 1 111 111_{three} = _____

Exercise 13

Figure out the base in each place value chart and fill in the missing place values for each

a. Base _____

			36		

b. Base _____

				7	

c. Base _____

			81		

d. Base _____

			64		

e. Base _____

		64			

f. Base _____

	10000				

g. Base _____

				12	

h. Base _____

				11	

Another way to write out the values in a place value chart for a given base is by using exponents. Each place value is a power of the base. When writing it out this way, the actual Base Ten values are not explicitly written out. The following is another way to write a place value chart for Base Seven:

7^{11}	7^{10}	7^9	7^8	7^7	7^6	7^5	7^4	7^3	7^2	7^1	7^0

Did you remember that 70 = 1? Any nonzero number raised to the zero power equals 1. For example, $3^0 = 1, 15^0 = 1, 1^0 = 1, 10^0 = 1, 546^0 = 1$ and so on.

? Exercise 14

Don't panic, but it's time to generalize just a little about the place value system for any given base. Consider Base n –yes, it's a variable just like in algebra! Look at the place value chart below and fill in the missing place values for Base n.

n^{11}				n^7				n^3			n^0

If you are converting very large numbers, it might be useful to write out a place value chart using the base written to a given exponent to save time and space.

For instance, let's say you were asked to convert the numeral $100\ 200\ 020\ 000\ 100_{\text{three}}$ to Base Ten. This represents a very large number. But there are a lot of zeros for most of the place values. In fact, this fifteen digit numeral only has four nonzero digits. So why bother writing out a chart with all those place values? Instead, we know each place value is a power of three, since this is a Base Three numeral. In Base Ten, this numeral is:

$$\begin{aligned} 1\times 3^{14}+2\times 3^{11}+2\times 3^{7}+1\times 3^{2} &= 1\times 4782969+2\times 177147+2\times 2187+1\times 9 \\ &= 4782969+354294+4374+9 \\ &= 5,141,646 \end{aligned}$$

? Exercise 15

$100\ 200\ 020\ 000\ 100_{\text{three}}$ can be written as $1\ \times 3^{14}\ +2\ \times 3^{11}\ +2\times 3^{7}\ +1\ \times 3^{2}$. This is called expanded notation. Pay close attention to the fact that the 1 in the third place from the right is 3 to the second (not the third) power. Write the following numerals in expanded notation:

a. $3\ 000\ 600\ 020\ 000\ 000_{\text{eight}}$ _____
b. $3\ 000\ 040\ 020\ 000\ 000_{\text{five}}$ _____
c. $400\ 030\ 000\ 000\ 002_{\text{eleven}}$ _____
d. $100\ 100\ 000\ 010\ 000_{\text{two}}$ _____

? Exercise 16

Refer to the numeral, $3\ 040\ 001\ 000\ 002_{\text{nine}}$. For each nonzero digit in the numeral, write the place value of that digit as a power of nine. *Do not write out the actual Base Ten numeral.*

a. *1* ____
b. *2* ____
c. *3* ____
d. *4* ____

If a number is written in expanded notation, you can reverse the process and write it as a number in the base used. For instance, $4\ \times 7^{13}\ +5\times 7^{10}\ +2\times 7^{8}\ +3\ \times\ 7^{3}+6\ \times\ 7^{0}$ can be written as the Base Seven numeral $40\ 050\ 200\ 003\ 006_{\text{seven}}$. NOTE: A number in the *ninth* place from the right is really the *eighth* power of the base, as shown by the placement of 2 in the numeral $40\ 050\ 200\ 003\ 006_{\text{seven}}$.

? Exercise 17

Rewrite the following numbers that are in expanded notation to a numeral in the base indicated:

a. $2\times 5^{9}\ +4\times 5^{8}\ +1\times 5^{6}\ +3\times 5^{2}$ in Base Five ____
b. $3\times 8^{10}\ +4\times 8^{7}\ +1\times 8^{3}\ +6\times 8^{2}$ in Base Eight ____
c. $7\times 12^{9}\ +8\times 12^{5}+4\ \times 12^{3}$ in Base Twelve ____
d. $2\times 3^{8}\ +1\times 3^{6}\ +1\times 3^{4}\ +2\times 3^{2}$ in Base Three ____

Exercise 18

a. If a Base Six numeral has eighteen digits, what is the place value of the first (leftmost) digit written as a power of six? ____

b. If a Base Nine numeral has twenty digits, what is the place value of the first (leftmost) digit written as a power of nine? ____

Take another look at some numerals in some bases besides Base Ten:

$1\ 100\ 101_{\text{two}}$	$212\ 201_{\text{three}}$	42101_{seven}	80034_{nine}

Did you notice that there are only 0's and 1's in Base Two numerals? That's because if there was a 2 or higher for one of the placeholders, it could be traded in for the next place value to the left. Using physical models at the beginning of this exercise set, two cups could be exchanged for a pint, two pints could be exchanged for a quart, etc. Although we only went up to the double-gallon which contained 32 cups, there is really no limit to the number of place values a numeral has in a place value system. Also, one doesn't have to come up with a new name for each place value (as in "heel bone" or "scroll" in Egyptian).

Think about any Base Ten numeral. There are ten possible digits (or symbols) for each placeholder in the numeral 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. There is no separate single digit for the number ten (10); it is made up of a 1 and 0, which means one group of ten and zero units! In Base Two, there are only two possible digits for each place value in the numeral: 0 and 1. There is no need or use for the symbol (or digit) "2" when writing a numeral in Base Two.

Exercise 19

How many possible digits can each place value in Base Three have? _____ What are those possible digits? ____

Exercise 20

How many possible digits can each place value in Base Four have? _____ What are those possible digits? ____

Exercise 21

How many possible digits can each place value in Base Five have? _____ What are those possible digits? ____

Exercise 22

How many possible digits can each place value in Base Six have? _____ What are those possible digits? ____

Exercise 23

How many possible digits can each place value in Base Eight have? _____ What are those possible digits? ____

Exercise 24

Can the digit (symbol) "7" be used in a Base Seven numeral? _____ Explain your answer.

Exercise 25

a. What is the lowest base in which the digit "6" may appear? _____

b. What is the highest base in which the digit "6" may appear? _____

Do you see a pattern developing? Just wait until you see what happens in Base Eleven, Twelve and Thirteen!

Exercise 26

How many possible digits for each place value do you need for Base Eleven? _____

Exercise 27

How many possible digits for each place value do you need for Base Twelve? _____

Exercise 28

How many possible digits for each place value do you need for Base Thirteen? _____

In Base Eleven, you need eleven different symbols (or digits) for possible placeholders. The problem we encounter is that we only have these ten recognizable digits –0,1,2,3,4,5,6,7,8 and 9. *Remember that 10 is made up of two separate digits –it isn't a single symbol!!* So we need to introduce a new symbol to represent ten in Base Eleven and higher. The convention is to use the letter "T". Similarly, we use the letter "E" to represent the number eleven in Base Twelve and higher and the letter "W" to represent the number twelve in Base Thirteen and higher. Since the second place value from the right represents the base you are working in, you never need a separate symbol to represent the value of the base or any number higher than the base. That's why there are only 0's and 1's and *no 2's* in Base Two and why there are only 0's, 1's, 2's, 3's and 4's and *no 5's* in Base Five, etc.

The next few problems are worked exactly like those you have done so far, except the bases are higher than ten. You will need to remember the values of the new "digits" T, E and W when working these problems.

144	12	1

Let's convert the numeral $TE5_{\text{twelve}}$ to Base Ten. To remember the place values, we might want to draw a Base Twelve place value chart for three digits as shown to the right.

Then, $TE5_{\text{twelve}} = 10 \times 144 + 11 \times 12 + 5 \times 1 = 1440 + 132 + 5 = \mathbf{1577}$.

Pay close attention to the difference between $2T9_{\text{eleven}}$ and 2109_{eleven}.

To convert $2T9_{\text{eleven}}$ to Base Ten, first consider the Base Eleven place value chart shown.

1331	**121**	**11**	**1**

$2T9_{\text{eleven}} = 2 \times 121 + 10 \times 11 + 9 \times 1 = 242 + 110 + 9 = \underline{361}$.

Now, let's convert 2109_{eleven} to Base Ten. Looking at the Base Eleven place value chart, 2109_{eleven} represents $2 \times 1331 + 1 \times 121 + 011 + 9 \times 1 = 2662 + 121 + 0 + 9 = 2792$

If someone told another person to convert "two, ten, nine, base eleven" as opposed to "two, T, nine, base eleven," it might not be clear what they meant. One person might write 2109_{eleven} (which would convert to 2792 in Base Ten, as illustrated above) whereas another person might write $2T9_{\text{eleven}}$ (which would convert to 361 in Base Ten, as illustrated above). It's important to remember to use the "T" to represent the number ten in Base Eleven and higher or it might be written as "10" which is two separate place values. $2T9_{\text{eleven}}$ is a three digit numeral and it is clear that there is a ten in the middle digit's place value. On the other hand, 2109_{eleven} is a four digit numeral –that is not a ten in the middle –and it represents a completely different number than $2T9_{\text{eleven}}$!

Let's try another one. We will convert TEW_{thirteen} to Base Ten. First, we should fill in a place value chart for Base Thirteen.

2197	169	13	1

Then, $TEW_{\text{thirteen}} = 10 \times 169 + 11 \times 13 + 12 \times 1 = 1690 + 143 + 12 = \underline{1845}$

? Exercise 29

Convert each of the following to a Base Ten numeral. Show your work.

a. $47E_{\text{twelve}} =$	b. $T74_{\text{eleven}} =$	c. $TTT_{\text{thirteen}} =$
d. $2034_{\text{twelve}} =$	e. $1025_{\text{eleven}} =$	f. $1028_{\text{thirteen}} =$

? Exercise 30

Convert each of the following to a Base Ten numeral. Show work

a. $110_{\text{two}} =$	e. $110_{\text{six}} =$	i. $110_{\text{eleven}} =$
b. $110_{\text{three}} =$	f. $110_{\text{seven}} =$	j. $110_{\text{twelve}} =$
c. $110_{\text{four}} =$	g. $110_{\text{eight}} =$	k. $110_{\text{thirteen}} =$
d. $110_{\text{five}} =$	h. $110_{\text{nine}} =$	l. $110_{\text{twenty}} =$

? Exercise 31

Without doing any computations, circle which numeral has the larger value. Explain your reasoning. Try to be mathematically clear in your explanation. Notice the sequence of digits is the same for each numeral. Only the base is different.

[Hint: It has to do with the place values.]

$$13\ 201\ 154\ 320\ 050\ 146_{\text{eight}}\ \textbf{OR}\ 13\ 201\ 154\ 320\ 050\ 146_{\text{eleven}}$$

? Exercise 32

Write $4000T00000E000_{\text{twelve}}$ in expanded notation

? Exercise 33

Write $6 \times 13^{14} + 10 \times 13^{11} + 11 \times 13^{7} + 12 \times 13^{2}$ as a base thirteen number in standard form

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2.4: Other Base Systems

*You will need: ****Base Two Blocks, Base Three Blocks, Base Four Blocks, Unit Blocks*** *(Material Cards 4 - 7),* ***Calculator***

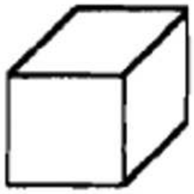
IMPORTANT OBSERVATION

The actual blocks are 3-dimensional. ***The units, longs and flats from the material cards are only shown in 2-dimensions, so they wouldn't be too cumbersome to make.*** *You have to imagine they are really made of cubes. The units are like dice. In base three, a long is 3 units together, so the dimensions are really 3 by 1 by 1 and a flat is made up of 3 longs, so the dimensions are really 3 by 3 by 1.*

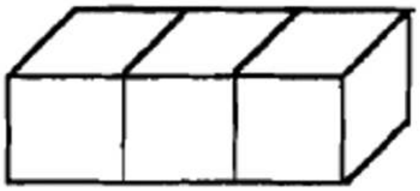
You can check out the base blocks from your instructor. Just send her an email requesting to borrow them.

Below is an amateur drawing of what a unit, long, flat and block really look like in Base Three.

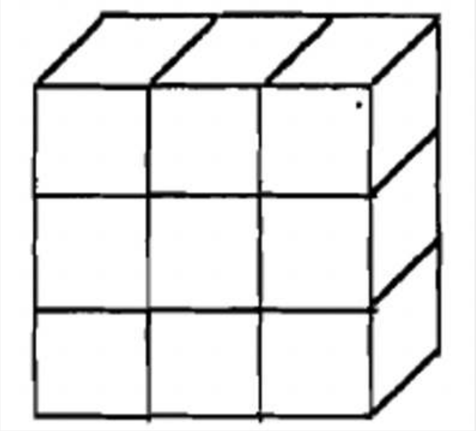
Base Three unit



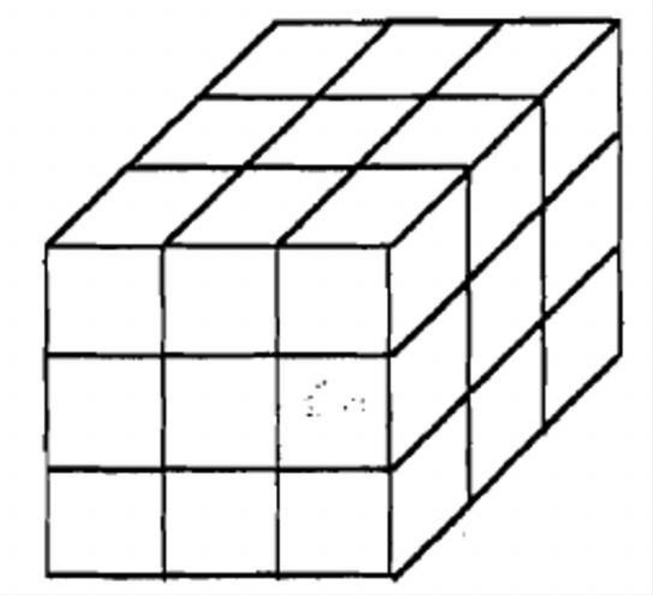
Base Three unit



Base Three unit



Base Three unit

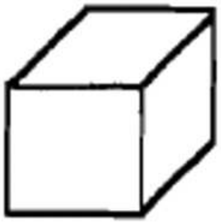


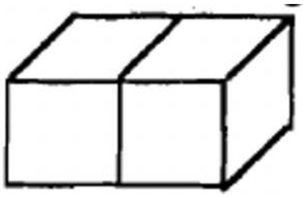
Since we only live in a three dimensional world, once we get past a block, we can't really draw in the 4th dimension or higher. Well, at least I can't. In Base **Three**, it takes **three** units to make a long, it takes **three** longs to make a flat and it takes **three** flats to make a block. In Base **Three**, the next level after a block is made by putting **three** blocks together and we call that a long block. **Three** long blocks put together is called a flat block, and **three** flat blocks put together is called a block block.

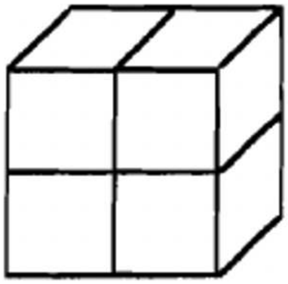
In Base Three, a unit is made up of 1 cube, a long is made up of 3 cubes, a flat is made up of 9 cubes, a block is made up of 27 cubes, a long block is made up of 81 cubes, a flat block is made up of 243 cubes, and a block block is made up of 729 cubes.

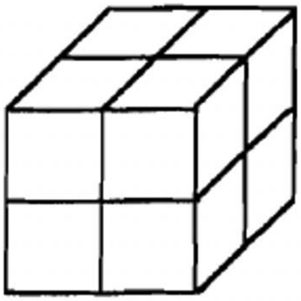
Base Two blocks are different. In Base **Two**, it takes **two** units to make a long, it takes **two** longs to make a flat and it takes **two** flats to make a block. In Base **Two**, the next level after a block is made by putting **two** blocks together and we call that a long block. **Two** long blocks put together is called a flat block, and **two** flat blocks put together is called a block block. Try to imagine what a unit, long, flat and block look like in Base Two before turning the page.

Below is an amateur drawing of a unit, long, flat and block in Base Two.


Base Two unit


Base Two long


Base Two flat


Base Two block

In base two, a unit is made up of 1 cube, a long is made up of 2 cubes, a flat is made up of 4 cubes, a block is made up of 8 cubes, a long block is made up of 16 cubes, a flat block is made up of 32 cubes, and a block block is made up of 64 cubes.

Exercise 1

Draw a picture of a Base Four unit, long, flat and block.

Exercise 2

Fill in the blanks.

a. In Base Four, it takes ____ units to make a long, it takes ____ longs to make a flat, it takes ____ flats to make a block, it takes ____ blocks to make a long block, it takes ____ long blocks to make a flat block, and it takes ____ flat blocks to make a block block.

b. In Base Four, a unit is made up of 1 cube, a long is made up of ______ cubes, a flat is made up of ______ cubes, a block is made up of ______ cubes, and a long block is made up of ______ cubes.

c. In Base Five, it takes ______ flats to make a block.

d. In Base Six, a flat is made up of ______ cubes.

Now that you have expertise in converting from other bases to Base Ten, we'll focus on converting from Base Ten to other bases. There is more than one way to accomplish this task. You will learn how to do it using two different approaches and then you can choose to use whichever method works best for you when working on the exercises. It is optimal if you understand and can convert using either method because as a teacher, one student may understand one explanation perfectly whereas another student might understand it better if it were explained quite differently.

Get out the blocks and work through these problems USING THE BLOCKS!

Before using either method, I want you to return to the process of exchanging like you did at the beginning of this exercise set. Instead of using the Base Two Models, this time we will use sets of Base Blocks in Base Two, Base Three and Base Four. You may have wood or plastic 3-dimensional sets in your classroom or you can make your own models using Material Cards 9-11. Remember that the material cards for the units, longs and flats are only two-dimensional. Keep the sets separate from each other. **The units are the same for any set of Base Blocks so you only need a total of one set of units.**

These next three exercises should be a piece of cake for you!

Exercise 3

Take out your set of Base Two Blocks. These consist of at least units, longs, flats, blocks, long blocks and flat blocks. How many units are in a long? _____ How many longs are in a flat? _____ How many flats are in a block? _____ How many blocks are in a long-block? _____ How many long-blocks are in a flat-block? _____

Exercise 4

Take out your set of Base Three Blocks. These consist of at least units, longs, flats and blocks. How many units are in a long? _____ How many longs are in a flat? _____ How many flats are in a block? _____

Exercise 5

Take out your set of Base Four Blocks. These consist of at least units, longs, flats and blocks. How many units are in a long? _____ How many longs are in a flat? _____ How many flats are in a block? _____

Place Value Chart for Base Blocks

FB	LB	B	F	L	U

Above is a place value chart **for all bases** when using Base blocks. **U** represents units, **L** represents longs, **F** represents flats, **B** represents blocks, **LB** represents long-blocks and **FB** represents flat-blocks. In general, there can be more than six place values, but six is sufficient to complete the exercises in this set. If you are using the Material Cards for Base Blocks, the Base Two blocks have all six types of blocks whereas the Base Three and Four blocks only have units, longs, flats and blocks.

Exercise 6

Imagine Maria had a baby and it was time for her six week check-up. It had been exactly six weeks since she had become a mother. You will be representing the number of days Maria has been a mother in Base Two, Base Three and Base Four using the Base Blocks.

a. To represent the number of days Maria has been a mom in Base Two, do the following using Base Two Blocks:

i. Using the units, count out how many days Maria has been a mom. You have _____ units.
ii. Trade in the units for as many longs as possible. You now have _____ long(s) and _____ unit(s).
iii. Trade in the longs for as many flats as possible. You now have _____ flat(s), _____long(s) and _____ unit(s).
iv. Trade in the flats for as many blocks as possible. You now have _____ block(s), _____ flat(s), _____ long(s) and _____ unit(s).
v. Trade in the blocks for as many long-blocks as possible. You now have _____ long-block(s), _____ block(s), _____ flat(s), _____ long(s) and _____ unit(s).
vi. Trade in the long-blocks for as many flat-blocks as you can. You now have _____ flat-block(s), _____ long-block(s), _____ block(s), _____ flat(s), _____ long(s) and _____ unit(s).
vii. Record the results from part vi. in the first row of the chart in part d below.

For parts b and c, make a series of exchanges like you did in part a, but use Base Three Blocks for part b and Base Four Blocks for part c. In other words, start with the same amount of units you had in part a.i. and begin by exchanging as many units for longs as possible. Then, trade as many longs for flats as possible, then flats for blocks, etc., until all possible exchanges have been made. You don't need to write down each individual step like you did for part a above just use the blocks and record the final results in part d.

b. Start with the same number of units you had in part a.i and make all possible exchanges using Base Three Blocks. When all trades have been made, record the results in the second row of the chart (Base Three) in part d below.

c. Start with the same number of units you had in part a.i and make all possible exchanges using Base Four Blocks. When all trades have been made, record the results in the third row of the chart (Base Four) in part d below.

d.

	FB	LB	B	F	L	U	Base
results from part a:							Two

	FB	LB	B	F	L	U	Base
results from part b:							Three
results from part c:							Four

Exercise 4 illustrates a way to change a numeral in Base Ten to one in a different base. For that exercise, you converted the Base Ten numeral 42 (six weeks of seven days each) to Base Two (101010_{two}), Base Three (1120_{three}) and Base Four (222_{four}). If you didn't get these answers, you should rework the problem using the physical models, and be careful about making accurate exchanges. Ask for help if you still have problems.

Exercise 7

Use the Base Blocks to convert the number of weeks in a year to a numeral in each of the bases given. You should first start out with the units and count out as many units as there are weeks in a year. Then, trade in units for as many longs as possible, next trade in longs for as many flats as possible and so on. Do this process for each base. Explain in words and/or show in a chart each step you took and what blocks you had at each step. State the exact number of each kind of block(s) you had when there were no more exchanges to be made. Finally, write the answer in the base requested on the space provided. Check your work by converting back to Base Ten. **Use the physical models (get out the blocks) to do these problems!**

a. Base Two ____

b. Base Three ____

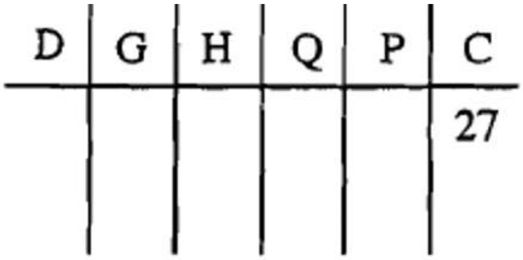
c. Base Four ____

If we only needed to convert relatively small numbers and we always had Base Blocks available in any base we needed to convert to, we could continue to convert to various bases by using the blocks and doing exchanges. But this is not always the case. Consider if you were asked to convert 999 to Base Four. This would be pretty time-consuming and tedious. So, now we will explore ways to convert without using manipulatives.

Let's back up a step by considering how, by using manipulatives, we obtained the answer to exercise 2 of Exercise Set 3. Basically, we converted the Base Ten numeral 27 to a Base Two numeral using cups (units), pints (longs), quarts (flats), half-gallons (blocks) and gallons (long-blocks).

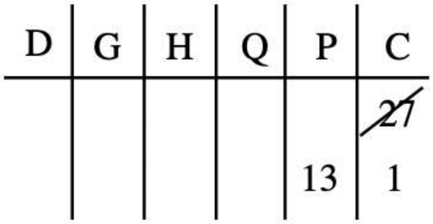
How did we do it?

1. Start by listing how many cups you start out with. See below.

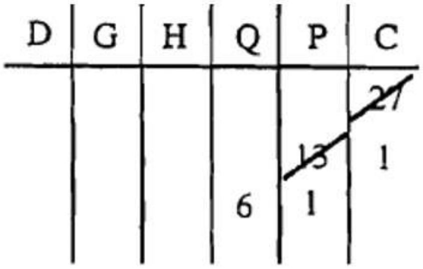


D	G	H	Q	P	C
					27

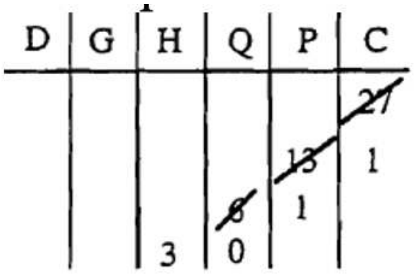
2. Next, to consolidate the cups into pints, we divide the number of cups (27 in Base Ten) by 2 (because we are converting to base Two) to get 13 pints (the quotient) with 1 cup left over (the remainder) *when you divide 27 by 2, 13 is the quotient and 1 is the remainder*. So, we now have 13 pints and 1 cup. Cross off 27 cups and show new quotient and remainder. Keep track in a chart as shown below.



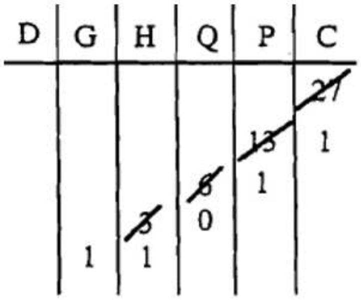
3. To consolidate the pints into quarts, we divide the number of pints (13) by 2 (the base) which gives 6 quarts (the quotient) and 1 pint left over (the remainder). Cross off the 13 and show new quotient and remainder. See below.



4. To consolidate the quarts into half-gallons, we divide the number of pints (6) by 2 (the base) which gives 3 half-gallons (the quotient) and zero quarts left over. See below for how to keep track.



5. To consolidate the half-gallons into gallons, we divide the number of half-gallons (3) by 2 (the base), giving us 1 gallon (the quotient) and 1 half-gallon left over. Keep track as shown below



At this point, we can't consolidate any further. As you do each division, the remainder must always be less than what you are dividing by, which is the base. Once you get a quotient that is also smaller than the base, you are done converting the number to the desired base. In this example, we were converting to base two. Therefore, we simply read the chart above to get the answer of 11011_{two}.

So, $27 = 11011_{\text{two}}$.

The above ***algorithm*** (or method) works to convert any Base Ten numeral to any other base. Below are steps:

1. Start by putting the number to convert in the unit's place. Divide the number by the base. The new *quotient* goes in the next place value to the left and the *remainder* goes in the unit's place.

2. Divide the quotient you just got by the base. The new *quotient* goes in the next place value to the left and the *remainder* goes in the old quotient's place.

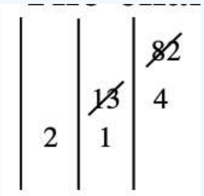
3. Repeat step 2 until the quotient is less than the base. Now write the numeral in the base you were converting to. Remember to write out the base as a number name ("two","three", etc.) to the right of the number!!!!! Lastly, but most importantly, CHECK YOUR Answer by converting back to a Base Ten numeral!

If you forgot how to use the calculator to easily find the quotient and remainder when doing division problems, look back at page 18 of Exercise Set 2 and practice exercise 8 again. Record each quotient and remainder in a careful, organized manner as you work through the problem. Then, at the end of the computation, it is easy to write the numeral in the new base. Notice how well this works in the next few examples.

✓ Example 1

Convert 82 to Base Six.

The chart where I keep track is below.



My computations were:

$82 \div 6 = 13$ rem. $\underline{4}$

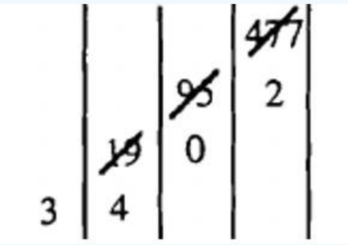
$13 \div 6 = 2$ rem. $\underline{1}$

Since the last quotient, $\underline{2}$, was less than the base, 6, this signaled the end of the problem. So 82 as a numeral in Base Six is recorded as $\mathbf{214_{six}}$. Check to see if 214_{six} really is 82.

? Exercise 2

Convert 477 to Base Five.

The chart where I keep track of the quotients/remainders is shown below.



My computations were:

$477 \div 5 = 95$ rem. $\underline{2}$

$95 \div 5 = 19$ rem. $\underline{0}$

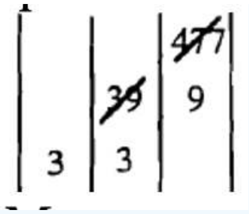
$19 \div 5 = 3$ rem. $\underline{4}$

So 477 as a numeral in Base Five is recorded as $\mathbf{3402_{five}}$. Check to see if 3402_{five} really is 477.

? Exercise 3

Convert 477 to Base Twelve.

The chart where I keep track of the quotients/remainders is shown below.



My computations were:

$477 \div 12 = 39$ rem. $\underline{9}$

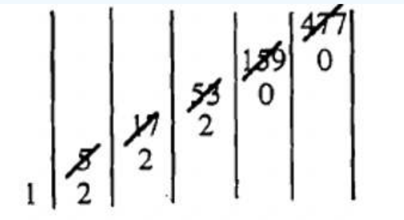
$39 \div 12 = 3$ rem. $\underline{3}$

So 477 as a numeral in Base Twelve is recorded as $\mathbf{339_{twelve}}$. Check to see if 339_{twelve} really is 477.

Example 4

Convert 477 to Base Three.

The chart where I keep track of the quotients/remainders is shown below.



My computations were:

$477 \div 3 = 159$ rem. $\underline{0}$

$159 \div 3 = 53$ rem. $\underline{0}$

$53 \div 3 = 17$ rem. $\underline{2}$

$17 \div 3 = 5$ rem. $\underline{2}$

$5 \div 3 = 1$ rem. $\underline{2}$

So 477 as a numeral in Base Three is recorded as $\mathbf{122200_{three}}$. Check to see if 122200_{three} really is 477.

Study the previous examples carefully before working through the next exercise. The key is to be organized in your work writing down each step in the process.

? Exercise 8

Convert each Base Ten numeral to the specified base and put a box around your answer. Then, check your answer by converting the answer back to Base Ten.

a. 200 to Base Four	b. 450 to Base Seven
c. 89 to Base Two	d. 264 to Base Three
e. 320 to Base Thirteen	

The algorithm we have been employing should make sense given the way we converted to other bases from Base Ten using the Base Blocks. This method is fairly straightforward and you do not need to know the actual place values as you fill in the place holders. You also do not know how many digits the numeral will have until you have finished the problem. There is another method of converting which we will now study. For this second method, we determine and write out the actual place values and figure out the number of digits the numeral will have as the first step of the problem.

$$\frac{\mathbf{2}}{36}\ \frac{}{6}\ \frac{}{1}$$

$$\frac{\mathbf{2}}{36}\ \frac{\mathbf{1}}{6}\ \frac{}{1}$$

$$\frac{\mathbf{2}}{36}\ \frac{\mathbf{1}}{6}\ \frac{\mathbf{4}}{1}$$

We will now convert 82 to Base Six using a new algorithm. We will look at the process of converting very differently. First, we consider the place values in a Base Six numeral. Since we only need to go up to 82, we do not need to show a place value greater than 82. Our first step is to write a place value chart for Base Six with the place values shown under blank spaces where we will fill in the digits for the Base Six numeral. This is shown to the right. Notice this will be a three digit numeral. Starting from the left, the first digit (in the 36's place) will indicate how many 36's are in 82. This is the division problem: $82 \div 36 = 2$, rem. 10. Thus, **2** will be the first digit and we have **2** __ __ six for our numeral thus far. We now have to take the remainder of 10 and do the same thing for the next place value. To determine how many 6's are in 10, we do this division problem: $10 \div 6 = 1$, rem. 4. This gives a **1** for the next digit in the numeral, so we have **2 1** __six and we are almost done. When you are down to the units place, the remainder goes in the units place. So 82 converts to 214_{six} which is the same answer we got when we computed it using the first algorithm. On the left, you can see the three steps taken as you fill in the chart on your way to writing the final answer in Base Six. Remember to check the answer!

$$\frac{\mathbf{3}}{125}\ \frac{}{25}\ \frac{}{5}\ \frac{}{1}$$

$$\frac{\mathbf{3}}{125}\ \frac{\mathbf{4}}{25}\ \frac{}{5}\ \frac{}{1}$$

$$\frac{\mathbf{3}}{125}\ \frac{\mathbf{4}}{25}\ \frac{\mathbf{0}}{5}\ \frac{}{1}$$

$$\frac{\mathbf{3}}{125}\ \frac{\mathbf{4}}{25}\ \frac{\mathbf{0}}{5}\ \frac{\mathbf{2}}{1}$$

Let's convert 477 to Base Five using this algorithm. First determine the place values for Base Five, noting you do not fill in place values higher than 477. This is shown on the top line to the right. We now know there are four digits in this Base Five numeral. Step 1 is to determine how many 125's are in 477 which is the division problem: 477 125 = 3, rem. 102. The sequence of steps for filling in the numeral is shown to the right. Notice the **3** in the first place value. Next, we must determine how many 25's are in 102 (the remainder). I hope you don't need a calculator for this one. There are 4, with a remainder of 2. So, **4** gets filled into the 25's place. We are left with a remainder of 2 and since there are no 5's in 2,a **0** is put in that place value. Next is the units place so the **2** gets deposited there. Hence, the answer is 3402_{five} which is exactly what we arrived at when using the first algorithm. Make sure you remember to check the answer!

$$\frac{}{128}\ \frac{}{64}\ \frac{}{32}\ \frac{}{16}\ \frac{}{8}\ \frac{}{4}\ \frac{}{2}\ \frac{}{1}$$

Let's convert 77 to a Base Two numeral using this algorithm. First, look at some Base Two place values shown above. The only digits in Base Two are 1's and 0's. You don't really have to do any division problems. The first digit will always be a 1 and it will be in the highest place value that can be subtracted from the number we are converting. 77 converts to a seven digit numeral with a 1 in the 64's place since 64 is the highest place value that can be subtracted from 77. Place a 1 in that place value and subtract (77 - 64 = 13). Take the new number, 13, and find the next highest place value from which 13 can be subtracted, putting a 0 in any place value greater than 13. This forces a 0 in the next two place value spots –32 and 16. Put a 1 in the 8's place since it is the highest place value that can be subtracted from 13. After subtracting 8 from 13, we have 5. Put a 1 in the 4's place since 4 can be subtracted from 5, leaving a 1. This forces a 0 in the place value for 2 and the remainder of 1 goes in the units place.

$$\begin{array}{r} 77 \\ -\ 64\ \checkmark \\ \hline 13 \\ -\ 8\ \checkmark \\ \hline 5 \\ -\ 4\ \checkmark \\ \hline 1 \\ -\ 1\ \checkmark \\ \hline 0 \end{array}$$

To the left is a way to keep track of each subtraction made. Simply put a 1 in each place value subtracted (note check marks) and a zero for all other place values. Hence, the answer is $\mathbf{1001101_{two}}$. This is easily checked: 64 + 8 + 4 + 1 = 77.

? Exercise 9

Convert the Base Ten numeral 155 to the base specified using the second algorithm presented.

a. Base Three	b. Base Twelve
c. Base Two	d. Base Five

? Exercise 10

Use any algorithm you prefer to convert 13595 to the base specified. SHOW WORK!

a. Base Eleven	b. Base Thirteen

? Exercise 11

Convert 838 to the base specified.

a. Base Twelve	b. Base Eight
c. Base Seven	d. Base Nine

Both methods shown for converting a Base Ten numeral to a different base work well. But the approaches are very different. It's best if you can both understand and do it either way. Eventually, you'll settle on doing it one way over the other. For years, I only did it using the second algorithm because that was the only way I knew how to do it. Now, I find the first method more interesting. Whichever way you choose to convert, it's very important for you to check your work by converting back to Base Ten.

? Exercise 12

Discuss the differences and advantages of each algorithm used to convert a Base Ten numeral to a different base. Which method do you prefer and why?

The last topic we will explore is how to count in bases other than Base Ten. In other words, how can we write down a sequence of counting numerals in a given base without converting each numeral from Base Ten one at a time? Also, you will learn how to take a numeral in a given base and, without any converting, state both the previous and next numeral in that same base.

Begin by studying patterns in the counting numerals in Base Ten. We'll start with zero although it really isn't a counting number. Read across one row at a time.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29

and so on for a while until we get to...

80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109
110	111	112	113	114	115	116	117	118	119

and again this goes on for a while until we get to...

980	981	982	983	984	985	986	987	988	989
990	991	992	993	994	995	996	997	998	999
1000	1001	1002	1003	1004	1005	1006	1007	1008	1009

and you know how this goes on and on and on, right?

Here are some things to pay attention to:

- The last digit always changes from one numeral to the next
- As you look across each row, only the last digit changes from one numeral to the next
- Notice how many numerals are written in a row before more than just the last digit changes from one numeral to the next
- When more than the last digit changes, note the last digit of the previous numeral and the last digit of the new numeral
- Notice what it takes to change from a one digit numeral to a two digit numeral, from a two digit numeral to a three digit numeral, from a three digit numeral to a four digit numeral, etc
- Notice the pattern of the first two, three or four digit numeral, etc

? Exercise 13

In Base Ten, how many consecutive numerals can you write in a row before having to change more than one digit? ____

? Exercise 14

In Base Ten, what is true of the numeral if more than the last digit changes when you write the next numeral? _____

? Exercise 15

In Base Ten, what is the largest eight digit numeral? ____

Exercise 16

In Base Ten, what is the smallest ten digit numeral? ____

Study this counting chart in Base Three. It starts with zero and counts to 95 which is 10112_{three}. Read across each row and then down the page. Try to recognize patterns similar to ones found in Base Ten. Then, answer the questions on the right side of the page.

0_{three}	1_{three}	2_{three}
10_{three}	11_{three}	12_{three}
20_{three}	21_{three}	22_{three}
100_{three}	101_{three}	102_{three}
110_{three}	111_{three}	112_{three}
120_{three}	121_{three}	122_{three}
200_{three}	201_{three}	202_{three}
210_{three}	211_{three}	212_{three}
220_{three}	221_{three}	222_{three}
1000_{three}	1001_{three}	1002_{three}
1010_{three}	1011_{three}	1012_{three}
1020_{three}	1021_{three}	1022_{three}
1100_{three}	1101_{three}	1102_{three}
1110_{three}	1111_{three}	1112_{three}
1120_{three}	1121_{three}	1121_{three}
1200_{three}	1201_{three}	1202_{three}
1210_{three}	1211_{three}	1212_{three}
1220_{three}	1221_{three}	1222_{three}
2000_{three}	2001_{three}	2002_{three}
2010_{three}	2011_{three}	2012_{three}
2020_{three}	2021_{three}	2022_{three}
2100_{three}	2101_{three}	2102_{three}
2110_{three}	2111_{three}	2112_{three}
2120_{three}	2121_{three}	2122_{three}
2200_{three}	2201_{three}	2202_{three}
2210_{three}	2211_{three}	2212_{three}
2220_{three}	2221_{three}	2222_{three}
10000_{three}	10001_{three}	10002_{three}
10010_{three}	10011_{three}	10012_{three}
10020_{three}	10021_{three}	10022_{three}

10100_{three}	10101_{three}	10102_{three}
10110_{three}	10111_{three}	10112_{three}

? Exercise 17

In Base Three, how many consecutive numerals can you write in a row before having to change more than one digit?

? Exercise 18

In Base Three, what is true of the numeral if more than the last digit changes when you write the next numeral?

? Exercise 19

In Base Three, what is the smallest seven digit numeral?

? Exercise 20

In Base Three, what is the largest seven digit numeral?

? Exercise 21

For each Base Three numeral, write the next consecutive numeral:

a. $1\ 202\ 010_{\text{three}}$:

b. $2\ 220\ 011_{\text{three}}$:

c. $1\ 010\ 102_{\text{three}}$:

d. $2\ 100\ 212_{\text{three}}$:

e. $2\ 120\ 222_{\text{three}}$:

? Exercise 22

For each Base Three numeral, write the numeral that precedes it:

a. $1\ 200\ 102_{\text{three}}$

b. $1\ 202\ 221_{\text{three}}$

c. $2\ 110\ 020_{\text{three}}$

d. $2\ 110\ 100_{\text{three}}$

? Exercise 23

What is wrong with the numeral, $1\ 022\ 301_{\text{three}}$

? Exercise 24

Starting with zero, count to 23 in Base Four by filling in the missing numerals.. Count across each row as was done in Base Ten and Base Three.

0_{four}		2_{four}	3_{four}
	11_{four}		

20_{four}			23_{four}
		112_{four}	

Check by picking a random numeral in the list. Take 112_{four} from the above list. Since you started counting from zero, that numeral should be the same as 22_{ten}, which is what 112_{four} converts to in Base Ten. Always check the last numeral too. In this case, the last numeral should convert to 23. Perhaps you are asked to count from 121 to 135 in Base Five. The first step would be to convert 121 to a Base Five numeral, which is 441_{five}. Then, start counting by writing each successive numeral. The fifteen numerals are listed:

$441_{\text{five}}, 442_{\text{five}}, 443_{\text{five}}, 444_{\text{five}}, 1000_{\text{five}}, 1001_{\text{five}}, 1002_{\text{five}}, 1003_{\text{five}}, 1004_{\text{five}}, 1010_{\text{five}}, 1011_{\text{five}}, 1012_{\text{five}}, 1013_{\text{five}}, 1014_{\text{five}}, 1020_{\text{five}}$

To check, take a random numeral in the middle, 1003_{five}, convert it and find it is 128. In the sequence, it should equal 128. Then, I check the last numeral 1020_{five} and make sure it converts to 135, which it does. This indicates that probably the rest are also correct.

? Exercise 25

Count from 1 to 20 in Base Two

? Exercise 26

Count from 61 to 68 in Base Eight

? Exercise 27

For each numeral, write the numeral that precedes it and also, its successor. Try to do it without converting! Afterwards, you can convert to check it

a. _____, 3027_{eight}, _____
b. _____, 1240_{five}, _____
c. _____, 101100_{two}, _____

? Exercise 28

For each base given, assume you had the designated number of blocks, flats, longs and/or units. State how to write the numeral in that base and also in base ten.

a. Base Three: 4 blocks, 5 flats, 1 long and 4 units

b. Base Eight: 19 flats, 7 longs and 13 units

c. Base Five: 1 block, 13 longs and 15 units

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2.5: Supplementary Topics

You will need: A ***Calculator***

So far, we've studied how to change a whole number in a different base to a whole number in Base Ten. It's also possible to work with a numeral that has a decimal point in another base. Look again at the chart for how the place value system worked for a number in Base n. The pattern continues to the left forever.

n^{11}	n^{10}	n^9	n^8	n^7	n^6	n^5	n^4	n^3	n^2	n^1	n^0

The model above is sufficient for a whole number. If you are looking at a number that does not have a decimal point, it is implied that the decimal point is to the right of the rightmost digit. We can stretch this model to include a number with decimal points. This is analogous to working with fractions. Following the pattern established above, the place values to the right of the decimal point, in order, are $n^{-1}, n^{-2}, n^{-3}, n^{-4}$,...

So, a way to view the various place values in base b is:

$$\underline{b^4}\ \underline{b^3}\ \underline{b^2}\ \underline{b^1}\ \underline{b^0} \bullet \underline{b^{-1}}\ \underline{b^{-2}}\ \underline{b^{-3}}\ \underline{b^{-4}}$$

That big dot in the middle is the decimal point. The place values still go infinitely to the left and also to the right. The above model only shows five places to the left of the decimal point and four places to the right of the decimal point.

Now, you need to recall from algebra what a negative exponent means. In case you've forgotten, here is the definition of a negative exponent and what it means to have an exponent of zero.

If $b \neq 0$ and for any n, $b^{-n} = \frac{1}{b^n}$ Example: $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$	If $b \neq 0$, then $b^0 = 1$ Example: $8^0 = 1$

Exercise 1

Simplify by first rewriting each problem so that there are no negative exponents. Simplify each fraction.

a. 4^{-1}	b. 4^{-2}	c. 4^{-3}
d. 3^{-1}	e. 5^{-2}	f. 2^{-3}

Let's figure out how to rewrite each of the following numerals in Base Ten.

Example 1

142.3_{five}

Solution

$$\begin{aligned} 142.3_{\text{five}} &= (1\times 5^2)+(4\times 5^1)+(2\times 5^0)+(3\times 5^{-1}) \\ &= (1\times 25)+(4\times 5)+(2\times 1)+(3\times \frac{1}{5^1}) \\ &= 25+20+2+\frac{3}{5} \\ &= 47\frac{3}{5} \end{aligned}$$

Exercise 2

Convert each numeral to Base Ten. **Show *each* of the following steps**:

Step 1: Write the numeral in expanded notation.

Step 2: Simplify each number that has a positive exponent; rewrite each number that has a negative exponent as a number with a positive exponent.

Step 3: Simplify each term (*terms are separated by addition signs*).

Step 4: Add the terms together. Simplify any fractions if necessary

a. $43.2_{\text{eight}} =$
b. $143.7_{\text{eleven}} =$
c. 1111.1_{two}
d. $123.2_{\text{four}} =$

Here is an example of converting to base ten when the numeral has more than one place after the decimal point.

✓ Example 2

24.31_{five}

Solution

$$\begin{aligned} 24.31_{\text{five}} &= (2\times 5^1)+(4\times 5^0)+(3\times 5^{-1})+(1\times 5^{-2}) \\ &= (2\times 5)+(4\times 1)+(3\times \frac{1}{5^{-1}})+(1\times \frac{1}{5^2}) \\ &= 10+4+\frac{3}{5}+\frac{1}{25} \end{aligned}$$

(To add fractions, you need a common denominator, which is fairly easy in bases since the denominator will be a power of the original base.)

$$= 14+\frac{15}{25}+\frac{1}{25}$$
$$= 14\frac{16}{25}$$

? Exercise 3

Convert each numeral to Base Ten. **Show each of the following steps**:

Step 1: Write the numeral in expanded notation

Step 2: Simplify each number that has a positive exponent; rewrite each number that has a negative exponent as a number with a positive exponent

Step 3: Simplify each term (*terms are separated by addition signs*)

Step 4: Add like terms together; to add fractions, get a common demominator (it will be a power of the original base.)

Step 5: Simplify further if necessary. Reduce any fractions if necessary

a. $43.21_{\text{six}} =$
b. $231.123_{\text{four}} =$
c. 156.12_{seven}
d. $111.101_{\text{two}} =$
e. 222.222_{three}
f. $T2E.0T_{\text{twelve}} =$

In order to really understand computer technology, you must have the ability to express numbers in base two and sixteen. At first, Base Two (called the Binary System) was used to express computer code. Base Two was the natural choice, since there are only two symbols, 0 and 1. At the most primitive level, electronic computers only know two things, off (0) and on (1). To get a basic feel for how some information gets read into a computer, imagine there are eight switches –switch 1, switch 2, switch 3,..., switch 8. How the computer responds depends on which of the eight switches are turned "on." There are actually 256 (2^8) different configurations possible.

Imagine that switch 7, switch 2 and switch 1 was turned on. We will use an 8-digit code to show this. It looks like an 8-digit numeral, where the rightmost digit represents switch 1 and the leftmost digit represents switch 8. For our example, there should be a 1 in the spots for switches 1, 2 and 7 and a 0 in the spots for switches 3, 4, 5, 6 and 8. The code looks like this: 01000011. This is read: "off-on-off-off-off-off-on-on." The code 01000011 is really in the format of a Base Two numeral and can be converted to Base Ten. When writing binary *code* for the computer, base two is implied, but the numeral is not written as a base two number. Code is written 01000011, whereas the base two numeral is written 1000011_{two}.

Exercise 4

Using this system of eight switches being either turned on or off, represent the binary code if the switches indicated are turned on. Then, write the base ten numeral (also called decimal numeral) represented by each binary number.

a. switches that are on: 1, 3, 5, 6:	Binary Code: ____ Base ten numeral it represents: ____
b. switches that are on: 2, 4, 5, 8:	Binary Code: ____ Base ten numeral it represents: ____
c. switches that are on: 3, 5, 7:	Binary Code: ____ Base ten numeral it represents: ____
d. No switches are on:	Binary Code: ____ Base ten numeral it represents: ____
e. All eight switches are one:	Binary Code: ____ Base ten numeral it represents: ____

The binary system is fairly simplistic in that all it comes down to is a sequence of on and off switches. Actually, at the present time, the two electrical states usually used in computers are low voltage and high voltage. Whether we think of positive and negative charges, on and off switches, low and high voltages, etc., the main idea is the same –there are only *two states*. Although binary is simple on one hand, the disadvantage of the binary system is that even small numbers are made up of lots of digits. For instance, the base ten numeral 50 is written as 110010_{two} which has 6 digits. The base ten (also called decimal) numeral 1025 is written as 10000000001_{two} which has eleven digits!

An intermediate system of numeration, using a larger base, was developed so that the code for the numerals could be written more compactly, with fewer digits. For convenience, a base that was a power of two (making conversion easy) and one that had a reasonable number (not too many) of primary symbols was used.

In the first example where switches 1, 2 and 7 were on, the binary code was 01000011. Anything that can be typed on a keyboard actually has a special 8-digit code like this one.

A standard code called ASCII (an acronym for "American Standard Code for Information Interchange") was established in 1968 by the American National Standards Institute. Each separate keyboard symbol (and 32 special control functions, such as the "return" key) was assigned a number from 0 to 127, giving 128 different possibilities. When you type the uppercase letter "C" on the keyboard, the computer codes this little piece of information as 01000011 –*more ASCII code is on page 54.*

If I typed my name "JULIE HARLAND" on the keyboard, the computer would code this with 13 eight-digit numerals –one 8-digit numeral for each letter and one for the space in the middle. If I wanted to print out and read the code, I would be looking at 104 digits! To make this task less cumbersome, we can have the computer convert each original 8-digit numeral (which is really just a base two numeral) to a special hexadecimal (base sixteen) numeral, which consists of only two, instead of eight, digits. Note that if I wanted to read the 8-digit numeral as a single-digit numeral, I'd need a base 256 system, which would require 256 different symbols –that would be way too complicated! Here is how the 8-digit binary code 01000011 is converted to the hexadecimal system, making it easier for me to read the code.

Each 8-digit binary code numeral is broken into two separate 4-digit numerals: 0100 0011. Each 4 digit numeral is converted from base 2 to base 16 (there are 24 , or 16 possibilities for a system with only 4 switches.) But in Base Sixteen, 16 different primary symbols are needed. WE **DO NOT USE** T for ten, E for eleven AND W for twelve IN BASE SIXTEEN! Instead, these are the 16 symbols used for counting in base 16: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. In other words, A stands for ten, B for eleven, C for twelve, D for thirteen, E for fourteen and F for fifteen. The first group of four digits converts from 0100 in base two to 4 in base ten. 4 in base ten is written as 4 in base sixteen. The second group of four digits converts from 0011 in base 2 to 3 in base ten. 3 in base ten is written as 3 in base sixteen. Therefore, the 8-digit numeral 01000011 is simply written as 43 in hexadecimal.

Base Eight (the octal system) is another common intermediate system used. One only needs to memorize eight symbols for this system (0, 1, 2, 3, 4, 5, 6, and 7). Base Sixteen is more compact and only requires one to memorize six more than the usual ten in base ten. Using A, B, C, D, E and F for 10,11,12,13,14 and 15 is fairly natural for most people to learn. On the other hand, going to a base 32 (the next power of 2) system would be fairly cumbersome because you'd have to incorporate 22 new symbols. We'll limit our discussion to hexadecimal, and not work in the octal system in this exercise set.

One more example:

✓ Example 3

If switches 8, 7, 5, 4 and 2 are switched on, write the code for this configuration in binary and in hexadecimal. What decimal numeral does this represent?

Solution

In binary, the code is ***11011010***.

The first 4-digit sequence, 1101, is thirteen in base ten–D in hex; the second 4-digit sequence, 1010, is ten in base ten–A in hex. Therefore, the code in hex is ***DA***. Convert the binary or the hex to base ten to get 218.

Remember: If writing a "code" then the base is not written it is implied and the binary code is simply written as 11011010 or in hex as DA. If I'm writing a base two numeral, write it as 11011010two or if writing as a hexadecimal numeral, write DAsixteen. ***Pay attention to the directions!***

? Exercise 5

Write the binary and hex code corresponding to these switches being turned on.

Note: You already wrote the binary code for these in exercise 4. To check that you write the correct hexadecimal code, convert the base sixteen numeral back to base ten and see if it matches what you wrote for base ten in exercise 4.

a. switches that are on: 1, 3, 5, 6	Binary Code: ____ Hex Code: ____
b. switches that are on: 2, 4, 5, 8	Binary Code: ____ Hex Code: ____
c. switches that are on: 3, 5, 7	Binary Code: ____ Hex Code: ____
d. No switches are on	Binary Code: ____ Hex Code: ____
e. All eight switches are on	Binary Code: ____ Hex Code: ____

? Exercise 6

Convert each hexadecimal code to binary code, and to a base ten numeral.

a. Convert 5E to binary code: ____	and to base ten: ____

b. Convert E5 to binary code: ____ and to base ten: ____
c. Convert 39 to binary code: ____ and to base ten: ____
d. Convert 1F to binary code: ____ and to base ten: ____
e. Convert 98 to binary code: ____ and to base ten: ____
f. Convert 2A to binary code: ____ and to base ten: ____
g. Convert 07 to binary code: ____ and to base ten: ____
h. Convert 40 to binary code: ____ and to base ten: ____

Part of the ASCII code chart (for only 28 of the 128 possible keyboard strokes) is shown below. The keyboard stroke, the decimal (base ten) numeral, the hexadecimal code and the binary code are given for these 28 keyboard strokes.

Key	Base Ten	Hex	Binary
<space>	32	20	00100000
!	33	21	00100001
,	44	2C	00101100
-	45	2D	00101101
.	46	2E	00101110
A	65	41	01000001
B	66	42	01000010
C	67	43	01000011
D	68	44	01000100
E	69	45	01000101
F	70	46	01000110
G	71	47	01000111
H	72	48	01001000
I	73	49	01001001
J	74	4A	01001010
K	75	4B	01001011
L	76	4C	01001100
M	77	4D	01001101
N	78	4E	01001110
O	79	4F	01001111
P	80	50	01010000
Q	81	51	01010001
R	82	52	01010010
S	83	53	01010011
T	84	54	01010100

Key	Base Ten	Hex	Binary
U	85	55	01010101
V	86	56	01010110
W	87	57	01010111

Remember that first numeral we wrote where switches 1, 2 and 7 were on? We wrote this as 01000011, and I mentioned that this is the ASCII code for the letter "C." Look up the code for "C" in the chart –is this starting to make a little more sense?

In any case, let's get back to how the code for "JULIE HARLAND" would look. We'll do it in both hex and binary, separating the code for each space or letter by a comma. Hex is easier, so let's do that first: 4A, 55, 4C, 49, 45, 20, 48, 41, 52, 4C, 41, 4E, 44. That's all there is to it. It's a code. The computer stores it in binary as: 01001010, 01010101, 01001100, 01001001, 01000101, 00100000, 01001000, 01000001, 01010010, 01001100, 01000001, 01001110, 01000100.

Note that I've only given you the ASCII code for some of the upper case letters of the alphabet. Code for a lower case letter is different from code for an upper case letter.

? Exercise 7

Decode the following messages. Part a is in binary and part b is in hex.

a. 01001001, 00100000, 01001100, 01001111, 01010110, 01000101, 00100000, 01001101, 01000001, 01010100, 01001000, 00100001 translates to: ______
translates to: _____

? Exercise 8

Write each of the following in hexadecimal code and in binary code. Show work.

a. **HELP!** Hex: ___ Binary: _____
b. **BE HAPPY.** *Note: I bet you can figure out the code for the "Y" if you try!* Hex: ___ Binary: _____
c. Write your first name in Hex and Binary. Be sure to include your work.

? Exercise 9

Convert the following Base Ten numerals to binary and to hexadecimal. In this case, you aren't being asked to write a code, so remember to write the base to the right and a little below each numeral as with converting to any other base. Show all work

a. 73
b. 122
c. 50
d. 250
e. 1000

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2.6: Homework

- Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)
- Start a new module on the front side of a new page and write the module number on the top center of the page.
- Answers without supporting work will receive no credit.
- Some solutions are given in the solutions manual.
- You may work with classmates but do your own work.

? HW #1

For each two sets noted, indicate whether or not the sets match. If they do, show a matching. If they do not, explain why not.

a. Small Blue A-blocks and Large Red A-Blocks

b. Yellow A-blocks and Circular A-blocks

? HW #2

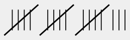
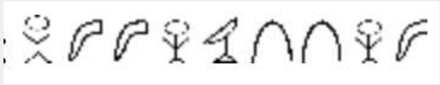
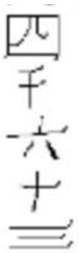
Describe a matching between the set of counting numbers, {1, 2, 3, ...} and the set of positive multiples of five, {5, 10, 15, ...}.

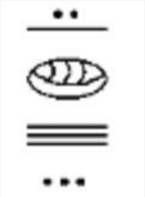
? HW #3

Show every possible one-to-one correspondence between the Small Blue A-blocks and the Small Red A-blocks. Use abbreviations or pictures to denote the blocks.

? HW #4

Convert each numeral to a Hindu-Arabic Base Ten numeral.

a. STROKE: | | | | | | | | | | | | | | | |
b. Tally: 
c. Roman: MCMLXII
d. Roman: DCCXLIV
e. Roman: $\mathrm{I\overline{V}\overline{C}CXDLI}$
f. Egyptian: 
g. Chinese 

h. Mayan 
i. 4032_{seven}
j. $T6W_{\text{thirteen}}$
k. $1\ 111\ 001\ 011_{\text{two}}$
l. 507_{nine}

? HW #5

Convert 342 to a numeral in the numeration system or base specified.

a. Roman	b. Base Seven	c. Egyptian
d. Base Two	e. Chinese	

? HW #6

Convert 838 to:

a. Base Twelve	b. Base Eight
c. Base Five	d. Mayan

? HW #7

Convert 13,595 to Base Twelve

? HW #8

Convert 120,258 to Mayan

? HW #9

Count from 620 to 630 in Base Five

? HW #10

State the numeral that comes just before:

a. $173\ 425\ 760_{\text{eleven}}$	b. $2\ 010\ 212\ 000_{\text{four}}$

HW #11

State the numeral that comes right after:

a. $539100TE_{\text{twelve}}$	b. $3\ 102\ 313\ 444_{\text{five}}$

HW #12

Answer true or false. If false, explain why

a. $2_{\text{four}} = 2$	b. $3_{\text{four}} = 3_{\text{twelve}}$	c. $10_{\text{twelve}} = 10_{\text{five}}$

HW #13

Using Base Three blocks, you had 7 flats, 10 longs and 5 units. What number does this represent in

a. Base **Three**?	b. Base **Ten**?

HW #14

Write each number shown in expanded notation as a numeral in the base specified.

a. $3 \times 7^8 + 6 \times 7^5 + 4 \times 7^4$ to Base Seven
b. $1 \times 3^{10} + 2 \times 3^9 + 2 \times 3^3$ to Base Three

HW #15

Write each number in expanded notation:

a. $200\ 050\ 030\ 000_{\text{nine}}$
b. $1\ 000\ 100\ 001\ 000_{\text{two}}$

HW #16

Write each numeral in expanded form. Then, convert each numeral to a Base Ten mixed numeral with the fraction simplified.

a. 43.3_{nine}	b. 35.12_{six}
c. 121.21_{three}	d. 333.333_{five}

HW #17

Rewrite from expanded form to a numeral in the appropriate base.

a. $3 \times 4^2 + 2 \times 4^0 + 3 \times 4^{-1} + 1 \times 4^{-3}$
b. $5 \times 11^3 + 10 \times 11^1 + 8 \times 11^{-1} + 1 \times 11^{-2}$
c. $1 \times 2^2 + 1x \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-4}$

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CHAPTER OVERVIEW

3: Addition and Subtraction

This is just

2 good

+ 2 be

———

4 gotten

My dad, John Browne, always made learning fun for me. It was always a game, whether it was learning all the states and capitols, the presidents in order or basic math facts. I never had to drill. I loved doing flashcards and competing with anyone to see who could shout out the answer first. Neither of my parents liked or did well in math, but they only shared the positives and the fun of learning to me. Now, I teach my dad arithmetic tricks for fun and he teaches them to anyone who will listen -- including me. Then, I say "But Dad, I taught you that trick last year!" We both know the best way to learn and then remember anything is to teach someone else. Thanks Dad.

3.1: Definition and Properties
3.2: Combining
3.3: Addition Algorithms
3.4: Subtraction
3.5: Subtraction Algorithms
3.6: Homework

Thumbnail: The abacus is a calculating tool that has been in use since ancient times and is still in use today. The abacus consists of a number of rows of movable beads or other objects, which represent digits. One of two numbers is set up, and the beads are manipulated to implement an operation involving a second number (e.g., addition), or rarely a square or cubic root. (Unsplash Lisense; **Crissy Jarvis via Unspash**)

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3.1: Definition and Properties

You will need: ***Coins*** *(Material Card 1),* ***A-Blocks*** *(Material Cards 2A-2E),* ***Centimeter Strips*** *(Material Cards 17A-17L)*

Whole numbers are made up of zero and the counting numbers 0, 1, 2, 3, 4, 5, . . . We begin this exercise set by exploring how to add two *whole numbers* together. To add 3 + 4, we find a set with three elements and a different set with four elements (having no elements in common with the first set). We combine the two sets and then count how many elements are in the union. This method is illustrated in the following exercise.

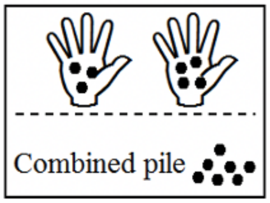
? Exercise 1

Take out the set of 25 coins from Material Card 1. Put three coins in one hand and call this set of coins set A. Put four different coins in your other hand and call this set of coins set B. Use appropriate set notation and coin abbreviations when completing the following:

a. A =	b. $n(A)$ =
d. $n(B)$ =	d. $n(B)$ =

Put the coins from both hands into one pile to form the union of A and B.

e. $A \cup B$	f. $nA \cup B$



So to add two numbers, 3 and 4, we must find a set A such that $n(A) = 3$ and a set B such that $n(B) = 4$. Then, 3 + 4 is how many elements are in $A \cup B$. In Exercise 1, since $n(A \cup B) = 7$, then 3 + 4 = 7. Of course, we already knew that from years of drill and practice, flashcards, etc. To the right is a picture which shows three coins in one hand and four in another. Combining the coins into one big pile, they can be counted to find the sum.

There is one restriction on the two sets chosen which you will discover by doing the next exercise.

? Exercise 2

We are going to try and find the answer to 3 + 6 by using the same method just employed. But this time, I want you to work with sets A and B as I define them here: A = {t, u, v} and B = {u, v, w, x, y, z}. Note that $n(A) = 3$ and $n(B) = 6$.

a. List the elements in $A \cup B$:
b. How many elements are in $A \cup B$? ____
c. Is 3 + 6 = n$A \cup B$? ____
d. Is n(A) + n(B) = n$A \cup B$? ____
e. How might the sets A and B be chosen so that the answers to both c and d would be yes? If you need a hint, look at what is in parentheses in the beginning paragraph of this exercise set!

By the way, since $n(A) = 3$ and $n(B) = 6$, part c and d are asking the same question!

Use your A-Blocks for the following exercises. As a reminder, the letter that is in **BOLD** type in each word below is the abbreviation used for the nine main subsets of A-Blocks. **R**ED, **B**LUE, **G**REEN, **Y**ELLOW, **S**MALL, **L**ARGE, **C**IRCLE,

TRIANGLE, S**Q**UARE (To denote a particular A-Block, use three letters in the format Size, Color, Shape — e.g., Large Blue Triangle is abbreviated LBT, Small Red Square is abbreviated SRQ, etc.)

? Exercise 3

a. Name two subsets of A-Blocks (each containing 6 elements) that could be used to find the sum of 6 and 6: ____ and _____

b. Illustrate how the two sets you chose will give the correct sum by listing both the elements in each set you named and the elements in the union of the two sets (using abbreviations). The number of elements in the union should be 12!

? Exercise 4

Since *n*(L) = 12 *and n*(T) = 8, let's find out if the subsets of **L** and **T** can be used to find the sum of 12 and 8.

a. List the elements in $(L \cup T)$:

b. n$L \cup T$ = ____

c. Is *n* $(L \cup T)$ equal to 12 + 8? ____

d. Why didn't it work to use the sets of Large A-Blocks and Triangular A-Blocks to find the sum?

A formal, more general definition of addition of whole numbers now follows.

Set Theory Definition of Addition: If A and B are two disjoint sets, that is $A \cap B = Ø$, then n(A) + n(B) = n($A \cup B$)**.**

Following are some examples of how to use the set theory definition of addition to add.

✓ Example 1

Use the set theory definition of addition to show that 3 + 2 = 5.

Solution

Let A = {x, y, z} and B = {m, r} Since n(A) = 3, n(B) =2, and $A \cap B = Ø$ then

3+2 = n(A)+n(B)	by substituting n(A) for 3 and n(B) for 2
= n($A \cup B$)	by the set theory definition of addition
= n({x, y, z, m, r})	by computing A
= 5	by counting the elements in A

Therefore, 3 +2 = 5.

✓ Example 2

Use the set theory definition of addition to show that 1 + 3 = 4

Solution

Let A = {x} and B {a, b, c}. Since n(A) = 1, n(B) = 3 and $A \cup B = Ø$, then

1 + 3 = n(A) + n(B)	by substituting n(A) for 1 and n(B) for 3
= n($A \cup B$)	by the set theory definition of addition
= n({x, a, b, c})	by computing $A \cup B$
= 4	by counting the elements in $A \cup B$

Therefore, 1 + 3 = 4

✓ Example 3

Use the set theory definition of addition to show that 0 + 4 = 4.

Solution

Let A = { } and B = {a, b, c, d}. Since n(A) = 0, n(B) = 4 and $A \cup B$ = Ø, then

0 + 4 = n(A) + n(B)	by substituting n(A) for 0 and n(B) for 4
= n($A \cup B$)	by the set theory definition of addition
= n({a, b, c, d})	by computing $A \cup B$
= 4	by counting the elements in $A \cup B$

Therefore, 0 + 4 = 4

The definition of addition allows us to verify some properties about addition. Example 3 verifies the property from Set Theory that states that for any set A, $A \cup \varnothing$ = A.

Let n(A)= a. Since the number of elements in the null set is zero, then a + 0 = a and 0 + a = a.

? Exercise 5

Use the set theory definition of addition to do the following additions. Write it out in detail! Follow the examples above.

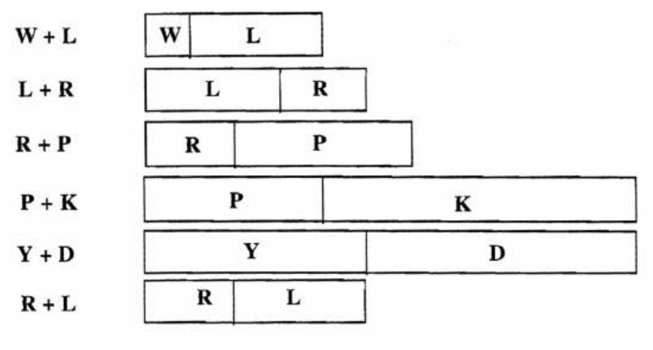
a. Show that 2 + 4 = 6
b. Show that 2 + 2 = 4
c. Show that 3 + 0 = 3

Take out your set of Centimeter Strips (C-Strips). There are twelve different kinds of strips, each having a different length. Below is a listing of the different kinds, with their dimensions, colors and the abbreviations we'll be using to refer to each. Let's agree to use the abbreviations in more than one context — a letter may refer to the actual physical strip itself *or* it might refer to the length of the strip. This isn't exactly perfect notation wise, but it will make the work that follows less cumbersome.

1 cm by 1 cm.................................. **W**HITE.................. **W**
1 cm by 2 cm....................................... **R**ED................... **R**
1 cm by 3 cm...................... **L**IGHT GREEN................... **L**
1 cm by 4 cm................................ **P**URPLE.................... **P**
1 cm by 5 cm.............................. **Y**ELLOW................... **Y**
1 cm by 6 cm....................... **D**ARK GREEN................... **D**
1 cm by 7 cm.................................. BLAC**K**................... **K**
1 cm by 8 cm................................ BROW**N**................... **N**
1 cm by 9 cm...................................... **B**LUE................... **B**
1 cm by 10 cm............................ **O**RANGE................... **O**
1 cm by 11 cm............................... **S**ILVER.................... **S**
1 cm by 12 cm........................... **H**OT PINK................... **H**

We will be using the C-Strips to discover and reinforce concepts and properties about addition. For many of the problems, we will be working with "trains", which are made by putting one or more strips end to end in a straight line. So the width will remain 1 cm but the length of the train will vary.

To designate a train consisting of more than one strip, we write an addition problem so that the C-Strip on the left is written first and the rest, if any, are written from left to right. Below, note the way trains are formed and how we denote each one.



Exercise 6

Use the C-Strips to make the train indicated. Then, find a C-Strip of equal length and fill in each blank with the **abbreviation** of that C-Strip

a. P + L =	b. S + W =	c. R + K =
d. B + W =	e. N + L =	f. L + W =
g. P + Y =	h. D + R =	i. B + L =
j. Y + P =	k. R + D	l. L + B =

m. What do you notice about j & g, k & h and l & i?

Parts j & g, k & h and l & i of Exercise 6 illustrate an important principle about whole numbers called the Commutative Property of Addition.

The **Commutative Property of Addition** states that if a and b are any two whole numbers, then $a + b = b + a$.

Using the definition of addition and **this important fact from Set Theory,**

$A \cup B = B \cup A$, we can show why the Commutative Property of Addition is true:

$a + b = n(A) + n(B) = n(A \cup B) = n(B \cup A) = n(B) + n(A) = b + a$

Most of us have been adding numbers together for years and years and so this property may seem obvious — it is second nature that the order in which we add two numbers is irrelevant. But if you take the opportunity to ask a child who is just learning to add the following two questions, even one right after the other, you may notice the child can do the first one quickly and easily but then struggles a little longer at the second one. That is usually the case if the child hasn't yet discovered the commutative property of addition. As adults, we take this property for granted.

First question: What is 7 + 2? Second question: What is 2 + 7?

Exercise 7

How do you think a child might figure out the answer to the first question and how might he or she figure out how to do the second question? Be specific and assume the answers haven't been memorized yet! Why might the second question be perceived as a harder problem?

When adding two numbers together, some people start with the first number and *count on* the second number. So if you think about 7 + 2, start with 7 and count two more in your head — eight, nine. For 2 + 7, start with 2 and count seven more in your head — three, four, five, six, seven, eight, nine. Although the answer is the same, 7 + 2 was quicker and easier to keep track of. Just think about the difference between 1000 + 1 and 1 + 1000 using this counting on method! Thank goodness for the commutative property of addition!

What if you had to add three numbers together? Actually, we only add two numbers together at a time and then add the third one to the sum of the first two numbers. We will use C-Strips to illustrate ways of adding three numbers together. Consider putting the C-Strips L, R and P together to form a train. Let's compute the sum of these three in two different ways, working inside parentheses first.

? Exercise 8

Fill in each blank with the correct abbreviation of a C-Strip by first computing the sum in the parentheses and putting that answer in the first blank. Then add once more to get the answer. An example with numbers: (5 + 3) + 9 = 8 + 9 = 17.

Exercise 8 illustrates another important principle about whole numbers called the Associative Property of Addition.

The **Associative Property of Addition** states that if *a* , *b* and *c* are any three numbers, then (a + b) + c = a + (b + c)

Using the definition of addition and **this important fact from Set Theory**, $(A \cup B) \cup C = A \cup (B \cup C)$, we show why the Associative Property of Addition is true: (a + b) + c = (*n*(A) + *n*(B)) + *n*(C) =*n*($A \cup B$) + *n*(C) = *n*($(A \cup B) \cup C$) = *n*($(A \cup B) \cup C$ = *n*(A) + *n*($(B \cup C)$) = *n*(A) + (*n*(B) + *n*(C)) = a + (b + c)

? Exercise 9

Using the model of Exercise 8 and the C-Strips, illustrate another application of the associative property of addition. Then write down an equivalent statement using numbers.

The root in the word "associative" is *associate*. Think about whether the middle number will associate with the first number or the last number for the first computation. As for adding numbers in your head, it may be a lot easier to do it one way instead of the other. For instance, to add 58 + 39 + 41, you could think (58 + 39) + 41 = 97 + 41 = 138 or you could think 58 + (39 + 41) = 58 + 80 = 138. 97 + 41 and 58 + 80 both equal 138, but that isn't obvious until after you add. I prefer to add the 39 and 41 together first. The associative property allows that. But again, the associative property is something that most adults take for granted and use but they don't really think about it.

Here is another example. Although the sum of the three numbers is the same for each problem, the way you do the computations are different.

(84 + 56) + 73	84 + (56 + 73)
140 + 73	84 + 129
213	213

? Exercise 10

Make up an example of your own showing an application of the associative property

It might be helpful to note that the **commutative *property implies there has been an* order *change.*** An order change means that the numbers written are in a different order. The **associative *property implies there has been a* parentheses *change.*** A parentheses change means there are different numbers in the parentheses and that the parentheses are now around different numbers.

? Exercise 11

State which property (Commutative or Associative Property of Addition) is being used in each equation. Ask yourself: Is the difference between the left and right side due to order (commutative property) or parentheses (associative property)?

a. ______	(99 + 76) + 38 = (76 + 99) + 38
b. ______	(65 + 22) + 56 = 56 + (65 + 22)
c. ______	(57 + 88) + 43 = 57 + (88 + 43)
d. ______	(a + b) + (c + d) = ((a + b) + c) + d

We use the commutative and associative properties of addition all the time to do simple arithmetic — of course, the arithmetic might not be quite so easy if we weren't allowed to use these properties. Imagine neither property existed and the only way to get the sum of three numbers was to add them in order from left to right. Consider this arithmetic problem: 17 + 59 + 83. Going from left to right, we first add 17 and 59 to obtain 76 and then add 83 to 76 to get the answer of 159. Okay, so that's not so difficult. But it becomes a breeze if we are allowed to use the commutative and associative properties of addition as shown below (three ways are shown using different steps):

(17 + 59) + 83 = 83 + (17 + 59) Commutative property

= (83 + 17) + 59 Associative property

= 100 + 59

= 159

(17 + 59) + 83 = (59 + 17) + 83 Commutative property

= 59 + (17 + 83) Associative property

= 59 + 100

= 159

(17 + 59) + 83 = 17 + (59 + 83) Associative property

= 17 + (83 + 59) Commutative property

= (17 + 83) + 59 Associative property

= 100 + 59

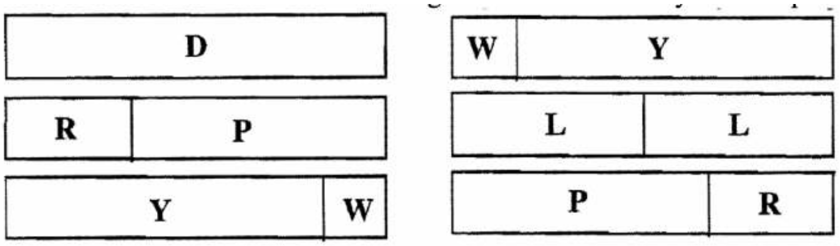
= 159

Of course, most of us probably automatically add the 83 and 17 in our head first without bothering to think about the properties. But it is because of these fundamental properties that we arrive at the correct answer by adding the numbers in any order or combination we find convenient.

Exercise 12

Use the commutative and associative properties to add (135 + 384) + 165. Assume you prefer to add 135 and 165 in your head. Show each step and state which property is being used at each step. **Then do it again** another way using different steps.

We say two trains are equal if they have the same length. For instance, look at the six trains shown below. All of these have the same length of 6 cm and so they are all equal.



Although the six trains shown above are equal, they are considered *different* because none of them have the exact same strips in exactly the same order. Here is a list of the six trains above: **D, W + Y, R + P, L + L, Y + W, P + R**. There are more trains having this same length but different from the ones listed so far. For instance, seven more are: **W + R + R + W, W + P + W, P + W + W, W + W + P, W + L + W + W, R + L + W, L + W + W + W**. In fact, there are several more trains that are equal in length to the Dark Green C-Strip than the 13 different ones listed so far . All of the trains listed translate into addition facts about combinations of numbers that add up to 6: W + Y = D translates to 1 + 5 = 6, W + P + W = D translates to 1 + 4 + 1 = 6, R + P = D translates to 2 + 4 =6, P + R = D translates to 4 + 2 = 6 and so on.

Exercise 13

Use the C-strips to list all the different trains that are equal in length to the Light Green C-strip. *Different trains have different strips or and/or a different order of strips.*

Exercise 14

Use the trains listed in Exercise 13 to write addition facts obtained about the number 3.

Exercise 15

Use the C-strips to list all the different trains that are equal in length to the Purple C-strip. *Different trains have different strips or and/or a different order of strips.*

Exercise 16

Use the trains listed in Exercise 15 to write addition facts obtained about the number 4

Listing *all* the different trains equal in length to the Dark Green C-Strip is quite a chore. We will define ***equivalent trains*** as equal trains that have the same make-up (same exact strips), but the strips are listed in a different order. For instance, **W + R + L, W + L + R, L + R + W, L + W + R, R + L + W** and **R + W + L** are all *equivalent* trains because each is made up of a red, light green and white C-strip. **W+L and L+W** are *equivalent* trains because they are all made up of exactly the same C-strips — a white C-Strip and a red C-Strip. **W+L, R+R, P, W+W+R and W+W+W+W** are equal but *nonequivalent* trains because they are composed of different strips. ***Nonequivalent trains* are equal trains made up of *different C-strips*. If they are made up of the same C-strips but are in a different order, they are equivalent!**

Exercise 17

List all ***non*equivalent** trains (note the last sentence in **bold** in the above paragraph) equal in length to the Dark Green C-Strip. That means only one of the six ones listed in the middle of the paragraph above (and it doesn't matter which one) should be written down in this list. Remember to include the train of one single strip. How many *non*equivalent trains equal in length to the Dark Green C-Strip are there?

Exercise 18

From your work in Exercise 17, list all the facts about what combinations of numbers add up to 6.

If you look at the C-Strips, it is clear that some are longer than others. We say that one C-Strip is longer than another C-Strip if there is a C-Strip that can be added to the second C-Strip to make a train equal in length to the longer C-Strip. For example, the Black C-Strip is longer than the Purple C-Strip because we can add the Light Green C-Strip to the Purple C-Strip to make a train equal in length to the Black C-Strip. We write **K** is longer than **P** because **K = P +** L.

Exercise 19

Verify that each of the named C-Strips are longer than the other C-Strip by filling in the blanks. Abbreviations are used.

a. **H** is longer than **O** because **H** = **O** + ___

b. **N** is longer than **L** because **N** = **L** + ___

c. **B** is longer than **W** because **B** = **W** + ___

d. **P** is longer than **R** because **P** = **R** + ___

e. **S** is longer than **K** because **S** = **K** + ___

f. **D** is longer than **Y** because **D** = **Y** + ___

The same principle just used with C-Strips applies when comparing two whole numbers.

Definition: The whole number m *is greater than* the whole number n if there is a whole number k such that m = n + k.

In mathematics, the symbol used to denote "is greater than" is: " > ". Therefore, " 7 > 5 " is read "Seven is greater than five".

Exercise 19 can be modified to express information about numbers. For instance, part a corresponds to the fact "12 is greater than 10 because 12 = 10 + 2" or if we want to write it symbolically, we write "12 > 10 because 12 = 10 + 2".

? Exercise 20

Verify each of the following (I've done part a for you):

a. 9 > 3 because $\underline{9 = 3 + 6}$	b. 30 > 22 because ____
c. 156 > 96 because ____	d. 80 > 0 because ____
e. 231 > 195 because ____	f. 987> 967 because ____

Now that you've worked a little with *greater than*, we may as well define *less than.*

Definition: The whole number r *is less than* the whole number s if there is a whole number k such that r + k = s.

In mathematics, the symbol used to denote "is less than" is: " < ". Therefore, " 4 < 9 " is read "Four is less than nine".

? Exercise 21

Verify each of the following (I've done part a for you):

a. 9 < 22 because $\underline{9 + 13 = 22}$	b. 30 < 99 because
c. 19 < 70 because	d. 0 < 32 because
e. 489 < 500 because	f. 65 < 201 because

? Exercise 22

"Less than" and "greater than" can be used to order numbers on a number line. Fill in each blank with *right* or *left*.

a. If m < n, then m is to the _____________ of n on the number line

b. If m > n, then m is to the _____________ of n on the number line.

? Exercise 23

Form two trains of C-Strips, m and n, that are equal. If p is a C-Strip or train of C-Strips of any length, is it always true than m + p = n + p?

Exercise 24

Take two different C-Strips or form a train of C-Strips, m and n, such that $m > n$. If p is a C-Strip or train of C-Strips of any length, is it always true than $m+p>n+p$?

Exercise 23 illustrates that the *additive property of equality* . If the same number is added to both sides of an equality, the equation remains true. Exercise 24 illustrates the *additive property of inequality*. If the same number is added to both sides of an inequality, the inequality remains true.

If you take any two trains, s and t, and measure them up by placing one next to the other, you will notice that either they are of equal length or one of them is longer that the other. This observation leads us to the *trichotomy law.*

Trichotomy law

If a and b are whole numbers, then exactly one of the following is true: a = b or a < b or a > b.

Exercise 25

Place the correct symbol (<, =, or >) between each pair of numbers. Verify if you are correct if you use the less than or greater than symbol (as in exercises 20 and 21).

a. 8 ____ 19 ____
b. 24 ____ 13 ____

Exercise 26

We began this exercise set by defining the whole numbers. Using set notation, express the whole numbers using the listing method: ____

Exercise 27

If two whole numbers are added together, will the sum always be a whole number? ____

Exercise 28

If two whole numbers are added together, is it possible to get more than one answer? ____

Your answers to Exercises 24 and 25 should confirm that the sum of two whole numbers is always a unique whole number. This is called the ***closure property of addition*** for whole numbers. We say the set of whole numbers is closed under addition because when you add any two whole numbers together, you get another unique whole number.

A set is closed under addition if the sum of any two elements in the set (these could be the same elements or two different elements) produces a unique element in the same set.

Addition is actually just an operation we perform on two members of a set. Other operations you are no doubt familiar with are subtraction, multiplication and division. There are other operations and there are new ones you can make up, but we'll leave that discussion for another time. If the operation is defined to be performed on exactly two elements, then it is called a binary operation.

In general, a set is closed under an operation if when you perform the operation on any two elements in the set, the answer is a unique element of the same set.

To show that a set is **not closed** under addition, you must give a **counterexample** showing that when you add two numbers in the set, you get a sum that is not in the set. The two numbers you choose can be the same number or different numbers.

✓ Example 1

Is the set {1, 2, 3, 4, 5} closed under addition?

Solution

Imagine you had two hats, each containing these five numbers in the set written on pieces of paper. So each hat contains the set {1, 2, 3, 4, 5}. If you pull a number out of each hat, you might get the same number twice or you might get different numbers. Some of the sums are 1 + 1 = 2, 2 + 3 = 5, 1 + 3 = 4, 3 + 3 = 6, etc. Note that the first three sums 2, 5 and 4 are in the set {1, 2, 3, 4, 5}. **But for a set to be closed, the sum of *every possible* two numbers in the set must yield a number in the original set.** Since 3 + 3 = 6, (where the two addends, 3 and 3, are in the original set, but the sum 6 is NOT in the original set), then the set {1, 2, 3, 4, 5} is not closed. There are any number of counterexamples you can use to show it is not closed. 3 + 3 = 6 will do. Or someone else might write 4 + 5 = 9. What are two other counterexamples you could use to show {1, 2, 3, 4, 5} is not closed under addition?

✓ Example 2

Is {4} closed under addition?

Solution

The only possibility to try is 4 + 4, which is 8, and 8 is not in the set. So the set is not closed. The counterexample to show it is not closed is 4 + 4 = 8.

✓ Example 3

Is {5, 10, 15, 20, . . .} closed under addition?

Solution

Start by trying a few examples: 5 + 5 = 10 which is in the set; 10 + 15 = 25 which is in the set (note the three dots which means the next few numbers are 25, 30 and 35. It looks like it is closed. But how can you convince someone that the sum of any two numbers in the set will always be in the original set. Note that the set is composed of multiples of 5. Note any two multiples of 5 can be written as 5 times something, like 5x and 5y. Let 5x and 5y denote two arbitrary elements in the original set. Adding them together, we get 5x + 5y = 5(x+y) which is another multiple of 5 (since it is 5 times something). So, you should be convinced that the sum of any two elements in the set is a multiple of 5, which is what the original set consisted of. Therefore, {5, 10, 15, 20, . . .} is closed under addition.

? Exercise 29

State whether or not each set below is closed under addition. If it is not closed, provide an counterexample showing two elements in the set whose sum is not in the set. In general, providing an example showing that something is not always true is called providing a *counterexample.* If a set is closed, prove and explain why.

a. {0}

b. {1}

c. {0,2,4,6,. . .}

d. {1,3,5,7,. . .}

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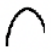
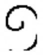
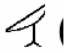
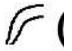
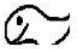
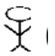
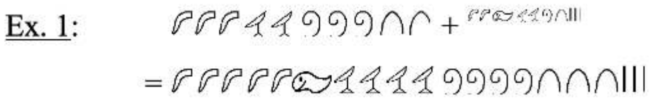
3.2: Combining

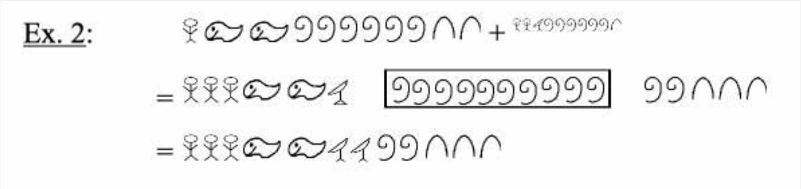
You will need: ***Base Blocks*** *(Material Cards 4-15)*

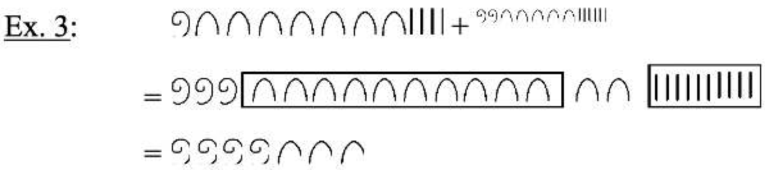
We will begin this exercise set by figuring out how to add in the Egyptian numeral system. You are reminded of the symbols and their Hindu-Arabic equivalents below:

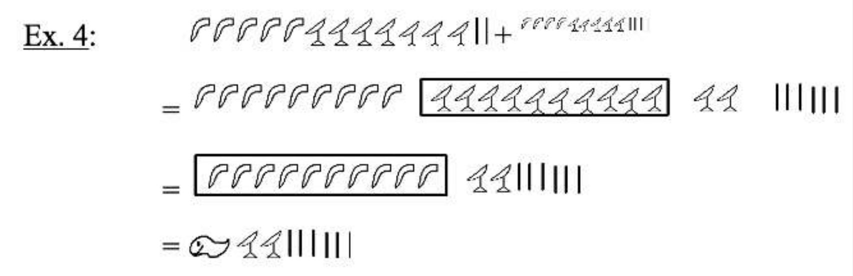
| (1)	 (10)	 (100)	 (1,000)	 (10,000)	 (100,000)	 (1,000,000)

Since the Egyptian numeration system is a basic additive system, the sum of two numbers is formed by simply *combining*the symbols from both numerals together. Because it is a Base Ten system, an exchange can be made any time there are ten of the same symbol by replacing ten of one symbol for the next higher symbol . To show that an exchange is being made, you can circle, put a box around or underline ten of the same symbol. In the examples that follow, the exchanges are indicated by putting a box around a group of ten symbols that are to be replaced with a new symbol at the next step. After each exchange, it may be necessary to make another exchange. Study the following examples of Egyptian addition problems. Try to think in Egyptian as you do these by simply following the rules of combining and exchanging as opposed to thinking about what numbers each numeral stands for in Hindu-Arabic. After working through each problem, you can always go back and check by first converting each of the numerals being added (the *addends*)to Hindu-Arabic, adding them together and then checking that the sum agrees with the answer obtained in Egyptian.





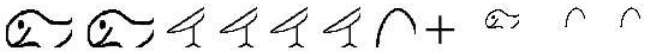




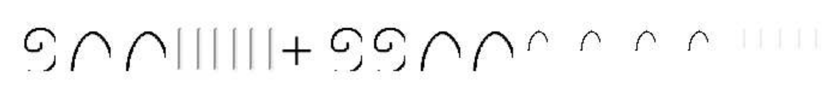
? Exercise 1

Add the following Egyptian numerals. Show all steps, indicating any exchanges made. Again, try to work through these by thinking only in Egyptian. You can always go back and check when you are done by doing them in Hindu-Arabic. Make up your own problems for part e and part f.

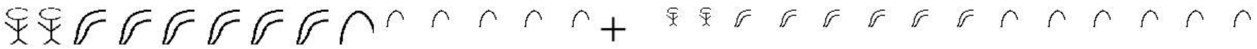
a.



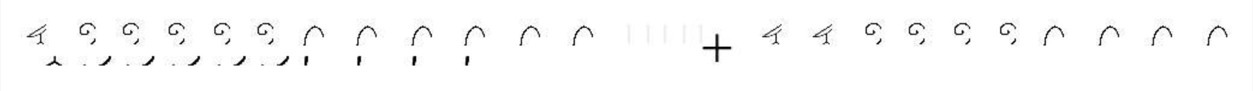
b.



c.



d.



Let's pretend that our current system of money was strictly in Base Ten so that the only currency we used were pennies, dimes, one dollar bills, ten dollar bills, hundred dollar bills, thousand dollar bills and so on. Instead of drawing pictures of the money, these abbreviations for each kind of coin or bill will be used: penny (A), dime (B), one dollar bill (C), ten dollar bill (D), hundred dollar bill (E), thousand dollar bill (F) and so on. Since most of you are considering entering the teaching profession, you probably won't need to handle anything higher than a thousand dollar bill. At least that's the case as I write this in 1999 but who knows what the future holds, right?

? Exercise 2

Show what exchanges could be made for each of the following:

a. AAAAAAAAAA = ____	b. CCCCCCCCCC = ____	c. FFFFFFFFFF = ____

? Exercise 3

Form each sum by combining the two addends into one lump. Make exchanges as necessary so that the sum is represented by the least number of coins or bills. Show all steps, indicating any exchanges made.

a. CCAADDDCB + BBBBBBBBBAAACC

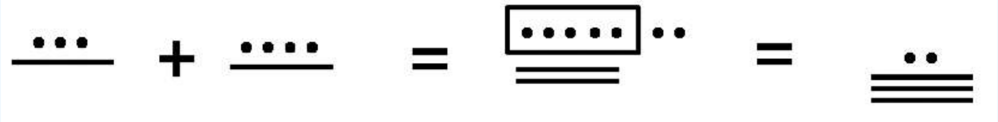
b. AAAACCCCCCCCCEEE + CCCCCCBBA

c. EEEEEEEEAAAAAAAA + EEEAA

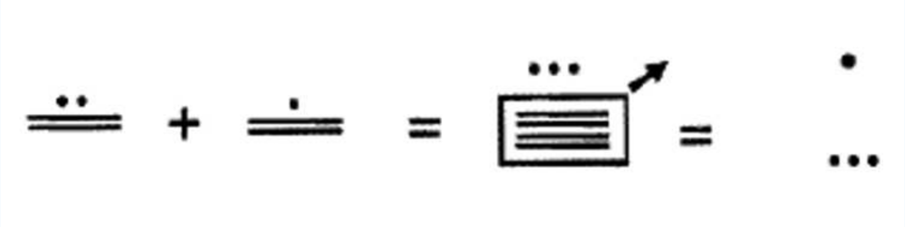
d. DDDDCCCCCBBBBB + DDDDDCCCCBBBBB

To add in the Mayan system, the first step is to combine the symbols at each level together. *The symbols at each level must be combined separately.* Then, at each level, look to see if any exchanges can be made. A group of five dots can always be exchanged for a line segment. Exchanges from one level to the next are a little trickier. A group of 20 at one level can be traded for one dot at the next level up except from the second to the third level *where a group of* ***18*** *at the second level can be traded for one dot at the third level*. Study the following examples of one level Mayan numerals being added together. Any time an exchange is being made, a circle or box will be put around what will be replaced at the next step. An arrow will be used to indicate if an exchange is being made from one level to the next level up. The arrow indicates a dot will be on the next level up in the next step in place of the amount boxed at the original level. See example 2 below: From step 2 to step 3, the four line segments on level one are replaced with a dot at level two.

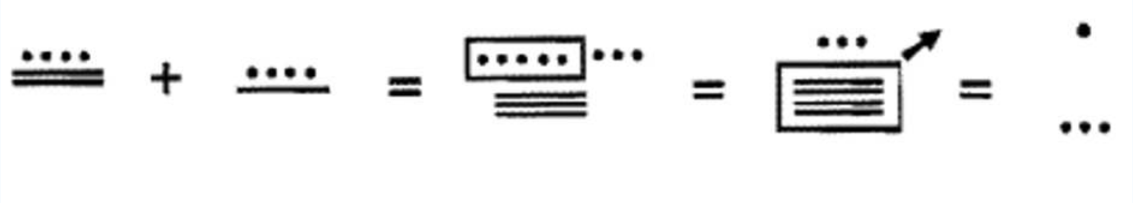
✓ Example 1



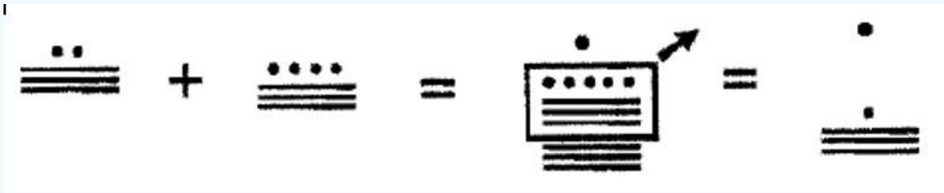
✓ Example 2



✓ Example 3



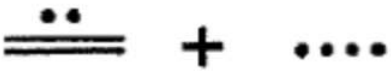
✓ Example 4



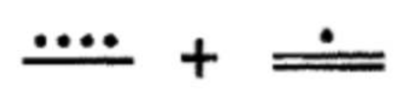
? Exercise 4

Add the following single level Mayan numerals together. Make any necessary exchanges and show all steps.

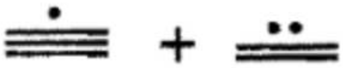
a.



b.

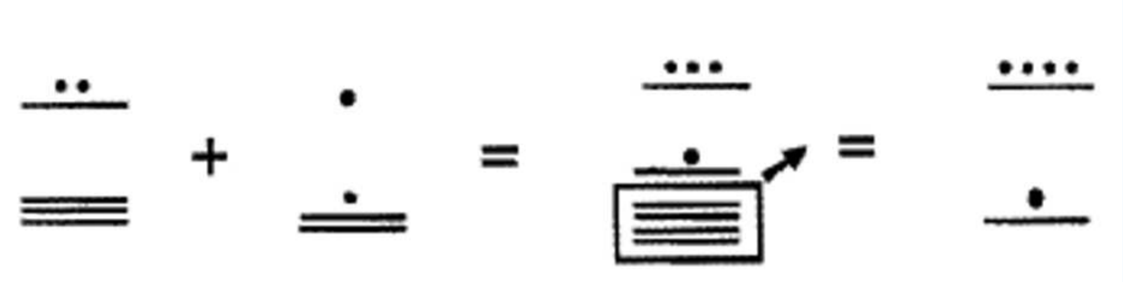


c.

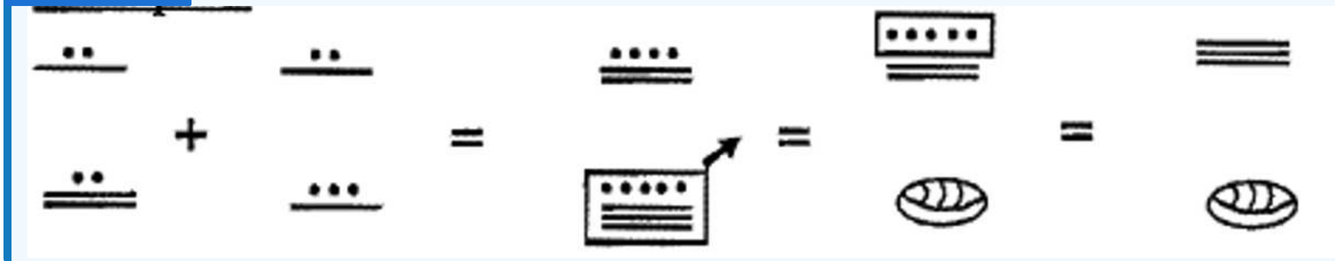


Study the following examples on this and the next page of adding Mayan numerals together. All of these are more than one level. Large spaces are used to differentiate between the different levels. Try to do them by yourself on your own paper.

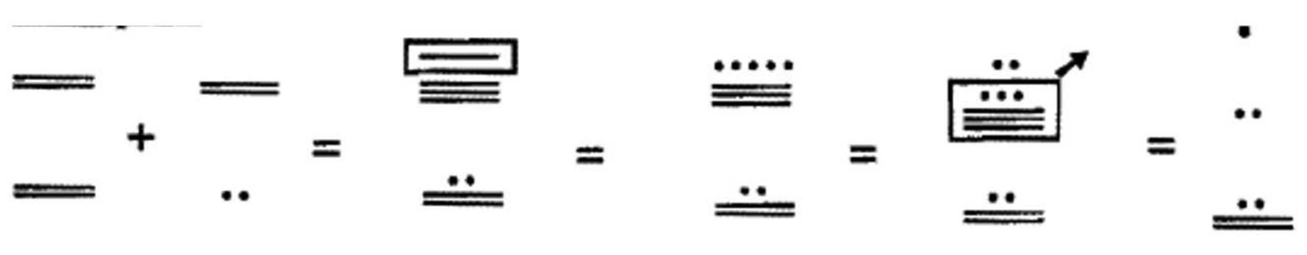
Example 1



Example 2

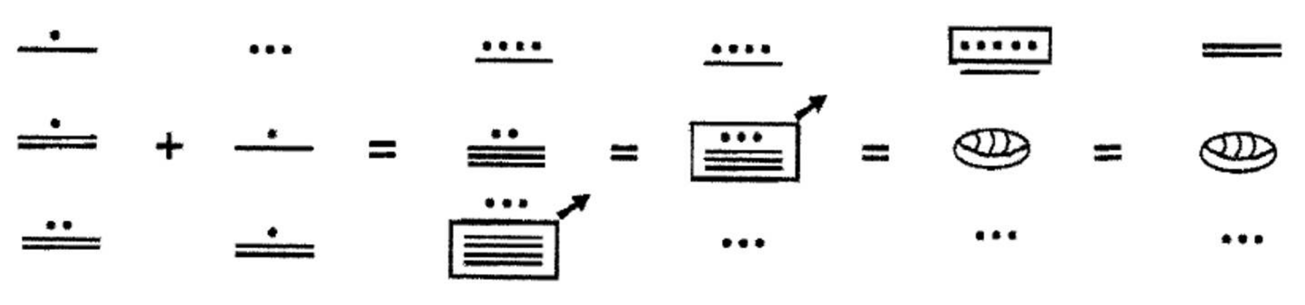


Example 3

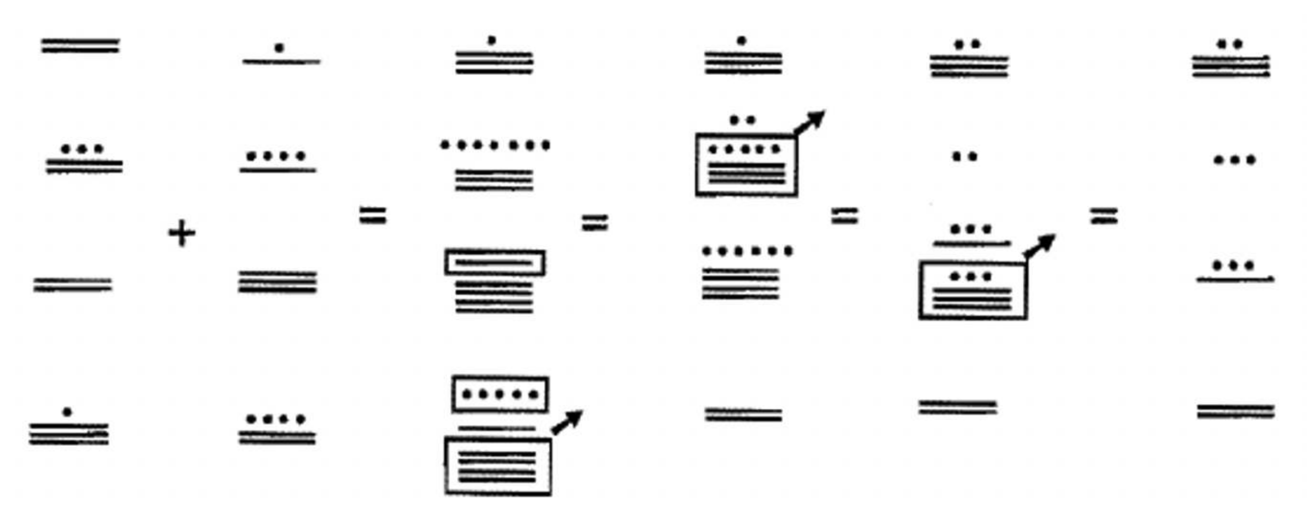


Remember, a group of 18 at the second level can be exchanged for a dot at the third level!

Example 4



Example 5

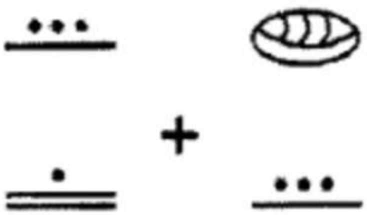


After doing each of the above examples solely in Mayan, I checked each problem by converting each addend to Hindu-Arabic, adding in Base Ten, converting the sum into Mayan and making sure the answer agreed with the answer I had gotten. When I checked Example 5 (76,896 + 46,754 = 123,650), a fairly complicated problem, it didn't check. So I reworked the problem until I got it right. It's a good idea to check your work carefully as you go along which requires strong doses of concentration, effort and patience. Otherwise, it's quite easy to make one or more mistakes during computation. Keep in mind that if all you do is convert to Hindu-Arabic, add and convert back to Mayan, you won't get credit because that would not show all of the steps involved, combining, exchanging, etc. that you are required to show when you do the exercises. Try each of the above examples on a separate piece of paper to make sure you are working them correctly. Note that the sequence of steps shown for each example might not be the only way to arrive at the correct answer.

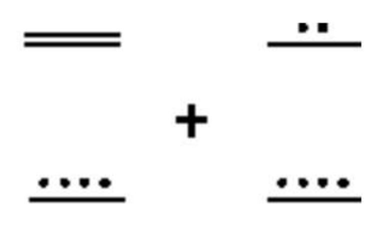
? Exercise 5

Add the following Mayan numerals together. Make any necessary exchanges and show all steps. It's a very good idea to check your work and that doesn't mean looking up the answer in the solutions! After adding in Mayan, convert the answer to Hindu-Arablic. Then, convert the original two numbers to Hindu-Arabic and add together and see if it matches the answer.

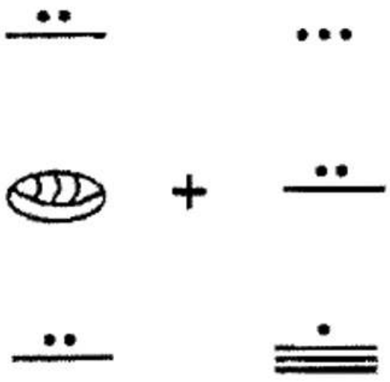
a.



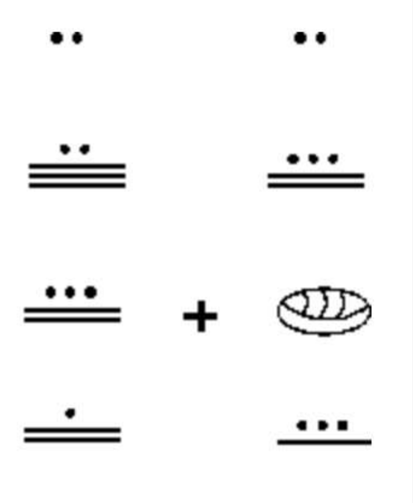
b.



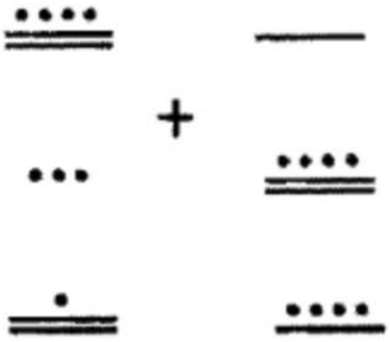
c.



d.



e.



? Exercise 6

Count out 15 unit blocks and make exchanges with base four blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base four, write 15 as a base four numeral on the space provided for Pile A below. Now, count out 13 more units and make exchanges with base four blocks. Put them in a pile, called Pile B. Since you've made exchanges in base four, write 13 as a base four numeral on the space provided for Pile B below. To add, simply combine the blocks from Pile

A and Pile B to form one big pile of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base four blocks and then write the number of units in the combined pile as a base four numeral on the space provided.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in a different base.

Below is the addition problem you have just performed in Base Four.

____________ + _____________ = ____________

Pile A Pile B Combined Pile

Exercise 7

Count out 15 unit blocks and make exchanges with base seven blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base seven , write 15 as a base seven numeral on the space provided for Pile A below. Now, count out 13 more units and make exchanges with base seven blocks. Put them in a pile, called Pile B. Since you've made exchanges in base seven , write 13 as a base seven numeral on the space provided for Pile B below. To add, simply combine the blocks from Pile A and Pile B to form one big pile of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base seven blocks and then write the number of units in the combined pile as a base seven numeral on the space provided below.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in a different base.

Below is the addition problem you have just performed in Base Seven

____________ + _____________ = ____________

Pile A Pile B Combined Pile

Exercise 8

Count out 15 unit blocks and make exchanges with base two blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base two, write 15 as a base two numeral on the space provided for Pile A below. Now, count out 13 more units and make exchanges with base two blocks. Put them in a pile, called Pile B. Since you've made exchanges in base two, write 13 as a base two numeral on the space provided for Pile B below. To add, simply combine the blocks from Pile A and Pile B to form one big pile of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base two blocks and then write the number of units in the combined pile as a base two numeral on the space provided.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in a different base.

Below is the addition problem you have just performed in Base Two.

____________ + _____________ = ____________

Pile A Pile B Combined Pile

Exercise 9

Count out 15 unit blocks and make exchanges with base nine blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base nine, write 15 as a base nine numeral on the space provided for Pile A below. Now, count out 13 more units and make exchanges with base nine blocks. Put them in a pile, called Pile B. Since you've made exchanges in base nine, write 13 as a base nine numeral on the space provided for Pile B below. To add, simply combine the blocks from Pile A and Pile B to form one big pile of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base nine blocks and then write the number of units in the combined pile as a base nine numeral on the space provided.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in a different base.

Below is the addition problem you have just performed in Base Nine.

____________ + _____________ = ____________

Pile A Pile B Combined Pile

Exercise 10

Count out 15 unit blocks and make exchanges with base three blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base three, write 15 as a base three numeral on the space provided for Pile A below. Now, count out 13 more units and make exchanges with base three blocks. Put them in a pile, called Pile B. Since you've made exchanges in base three, write 13 as a base three numeral on the space provided for Pile B below. To add, simply combine the blocks from Pile A and Pile B to form one big pile of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base three blocks and then write the number of units in the combined pile as a base three numeral on the space provided below.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in a different base.

Below is the addition problem you have just performed in Base Three

____________ + _____________ = ____________

Pile A Pile B Combined Pile

Exercise 11

Count out 15 unit blocks and make exchanges with base eight blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base eight, write 15 as a base eight numeral on the space provided for Pile A below. Now, count out 13 more units and make exchanges with base eight blocks. Put them in a pile, called Pile B. Since you've made exchanges in base eight, write 13 as a base eight numeral on the space provided for Pile B below. To add, simply combine the blocks from Pile A and Pile B to form one big pile of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base eight blocks and then write the number of units in the combined pile as a base eight numeral on the space provided below.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in a different base.

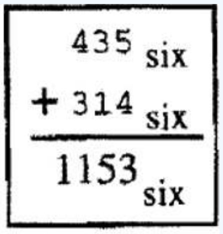
Below is the addition problem you have just performed in Base Eight.

____________ + _____________ = ____________

Pile A Pile B Combined Pile

Exercise 12

Write the six addition problems from exercises 6 -11 in a vertical format like the one in Base Six shown below. Study this Base Six problem as well as all six problems from the previous six exercises. Try to figure out a way to do the addition problems using paper and pencil instead of by using the Base Blocks. In other words, try to come up with your own algorithm (method) for doing addition in other bases. Explain your method and show a few examples. In the next Exercise Set, you will be learning algorithms for addition.



$$\begin{array}{r} 435_{\text{six}} \\ +\,314_{\text{six}} \\ \hline 1153_{\text{six}} \end{array}$$

Let m < b and n < b. Let's define the word "complement" of a number, m_b, in a given base b to be the number, n_b, such $m_b + n_b$ $= 10_b$. *(Don't read 10b as "ten, base b").* In other words, think of mb as a number of single units less than the base, b. **Then the complement, nb, would be the number of single units it takes to add to mb such that the sum of mb and nb could be traded in for exactly one long.** *Note: Complement as defined here has nothing to do with the word complement as we defined and used it in Set Theory!* The examples on the next page should clarify this definition.

Here are some examples of complements. The base must be specified.

In Base Ten: 3 & 7 are complements since 3 + 7 = 10

4 & 6 are complements since 4 + 6 = 10

8 is the complement of 2 since 8 + 2 = 10

5 is the complement of 5 since 5 + 5 = 10

In Base Eight: the complement of 5_{eight} is 3_{eight} since $5_{\text{eight}} + 3_{\text{eight}} = 10_{\text{eight}}$

4_{eight} & 4_{eight} are complements since $4_{\text{eight}} + 4_{\text{eight}} = 10_{\text{eight}}$

2_{eight} & 6_{eight} are complements since $2_{\text{eight}} + 6_{\text{eight}} = 10_{\text{eight}}$

7_{eight} is the complement of 1_{eight} since $7_{\text{eight}} + 1_{\text{eight}} = 10_{\text{eight}}$

? Exercise 13

Write the complement of each number:

a. 7_{nine}	b. 5_{seven}	c. 2_{tweleve}
d. 3_{eleven}	e. 2_{six}	f. 1_{two}
g. 2_{three}	h. 1_{four}	i. 3_{five}

Although not necessarily a conscious effort, many people use complements and the associative property of addition to add. For instance, some people use the following strategy to add in Base Ten. Let's say the problem was to add 8 and 6. A person might think "I have 8 and it only takes 2 more to get to 10. If I take 2 from the 6 and add it to the 8, then I have 10 + 4 which is 14" In other words, the problem 8 + 6 becomes 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14. Of course, they do it more automatically than it looks when written down.

The same type of strategy can be used in other bases. Study the following examples. It might help if you use units and longs in the Base Blocks.

$$4_{\text{nine}} + 8_{\text{nine}} = 4_{\text{nine}} + (5_{\text{nine}} + 3_{\text{nine}})$$
$$= (4_{\text{nine}} + 5_{\text{nine}}) + 3_{\text{nine}}$$
$$= 10_{\text{nine}} + 3_{\text{nine}}$$
$$= 13_{\text{nine}} \text{ (which is 1 long and 3 units in base nine)}$$

$$T_{\text{twelve}} + 9_{\text{twelve}} = T_{\text{twelve}} + (2_{\text{twelve}} + 7_{\text{twelve}})$$
$$= (T_{\text{twelve}} + 2_{\text{twelve}}) + 7_{\text{twelve}}$$
$$= 10_{\text{twelve}} + 7_{\text{twelve}}$$
$$= 17_{\text{twelve}} \text{ (which is 1 long and 7 units in base twelve)}$$

$$4_{\text{six}} + 4_{\text{six}} = 4_{\text{six}} + (2_{\text{six}} + 2_{\text{six}})$$
$$= (4_{\text{six}} + 2_{\text{six}}) + 2_{\text{six}}$$
$$= 10_{\text{six}} + 2_{\text{six}}$$
$$= 12_{\text{six}} \text{ (which is 1 long and 2 units in base six)}$$

Exercise 14

Use your Base Blocks or the definition of complements to do the following addition problems:

a. $7_{\text{eleven}} + 8_{\text{eleven}}$	b. $4_{\text{five}} + 3_{\text{five}}$
c. $2_{\text{eight}} + 7_{\text{eight}}$	d. $9_{\text{thirteen}} + 5_{\text{thirteen}}$
e. $3_{\text{four}} + 3_{\text{four}}$	f. $6_{\text{seven}} + 4_{\text{seven}}$
g. $2_{\text{three}} + 2_{\text{three}} =$	h. $1_{\text{two}} + 1_{\text{two}}$
i. $3_{\text{six}} + 5_{\text{six}}$	j. $7_{\text{nine}} + 5_{\text{nine}}$

Now that you can add single digit numbers in any base, you can make up addition tables. Below are addition tables in Base Five and Base Eight. Since Base Five has five digits, its addition table should be a five by five grid and since Base Eight has eight digits, its addition table should be a eight by eight grid. Notice how symmetry, the commutative property and the property that x + 0 = x for all x makes much of the table easy to fill in. When making addition tables (and later when making multiplication tables) in different bases, you should label what kind of a table it is. If you do that, then the convention will be that it is not necessary to write out the word for what base it is to the right of each numeral.

Table 3.2.1: Base Five Addition Table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Table 3.2.2: Base Eight Addition Table

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Exercise 15

Make up addition tables in the base specified.

a. *Base Three Addition Table*
b. *Base Six Addition Table*

c. *Base Two Addition Table*

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3.3: Addition Algorithms

You will need:A ***Calculator, Base Blocks*** *(Material Cards 4-15)*

There is no one algorithm that everyone uses for adding numbers together. A teacher should be familiar with several methods so that the most number of students can be reached. What's important is that students understand what addition means and knows how to add numbers together one way or another. If a student figures out an algorithm that works on their own, I think that's terrific. It shows creativity and the student probably won't forget it. My hope is that more and more teachers realize many paths may lead to solving a problem correctly and that there is usually no reason to insist students work a problem in a particular way. There are many ways to solve a problem and diversity should be encouraged. This course is a little different because you are presumably interested in becoming a teacher. Therefore, it is important for you to learn as many ways as possible to work problems as well as discover some methods of your own. In this section, you will be learning a variety of algorithms for adding and should be able to work through and explain the procedures for working a problem using any of the methods presented. You will learn some new methods for addition and will be using them to add numbers in Base Ten as well as in other bases.

The DOT METHOD for adding columns of single digits

Here is a method for adding several digits together without having to keep a large running total in your head. The idea is that you only add two single digits together at a time. Any time you add and get a number over 9, you put down a dot and only keep the unit's digit in your memory. The dot stands for the ten. For instance, since 6 + 7 = 13 (which is 10 + 3), you write a dot and think 3. *Think 6 + 7 = Dot 3*. In the example below, to add eight numbers together, a common approach would be to add the first two digits together, then keep adding on the next digit until all eight digits have been added. In other words, someone might think: 6 + 7 is 13, 13 + 8 is 21, 21 + 3 is 24, 24 + 5 is 29, 29 + 9 is 38, 38 + 4 is 42 and 42 + 7 is 49. Although this yields the correct answer, it's easy to lose track along the way. With the DOT METHOD shown, a dot is placed by a number if after it is added on, the sum is ten or more. At each step of the way, I've written in parentheses what number you think of after adding each digit. To start, see the eight digits to be added. Step 1: 6 + 7 = 13 so write a dot and think 3; Step 2: 3 + 8 = 11 so write a dot and think 1; Step 3: 1 + 3 = 4 so think 4; Step 4: 4 + 5 =9 so think 9; Step 5: 9 + 9 = 18 so write a dot and think 8; Step 6: 8 + 4 = 12 so write a dot and think 2; Step 7: 2 + 7 = 9 so write a 9 under the bar since all digits have been added. Count the dots (4), which stands for how many tens (40) and write in front of the 9. The answer is 49. Try it on your own now!

Start	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7
6	6	6	6	6	6	6	6
7	7 · (3)	7·	7·	7·	7·	7·	7·
8	8	8 · (1)	8·	8·	8·	8·	8·
3	3	3	3 (4)	3	3	3	3
5	5	5	5	5 (9)	5	5	5
9	9	9	9	9	9 · (8)	9·	9·
4	4	4	4	4	4	4 · (2)	4·
+7	+7	+7	+7	+7	+7	+7	+7 (9)
							49

7·
8·
3
5
9·
4·
+7
49

In the example just shown on the previous page, every single step, including what you think is shown. In reality, this is a quick algorithm and if you looked at the problem after someone did it, the work on the left is all you would see. To the right are are two more examples. Actually, each is the same sum of numbers written in a different order, which is another way to check addition! *A quick check when adding only two numbers is to reverse the order and add again.*

3 9· 6 5· 4 7· 7· +8 49	6 3 7· 9· 7· 5 8· +4 49

Exercise 1

Use the dot method to add the following columns of digits shown to the right.

3 9 4 5 2 6 8 6 +7	6 7 9 3 5 8 2 2 +9	2 9 6 4 2 5 7 9 +3	5 8 7 6 3 9 2 5 +7

By the way, you could put the dots to the left or right of the digit, do whatever is most comfortable for you. If you are adding without paper and pencil, use your fingers to keep track of the dots! Instead of writing a dot, put up a finger. The beauty of this addition is you never have to keep a number higher than nine in your head!

? Exercise 2

The dot method could also be used to add several digits written in a horizontal format. The dots could be placed *over* the numerals. Try it on the following problems. Make up your own problems for part d and part e.

a. 7 + 6 + 4 + 5 + 9 + 6 + 8 + 3 + 4 + 9 + 8 = ____

b. 4 + 9 + 6 + 2 + 9 + 7 + 7 + 6 + 8 + 4 + 8 = ____

c. 7 + 9 + 8 + 5 + 9 + 4 + 2 + 6 + 9 + 8 + 7 = ____

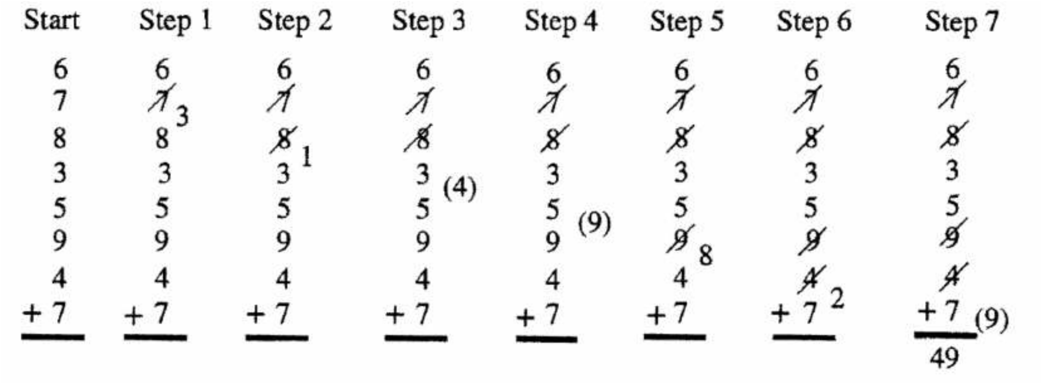
d.

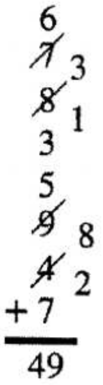
e.

The Scratch Method

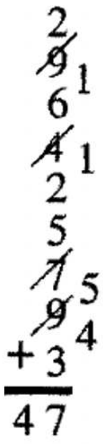
The **Scratch Method** is similar to the dot method. Any time you add and get a number over nine, you *scratch off* the last digit added. The scratch, like the dot, will represent ten. Just as in the dot method, you only keep the unit's digit in your memory or as an alternative, write the unit's digit to the right and just below the digit just scratched off. This can be done in the dot method also. I'll illustrate the steps involved by writing down the unit's digits this time. When the sum is less than ten, it is kept in your head. Parentheses will be used to denote what I am thinking in my head. The following problem is exactly the same one I did before when using the dot method. To start, see the eight digits to be added. Step 1: 6 + 7 = 13 so scratch off the 7 and write 3; Step 2: 3 + 8 = 11 so scratch off the 8 and write 1; Step 3: 1 + 3 = 4 so think 4; Step 4: 4 + 5 =9 so think 9; Step 5: 9 + 9 = 18 so scratch off the

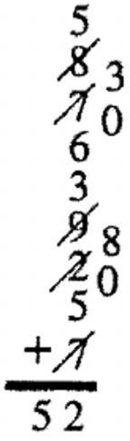
9 and write 8; Step 6: 8 + 4 = 12 so scratch off the 4 and write 2; Step 7: 2 + 7 = 9 so write a 9 under the bar since all digits have been added. Count the scratches (4), which stands for how many tens (40) and write in front of the 9. The answer is 49. Try the problem on your own.

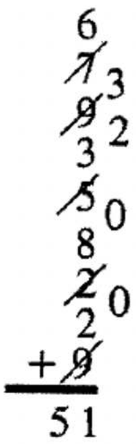




The above problem shows every single step, including what you think as you go along. If you were to look at the problem after someone did it, it would look like what you see on the left. More examples are shown on the right. Keeping the unit's digit in memory is faster than writing them down at each step. Try it both ways and ten adopt what's easiest and/or faster for you.







? Exercise 3

Use the scratch method to add the following columns of digits. Make up your own problem with at least 12 numbers for part e.

a. $\begin{array}{r}3\\9\\3\\6\\5\\8\\6\\\underline{+7}\end{array}$	b. $\begin{array}{r}5\\3\\9\\6\\7\\4\\2\\\underline{+7}\end{array}$	c. $\begin{array}{r}8\\8\\5\\7\\9\\7\\6\\\underline{+3}\end{array}$	d. $\begin{array}{r}7\\6\\4\\3\\8\\9\\7\\\underline{+9}\end{array}$	e.

The Dot and Scratch Methods can be used to add numerals with more than one digit. Add one column at a time, leaving a little space between the columns for dots if you use the Dot Method. To the right is an example of adding using the Scratch method with the usual method of adding from right to left and carrying. In this case, I am keeping each unit's digit in my head as opposed to writing each digit down after each scratch. The first column has four scratches, so the ***4*** is carried to column 2. *NOTE: Unless you practice several problems using these methods, it probably won't seem easier to add using one of these ways than whatever method you are used to using. But if you practice, you will probably become a whiz at addition and never go back to the old way!*

$$\begin{array}{rcccc}
 & & \mathbf{5} & \mathbf{4} & \\
 & & \cancel{6} & \cancel{9} & 5 \\
 & & 7 & \cancel{8} & \cancel{8} \\
 & & \cancel{7} & 7 & 4 \\
 & & \cancel{8} & \cancel{6} & \cancel{3} \\
 & & \cancel{9} & 5 & 4 \\
 & & \cancel{9} & \cancel{8} & \cancel{9} \\
+ & & 6 & \cancel{5} & \cancel{7} \\
\hline
 & 5 & 7 & 2 & 0
\end{array}$$

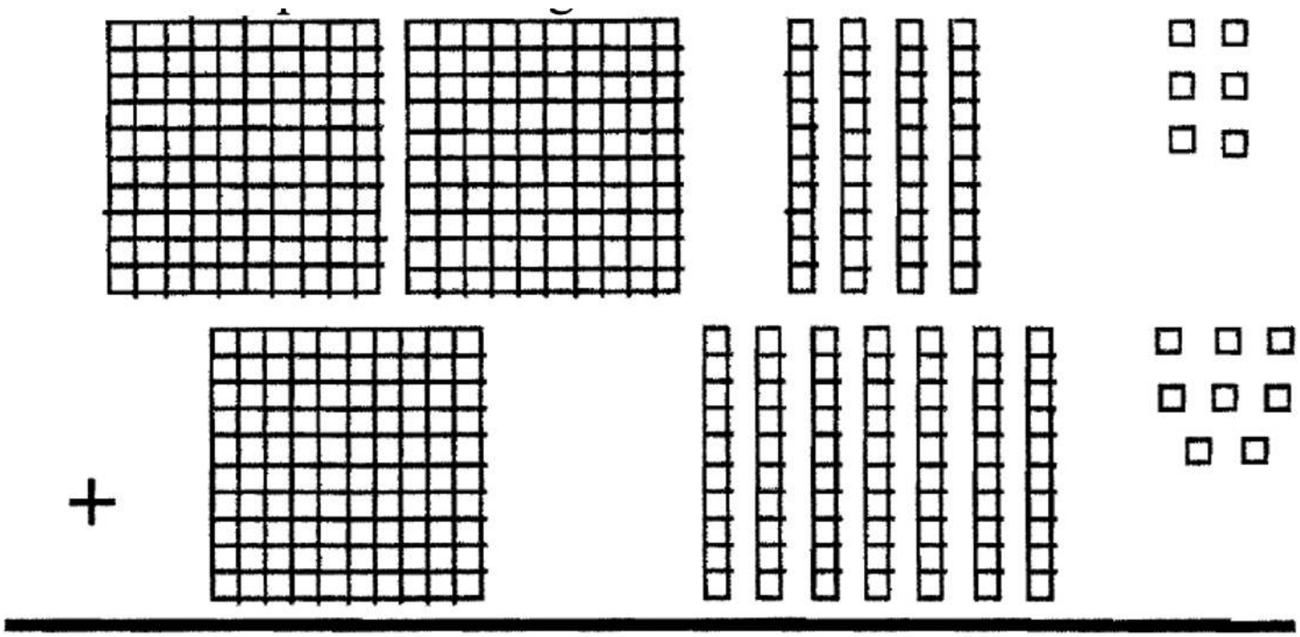
The most common addition algorithm is the *Right To Left Standard Addition Algorithm*, often referred to as the *Standard Addition Algorithm*. This is the one almost everyone grew up learning. You start on the right and add the digits. The unit's digit is put down under the line and the ten's digit is carried to the top of the next column to the left. Get out your Base Ten Blocks now to understand what is really going on.

Consider the addition problem 246 + 178. From our knowledge of place value, we know that 246 is 2 100 + 4 10 + 6 (or 200 + 40 + 6) and 178 is 1 100 + 7 10 + 8 (or 100 + 70 + 8). Using Base Ten blocks, 246 would be represented with 2 flats, 4 longs and 6 units whereas 178 would be represented with 1 flat, 7 longs and 8 units. So, the addition problem can be thought of in the following way:

Using Base Blocks

$$\begin{array}{rrr}
2 \text{ flat(s)} & +4 \text{ long(s)} & +6 \text{ unit(s)} \\
+1 \text{ flat(s)} & +7 \text{ long(s)} & +8 \text{ unit(s)} \\
\hline
\end{array}$$

This addition problem using blocks is shown below:



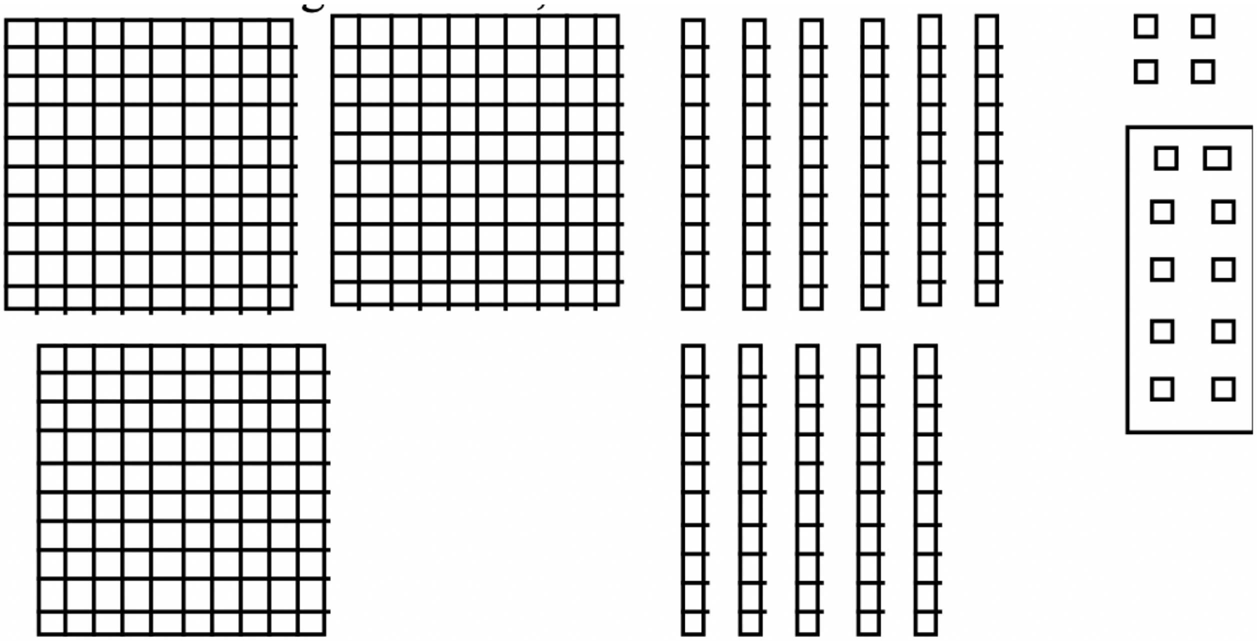
If we add, we get:

$$\begin{array}{rrr} 2 \text{ flat(s)} & +4 \text{ long(s)} & +6 \text{ unit(s)} \\ +1 \text{ flat(s)} & +7 \text{ long(s)} & +8 \text{ unit(s)} \\ \hline 3 \text{ flat(s)} & +11 \text{ long(s)} & +14 \text{ unit(s)} \end{array}$$

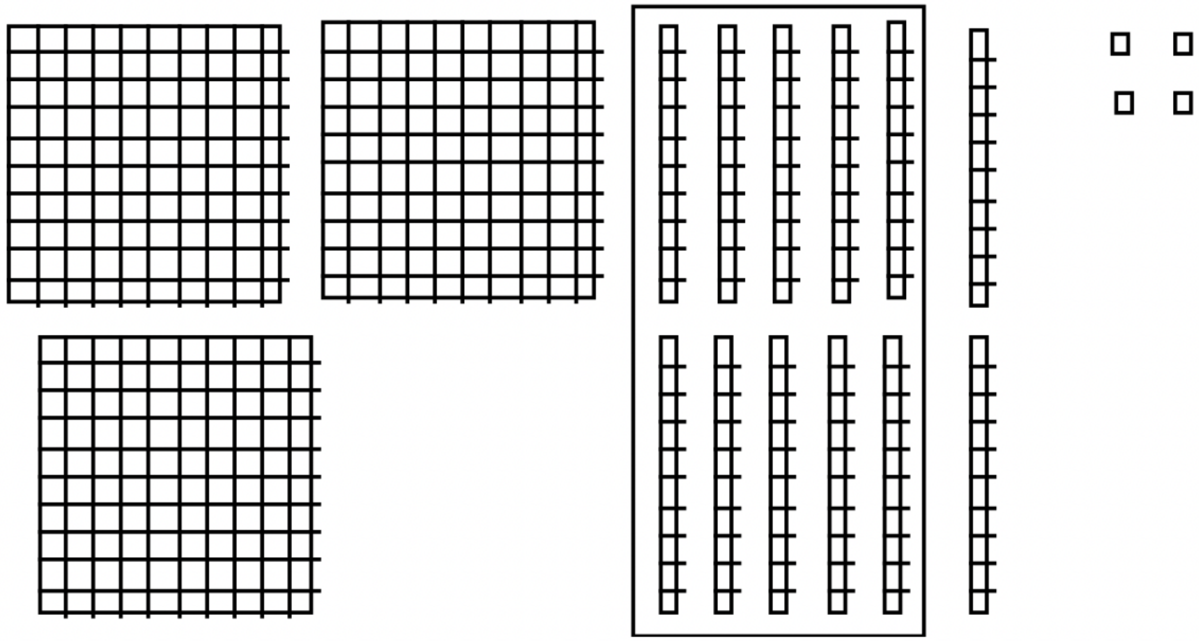
Trades can now be made. Ten of the 14 units can be exchanged for a long. That leaves us with 3 flats, 12 longs and 4 units. The sum and trades are illustrated with blocks on the next page.

After trading, sum = 3 flat(s) + 12 long(s) + 4 unit(s)

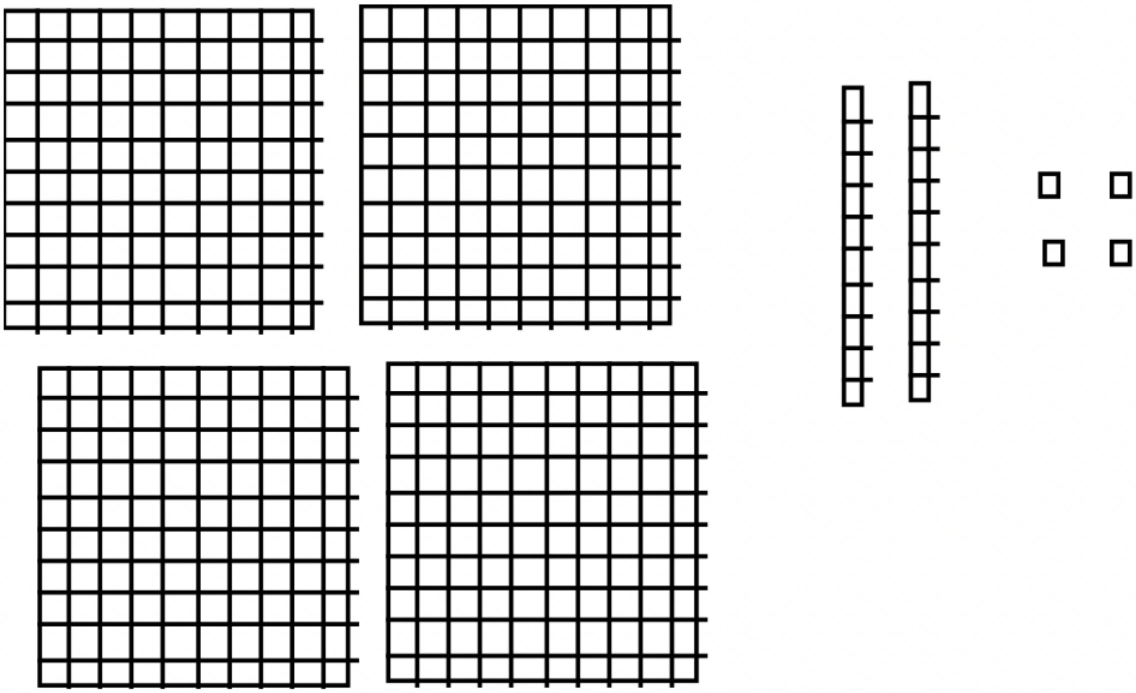
After combining the blocks, the sum is shown below.



The next step is to trade in ten units for a long. I've put a box around the units to be traded above. This yields 3 flats, 12 longs and 4 units as shown below.

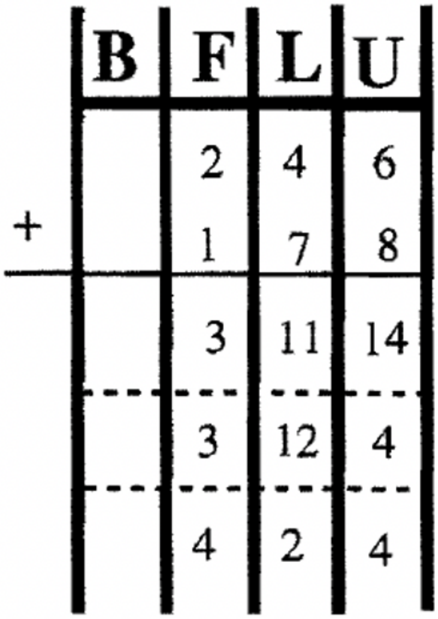


Now ten of the 12 longs can be exchanged for a flat. I've put a box around ten of the longs to be traded above. After the exchange, the sum is 4 flats, 2 longs and 4 units, which is 424. This is illustrated below.



After making all traded, the final sum = 4 flat(s) + 2 long(s) + 4 unit(s)

We're in Base Ten, so the answer is 424. Therefore, 246 + 178 = 424.



	B	F	L	U
		2	4	6
+		1	7	8
		3	11	14
		3	12	4
		4	2	4

In the previous example of adding 246 + 178 using the Base Blocks, ten longs could have been exchanged for a flat at the same time that ten units were exchanged for a long. I only did one trade at a time because in the Standard Addition Algorithm, we don't combine all the blocks (or place values) at once. First, the units are added together to get 14. The 4 is written down in the unit's column and the other 10 units are written as 1 long in the next column to signify that there is another long. Now, there are 1 + 4 + 7 longs to add up, which is 12. The 2 is written down in the longs column and the other 10 units are written as 1 flat in the next column to signify that there is another flat. Finally, there are 1 + 2 + 1 flats to add up, which is 4. So the sum is 4 flats, 2 longs and 4 units, which is 424. The exchanges being made is what carrying is all about. To the right is another way of keeping track of blocks, flats, longs and units. Notice the trades being made at each step. If the largest addend has x digits, I make x + 1 columns to allow for carrying. In this problem, there are three digits for each addend, so I made four columns to allow for a possible block being made.

Now, let's perform this basic algorithm in other bases. If you are adding in Base Seven, the trick here is to recall basic addition facts in Base Seven. You need to know your addition tables. Remember $5_{\text{seven}} + 6_{\text{seven}} = 11_{\text{ten}}$, which is 14_{seven} (which is one long and four units in Base Seven). Study the examples below. To help visualize the trades, the problems are first done using a chart with the sums *written in Base Ten* to keep track of the units, longs, flats, etc. In exchanging blocks, the conversion to the proper unit is accomplished. Then, the same problems are done without using the charts and using the traditional carrying method –here, you add in the base given as you go along. Try these five problems on your own before going on to the next exercise.

<table>
<tr>
<th>45_{seven}
$\underline{+36_{\text{seven}}}$</th>
<th>63_{eight}
$\underline{+45_{\text{eight}}}$</th>
<th>23_{four}
$\underline{+12_{\text{four}}}$</th>
<th>87_{twelve}
$\underline{+6T_{\text{twelve}}}$</th>
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$\underline{+111_{\text{two}}}$</th>
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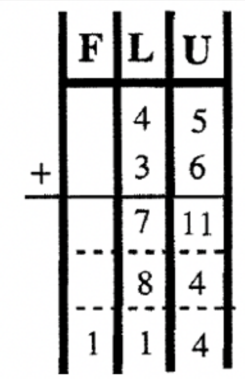
<td>
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<tr><td>+</td><td></td><td>3</td><td>6</td></tr>
<tr><td></td><td></td><td>7</td><td>11</td></tr>
<tr><td></td><td></td><td>8</td><td>4</td></tr>
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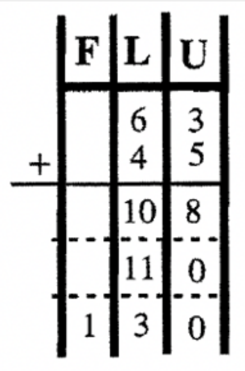
114_{seven}</td>
<td>
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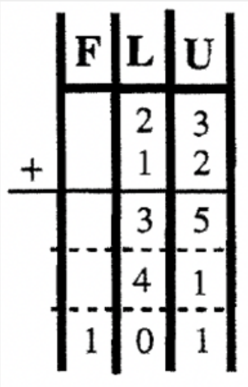
<td>
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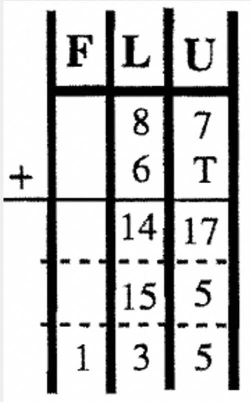
101_{four}</td>
<td>
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<tr><td></td><td></td><td>14</td><td>17</td></tr>
<tr><td></td><td></td><td>15</td><td>5</td></tr>
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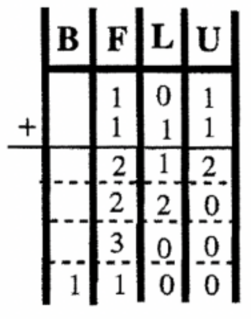
135_{twelve}</td>
<td>
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<tr><td>+</td><td></td><td>1</td><td>1</td><td>1</td></tr>
<tr><td></td><td></td><td>2</td><td>1</td><td>2</td></tr>
<tr><td></td><td></td><td>2</td><td>2</td><td>0</td></tr>
<tr><td></td><td></td><td>3</td><td>0</td><td>0</td></tr>
<tr><td></td><td>1</td><td>1</td><td>0</td><td>0</td></tr>
</table>
1100_{two}</td>
</tr>
</table>

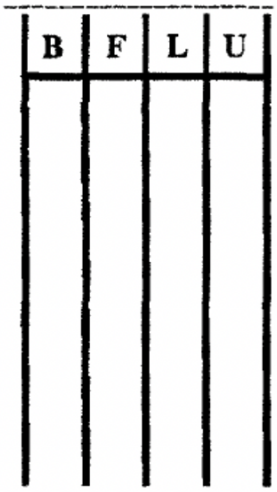
The same examples are worked below using the traditional carrying algorithm.

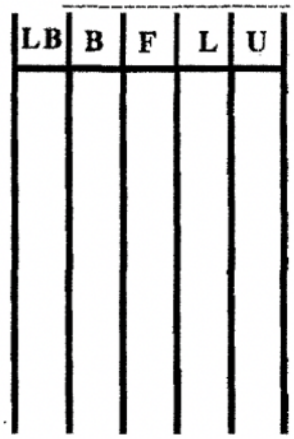
$\begin{array}{r} 1 \\ 45_{\text{seven}} \\ \underline{+36_{\text{seven}}} \\ 114_{\text{seven}} \end{array}$	$\begin{array}{r} 1\ \ 1 \\ 63_{\text{eight}} \\ \underline{+45_{\text{eight}}} \\ 130_{\text{eight}} \end{array}$	$\begin{array}{r} 1 \\ 23_{\text{four}} \\ \underline{+12_{\text{four}}} \\ 101_{\text{four}} \end{array}$	$\begin{array}{r} 11 \\ 87_{\text{twelve}} \\ \underline{+6T_{\text{twelve}}} \\ 135_{\text{twelve}} \end{array}$	$\begin{array}{r} 101_{\text{two}} \\ \underline{+111_{\text{two}}} \\ 1100_{\text{two}} \end{array}$

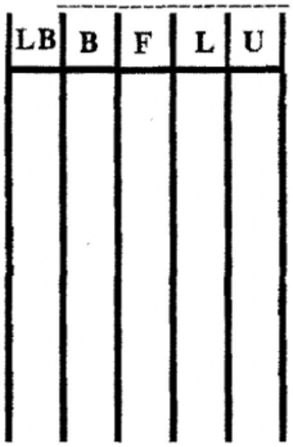
Exercise 4

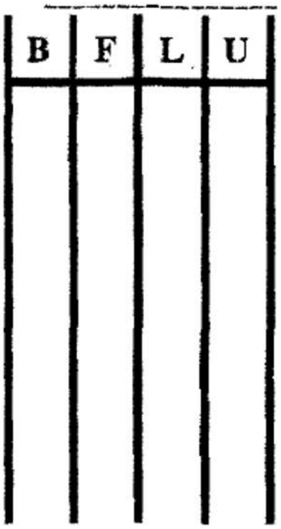
Add the following. Do each problem using charts as shown in the previous examples. Use your Base Blocks to visualize the problem further.

a. $\begin{array}{rrrl} 7 & E & 1 & _{\text{thirteen}} \\ +5 & 8 & 4 & _{\text{thirteen}} \end{array}$	b. $\begin{array}{rrrrl} 1 & 1 & 0 & 1 & _{\text{two}} \\ +1 & 0 & 0 & 1 & _{\text{two}} \end{array}$	c. $\begin{array}{rrrrl} 3 & 2 & 0 & 4 & _{\text{five}} \\ +4 & 0 & 1 & 3 & _{\text{five}} \end{array}$	d. $\begin{array}{rrrl} 6 & 1 & 2 & _{\text{nine}} \\ +4 & 5 & 6 & _{\text{nine}} \end{array}$

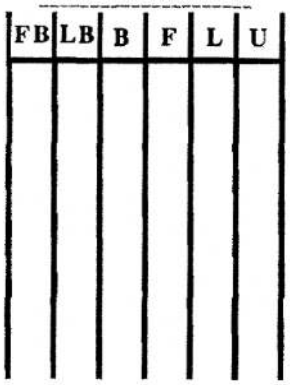


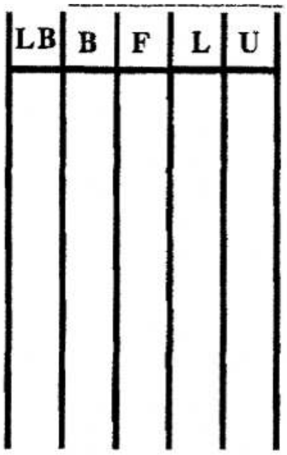


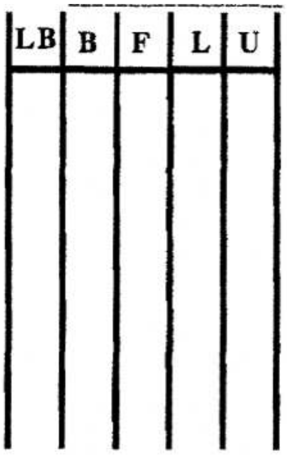


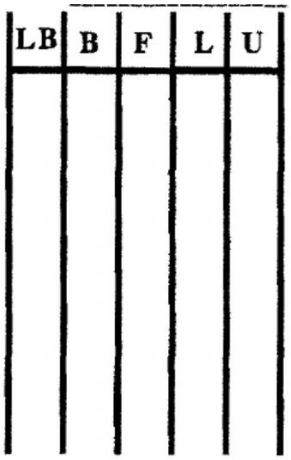


e. $\begin{array}{rrrrrl} 1 & 0 & 1 & 1 & 1 & _{\text{two}} \\ +1 & 1 & 1 & 0 & 1 & _{\text{two}} \end{array}$	f. $\begin{array}{rrrrl} 2 & 2 & 1 & 2 & _{\text{three}} \\ + & 2 & 2 & 2 & _{\text{three}} \end{array}$	g. $\begin{array}{rrrrl} 4 & 6 & 1 & 3 & _{\text{twelve}} \\ +5 & T & 3 & 9 & _{\text{twelve}} \end{array}$	h. $\begin{array}{rrrrl} 4 & 3 & 4 & 3 & _{\text{seven}} \\ +4 & 1 & 4 & 5 & _{\text{seven}} \end{array}$









Exercise 5

Eventually, you should be able to do the above problems without the use of charts or manipulatives. *You can always **think** in terms of blocks as you work them.* Add the following using the traditional carrying algorithm. Practice until you feel confident and are proficient at adding without using charts or manipulatives.

a. $\begin{array}{rrr} 7 & E & 1_{\text{thirteen}} \\ +5 & 8 & 4_{\text{thirteen}} \end{array}$	b. $\begin{array}{rrrr} 1 & 1 & 0 & 1_{\text{two}} \\ +1 & 0 & 0 & 1_{\text{two}} \\ \hline \end{array}$	c. $\begin{array}{rrrr} 3 & 2 & 0 & 4_{\text{five}} \\ +4 & 0 & 1 & 3_{\text{five}} \\ \hline \end{array}$	d. $\begin{array}{rrr} 6 & 1 & 2_{\text{nine}} \\ +4 & 5 & 6_{\text{nine}} \end{array}$
e. $\begin{array}{rrrrr} 1 & 0 & 1 & 1 & 1_{\text{two}} \\ +1 & 1 & 1 & 0 & 1_{\text{two}} \end{array}$	f. $\begin{array}{rrrr} 2 & 2 & 1 & 2_{\text{three}} \\ + & 2 & 2 & 2_{\text{three}} \\ \hline \end{array}$	g. $\begin{array}{rrrr} 4 & 6 & 1 & 3_{\text{twelve}} \\ +5 & T & 3 & 9_{\text{twelve}} \\ \hline \end{array}$	h. $\begin{array}{rrrr} 4 & 3 & 4 & 3_{\text{seven}} \\ +4 & 1 & 4 & 5_{\text{seven}} \\ \hline \end{array}$

Another way to add is by using expanded notation. Our first example, 246 + 178, can be written as (200 + 40 + 6) + (100 + 70 + 8). Using the commutative and associative properties, this sum can be written as (200 + 100) + (40 + 70) + (6 + 8) or 300 + 110 + 14 = 300 + (100 + 10) + (10 + 4) = (300 + 100) + (10 + 10) + 4 = 400 + 20 + 4 = 424. When written out like this, it is perhaps more clear what is really being added as opposed to doing it by rote without thinking about the place value of each digit.

? Exercise 6

6. Add by using expanded notation. Show all steps.

a. 43 + 47

b. 88 + 54

? Exercise 7

7. Suppose two different students add two numbers together as shown in the two problems to the right. Both have very similar methods. Explain what each student is doing and why it makes sense. Then, make up two more problems and add using one of these methods.

$\begin{array}{r} 859 \\ +467 \\ \hline 16 \\ 110 \\ +1200 \\ \hline 1326 \end{array}$	$\begin{array}{r} 859 \\ +467 \\ \hline 1200 \\ 110 \\ +16 \\ \hline 1326 \end{array}$

Below is the work of three different students doing the same addition problem. Figure out what each student is doing to get the answer before reading on.

$\begin{array}{r} 859 \\ +467 \\ \hline 1200 \\ 1310 \\ 1326 \end{array}$	$\begin{array}{r} 859 \\ +467 \\ \hline 1259 \\ 1319 \\ 1326 \end{array}$	$\begin{array}{r} 859 \\ +467 \\ \hline 866 \\ 926 \\ 1326 \end{array}$

The first student starts by adding the hundreds together (800 + 400 = 1200). Next, the tens are added together (50 + 60 = 110) and that answer is added on (1200 + 110 = 1310). Finally, the ones are added (9 + 7 = 16) and that is added on to get the answer (1310 + 16 = 1326). The second student takes the first number and adds on the hundreds of the second number (859 + 400 = 1259). Next, the tens of the second number is added on (1259 + 60 = 1319). Finally, the ones are added on (1319 + 7 = 1326). The third student starts with the first number and adds on the ones of the second digit first (859 + 7 = 866). Second, the tens digit of the second number is added on (866 + 60 = 926). Finally, the hundreds is added on (926 + 400 = 1326). Someone might be more inclined to use one of these methods if they are adding in their head.

? Exercise 8

Make up your own problem and solve using the three methods explained above. Explain the method and steps used to arrive at the correct answer.

Another method used for adding is the *Left to Right Addition Algorithm*. Some of the last few examples were actually employing Left to Right algorithms. In expanded notation, you could add left to right or right to left. Many people find the Left to Right

Addition Algorithm easier because there is no carrying. There are a couple of ways to perform this method. Start by adding the leftmost column of digits first. As you move to the next column to the right, you add the digits. If the sum is more than 9, write the unit's digit under that column and underline the digit already put down immediately to the left. *This is similar to carrying but you don't need to move up to the next column –it gets added on to the answer* ***later***. Continue until you've added the unit's digits. Then go back and add one to all the digits that are underlined. Can you understand why? The following example takes you step by step using this algorithm.

Step 1: Starting in the leftmost column, add the digits (3 and 4). Since 3 + 4 = 7, write 7 under that leftmost column.	$\begin{array}{rrrr} 4 & 6 & 3 & 7 \\ +3 & 8 & 2 & 8 \\ \hline 7 & & & \end{array}$
Step 2: Add the digits in the next column to the right. Since 6 + 8 = 14, write the 4 under that column and underline the digit to the left ($\underline{7}$).	$\begin{array}{rrrr} 4 & 6 & 3 & 7 \\ +3 & 8 & 2 & 8 \\ \hline \underline{7} & 4 & & \end{array}$
Step 3: Add the digits in the next column to the right. Since 3 + 2 is 5, write a 5 under that column.	$\begin{array}{rrrr} 4 & 6 & 3 & 7 \\ +3 & 8 & 2 & 8 \\ \hline \underline{7} & 4 & 5 & \end{array}$
Step 4: Add the digits in the next column to the right. Since 7 + 8 = 15, write the 5 under that column and underline the digit to the left ($\underline{5}$).	$\begin{array}{rrrr} 4 & 6 & 3 & 7 \\ +3 & 8 & 2 & 8 \\ \hline \underline{7} & 4 & \underline{5} & 5 \end{array}$
Step 5: The last step is to rewrite the answer to the problem by adding a 1 to any digit that is underlined. Thus the answer is 8465.	$\begin{array}{rrrr} 4 & 6 & 3 & 7 \\ +3 & 8 & 2 & 8 \\ \hline \underline{7} & 4 & \underline{5} & 5 \\ 8 & 4 & 6 & 5 \end{array}$

Note: Step 5 is what the problem looks like when it's done, as shown in the examples below. Study and then try the following examples on your own before going on to the next exercise.

$\begin{array}{r} 6483 \\ +5734 \\ \hline 1\underline{1}\underline{1}17 \\ 12217 \end{array}$	$\begin{array}{r} 5417 \\ +3971 \\ \hline \underline{8}388 \\ 9388 \end{array}$	$\begin{array}{r} 63925 \\ +41738 \\ \hline 10\underline{4}6\underline{5}3 \\ 105663 \end{array}$	$\begin{array}{r} 787878 \\ +65656 \\ \hline \underline{74242}4 \\ 853534 \end{array}$

❓ Exercise 9

a. $\begin{array}{r} 5386 \\ +6723 \\ \hline \end{array}$	b. $\begin{array}{r} 65381 \\ +46082 \\ \hline \end{array}$	c. $\begin{array}{r} 6789 \\ +9879 \\ \hline \end{array}$	d. $\begin{array}{r} 70426 \\ +57908 \\ \hline \end{array}$

In the Left to Right Algorithm, anytime a 9 is underlined, **you must**_*continue the underline* to include the digit to the left of the nine (if there is a digit to its left). If the digit to the left is a 9, continue the underline to include the digit to its left. Keep underlining until you underline a digit that is not a nine. Then, when you go back, add 1 to the underlined number, (which will now be more than one digit). Study and then practice the following four examples on your own before attempting the next exercise.

$\begin{array}{r} 4672 \\ +5826 \\ \hline \underline{9}498 \\ \\ 10498 \end{array}$	$\begin{array}{r} 8468 \\ +5538 \\ \hline 1\underline{399}6 \\ \\ 14006 \end{array}$	$\begin{array}{r} 5798 \\ +4605 \\ \hline \underline{939}3 \\ \\ 10403 \end{array}$	$\begin{array}{r} 35776 \\ +64525 \\ \hline \underline{9929}1 \\ \\ 100301 \end{array}$

Using this technique, the first example adds 1 to 9 to get 10. In the second example, 1 is added to 399 to get 400. In the third example, 1 is added to 9 to get 10 and 1 is added to 39 to get 40. In the fourth example, 1 is added to 99 to get 100 and 1 is added to 29 to get 30.

Exercise 10

Use the Left to Right Algorithm to add the following numbers using the above technique.

a. $\begin{array}{r}7658\\ \underline{+1147}\end{array}$	b. $\begin{array}{r}4804\\ \underline{+5659}\end{array}$	c. $\begin{array}{r}5679\\ \underline{+7350}\end{array}$	d. $\begin{array}{r}98765\\ \underline{+7238}\end{array}$

On the next page are some addition problems in other bases. The Left to Right Addition Algorithm is being used. *Be careful to pay attention to the base*. For example, in Base Six, any time you add and get a number higher than 5 (which is the highest digit in Base Six), you only write down the unit's digit under that column and underline the digit to its left. In Base Three, any time you add and get a number higher than 2 (which is the highest digit in Base Three), you only write down the unit's digit under that column and underline the digit to its left. In Base Twelve, any time you add and get a number higher than E (which is the highest digit in Base Twelve), you only write down the unit's digit under that column and underline the digit to its left. Make sure you understand and can do these next examples successfully on your own before going on to the next paragraph and examples.

$\begin{array}{r}423_{\text{six}}\\ \underline{+503}_{\text{six}}\\ 13\underline{2}0_{\text{six}}\\ 1330_{\text{six}}\end{array}$	$\begin{array}{r}839_{\text{twelve}}\\ \underline{+E58}_{\text{twelve}}\\ 17\underline{8}5_{\text{twelve}}\\ 1795_{\text{twelve}}\end{array}$	$\begin{array}{r}580_{\text{nine}}\\ \underline{+723}_{\text{nine}}\\ 1\underline{3}13_{\text{nine}}\\ 1413_{\text{nine}}\end{array}$	$\begin{array}{r}1011_{\text{two}}\\ \underline{+1011}_{\text{two}}\\ 10\underline{00}0_{\text{two}}\\ 10110_{\text{two}}\end{array}$	$\begin{array}{r}2012_{\text{three}}\\ \underline{+1112}_{\text{three}}\\ 10\underline{12}1_{\text{three}}\\ 10201_{\text{three}}\end{array}$

Below are some examples where you have to do some continuous underlining similar to examples shown previously in Base Ten. Pay close attention to the base. If you *underline* a 5 in Base Six, you must continue underlining the digit to its left until you underline a digit less than 5! If you *underline* a 2 in Base Three, you must continue underlining the digit to its left until you underline a digit less than 2! If you *underline* an E in Base Twelve, you must continue underlining the digit to its left until you underline a digit less than E!. Study and practice the five examples below on your own before attempting the next exercise.

$\begin{array}{r}324_{\text{six}}\\ \underline{+132}_{\text{six}}\\ \underline{45}0_{\text{six}}\\ 500_{\text{six}}\end{array}$	$\begin{array}{r}367_{\text{twelve}}\\ \underline{+35E}_{\text{twelve}}\\ \underline{6E}6_{\text{twelve}}\\ 706_{\text{twelve}}\end{array}$	$\begin{array}{r}528_{\text{nine}}\\ \underline{+367}_{\text{nine}}\\ \underline{88}6_{\text{nine}}\\ 1006_{\text{nine}}\end{array}$	$\begin{array}{r}1011_{\text{two}}\\ \underline{+1110}_{\text{two}}\\ 1\underline{01}01_{\text{two}}\\ 11001_{\text{two}}\end{array}$	$\begin{array}{r}22021_{\text{three}}\\ \underline{+1002}_{\text{three}}\\ \underline{2}0\underline{02}0_{\text{three}}\\ 100100_{\text{three}}\end{array}$

Exercise 11

Use the Left to Right Addition Algorithm to add the following. Pay careful attention to the Base!!!

a. $\begin{array}{r}514_{\text{six}}\\ \underline{+342_{\text{six}}}\end{array}$	b. $\begin{array}{r}835_{\text{eleven}}\\ \underline{+658_{\text{eleven}}}\end{array}$	c. $\begin{array}{r}473_{\text{eight}}\\ \underline{+473_{\text{eight}}}\end{array}$	d. $\begin{array}{r}1111_{\text{two}}\\ \underline{+1010_{\text{two}}}\end{array}$	a. $\begin{array}{r}2034_{\text{five}}\\ \underline{+1112_{\text{five}}}\end{array}$

f. $\begin{array}{r}7E1_{\text{thirteen}}\\ \underline{+584_{\text{thirteen}}}\end{array}$	f. $\begin{array}{r}1101_{\text{two}}\\ \underline{+1001_{\text{two}}}\end{array}$	f. $\begin{array}{r}3204_{\text{five}}\\ \underline{+4013_{\text{five}}}\end{array}$	f. $\begin{array}{r}612_{\text{nine}}\\ \underline{+456_{\text{nine}}}\end{array}$
j. $\begin{array}{r}10111_{\text{two}}\\ \underline{+11101_{\text{two}}}\end{array}$	k. $\begin{array}{r}2212_{\text{three}}\\ \underline{+222_{\text{three}}}\end{array}$	l. $\begin{array}{r}4613_{\text{twelve}}\\ \underline{+5T39_{\text{twelve}}}\end{array}$	m. $\begin{array}{r}4343_{\text{seven}}\\ \underline{+4145_{\text{seven}}}\end{array}$

Exercise 12

List all bases between two and thirteen in which each of the following addition problems are valid.

a. $\begin{array}{r}403\\ \underline{+542}\\ 1245\end{array}$	b. $\begin{array}{r}729\\ \underline{+526}\\ W52\end{array}$	c. $\begin{array}{r}E83\\ \underline{+1T3}\\ 1166\end{array}$	d. $\begin{array}{r}1011\\ \underline{+1111}\\ 2122\end{array}$	e. $\begin{array}{r}2012\\ \underline{+1011}\\ 3023\end{array}$

Exercise 13

Someone started to do each of these addition problems using the Standard Right to Left Algorithm. Figure out which base each addition problem is in and finish the computation.

a. $\begin{array}{r}64\\ \underline{+46}\\ 2\end{array}$	b. $\begin{array}{r}53\\ \underline{+28}\\ 2\end{array}$	c. $\begin{array}{r}21\\ \underline{+12}\\ 0\end{array}$	d. $\begin{array}{r}57\\ \underline{+66}\\ 3\end{array}$

Exercise 14

Someone was adding 47 + 68 in her head and said out loud, "47 + 70 = 117 and two less is 115." Explain her reasoning.

Exercise 15

Another person was adding 47 + 68 in his head and said out loud "40 + 60 is 100 and 8 + 7 is 15, so the answer is 115." Explain his reasoning.

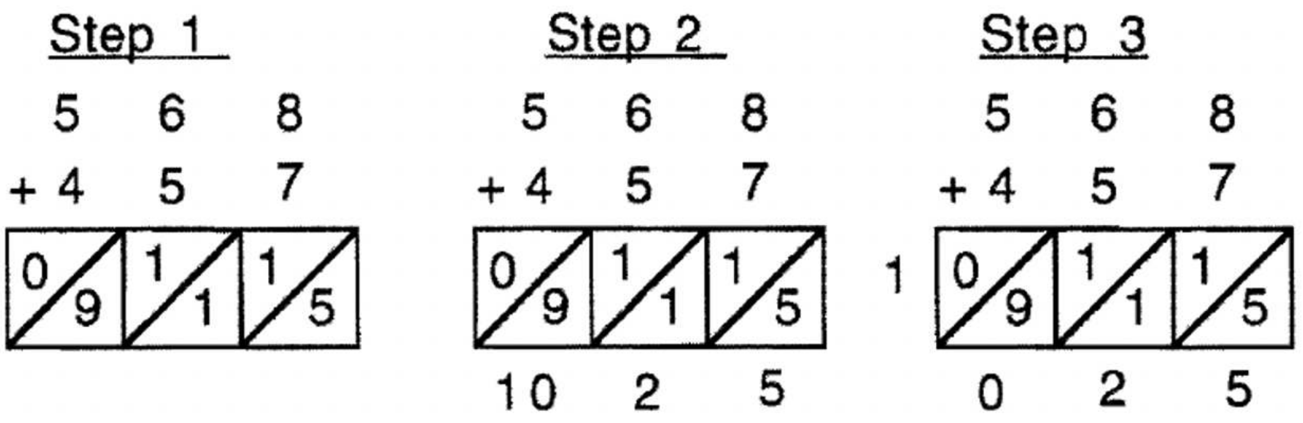
Exercise 16

The methods in exercises 14 and 15 can be called Break-Apart Methods. You break apart one or both of the addends using the place value of the number. Mentally compute 97 + 88 and then explain the method you used.

Exercise 17

Are there any addition tricks you use? Extra credit for sharing on the Forum

Below is another addition algorithm, called the Lattice Method for Addition, used for adding two numbers together. First, add down the columns, then down the diagonals. The addition problem is 568 + 457 and the answer is 1,025. See if you can understand how to do it and understand why it works. We'll use the lattice method again when we do multiplication.



Exercise 18

Use the lattice method to compute 456 + 659. Show your work.

Exercise 19

Use the lattice method to compute the following:

a. $12_{\text{six}} + 25_{\text{six}}$	b. $T4E_{\text{thirteen}} + 190_{\text{thirteen}}$

Learning to estimate is a very useful skill. The idea is to convert the actual numbers in the problem to simpler numbers that are easy to compute mentally. In exercise 14, 47 + 68 is close to adding 50 + 70, which is 120. 120 is pretty close –it's within 5% of the exact answer of 115. Even if you need to know the exact answer, if you do a quick estimate, you can usually tell if you're in the ball park. Sometimes estimating is all that is really necessary. For instance, if you are shopping for groceries and have a limited amount of cash on hand to pay for them, you might want to mentally add up what you spend as you go along. The quickest way to estimate is to round. And if you want to make sure you don't go over your allotted amount, you can always just round up. Let's say you had ten items in the cart for the following amounts:

$6.75	$3.23	$1.25	$7.18	$2.98	$1.89	$1.50	$2.45	$3.69	$.76

There are many ways you could choose to round and add –you could round to the nearest dollar, or up to the nearest dollar (to make sure you don't go over) or maybe to the nearest 50 cents. I am going to assume you know how to round numbers already. And you should be able to add in your head using your fingers and the Dot Method. The following are three examples of how you might get an estimate of what the grocery bill will be.

Rounding to the nearest dollar: 7 + 3 + 1 + 7 + 3 + 2 + 2 + 2 + 4 + 1 = 32
Rounding up to the nearest dollar: 7 + 4 + 2 + 8 + 3 + 2 + 2 + 3 + 4 + 1 = 36
Rounding to the nearest 50 cents: 7 + 3 + 1.5 + 7 + 3 + 2 + 1.5 + 2.5 + 3.5 + 1 = 32

The actual sum is $31.68 which is extremely close to our rough estimate of $32 which we got rounding both to the nearest dollar and to the nearest 50 cents. If you are working with higher priced items, you might round to the nearest ten or hundred dollars, etc.

? Exercise 20

Mentally estimate the cost of the grocery bill containing the following priced items. Explain how you did it. Then, compare your answer to the actual sum.

$4.67	$8.21	$9.53	$5.33	$2.79	$1.89	$2.14	$4.65	$5.14	$.83

? Exercise 21

Pretend you are going shopping and you buy ten items where each item is less than $10. List the actual cost of each item (make these up) and estimate the total. Then, compute the actual cost of the ten items.

? Exercise 22

When do you think you might get too high or low of an estimate? ***Give an example.***

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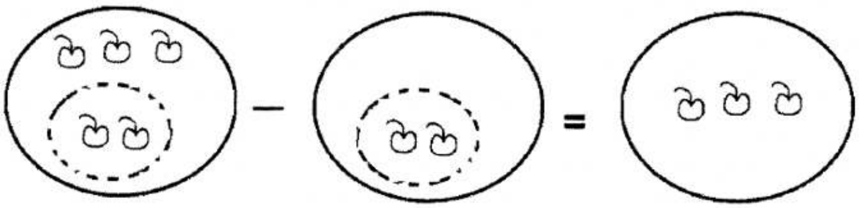
3.4: Subtraction

You will need: A ***Calculator, Base Blocks*** *(Material Cards 4-15)* ***C-Strips*** *(Material Cards 16A-16L)*

Exercise 1

How would you use manipulative to explain how to do subtraction to a young child?

There are two distinct approaches to subtraction. The one most of us are familiar with is the *Take-Away Method*. A typical way someone might introduce the idea of subtraction is by saying "If I put five apples in the bowl and then *take away* two of the apples, how many are left in the bowl?" The illustration below shows this as a subtraction problem where after two apples are removed from the bowl, there are three apples remaining.



The above problem illustrates the Take-Away Approach to Subtraction.

Subtraction Vocabulary: For **x – y = z**, **x** is called the *minuend*, y is called the *subtrahend*, and z (the answer) is called the *difference*.

Exercise 2

For the subtraction problem 7 – 3, State the

Definition: Subtraction of Whole Numbers using Set Theory

If B is a subset of A, then n(A) – n(B) = n(A – B)

Exercise 3

In your own words, explain how to use the definition of subtraction. What are the steps involved?

Examples of how to do subtraction using the Set Theory Definition:

Example 1

Use the set theory definition of subtraction to show that 5 – 2 = 3.

Let A = {v, w, x, y, z} and B = {w, z}. Since n(A) = 5, n(B) = 2 and B ⊆ A,

$$\begin{aligned} 5-2 &= n(A)-n(B) && \text{by substituting } n(A) \text{ for 5 and } n(B) \text{ for 2} \\ &= n(A-B) && \text{by the set theory definition of subtraction} \\ &= n(\{v,x,y\}) && \text{by computing } A-B \\ &= 3 && \text{by counting the elements in } A-B \end{aligned}$$

Therefore, 5 – 2 = 3.

Exercise 2

Use the set theory definition of subtraction to show that 6 – 1 = 5.

Let A {1, 2, 3, 4, 5, 6} and B = {4}. Since n(A) = 6, n(B) = 1 and B ⊆ A,

$$\begin{aligned} 6–1 &= n(A)–n(B) && \text{by substituting } n(A) \text{ for 6 and } n(B) \text{ for 1} \\ &= n(A–B) && \text{by the set theory definition of subtraction} \\ &= n(\{1,2,\mathbf{3},5,6\}) && \text{by computing } A–B \\ &= 5 && \text{by counting the elements in } A–B \end{aligned}$$

Therefore, 6– 1 = 5.

Exercise 3

Use the set theory definition of subtraction to show that 3 – 0 = 3.

Let A = {x,y, z} and B= {}. Since n(A)=3, n(B)=0 and B ⊆ A,

$$\begin{aligned} 3–0 &= n(A)–n(B) && \text{by substituting } n(A) \text{ for 3 and } n(B) \text{ for 0} \\ &= n(A–B) && \text{by the set theory definition of subtraction} \\ &= n(\{x,y,z\}) && \text{by computing } A–B \\ &= 3 && \text{by counting the elements in } A–B \end{aligned}$$

Therefore, 3 – 0 = 3.

Exercise 4

For each subtraction problem below, provide two sets that allow you to use the definition of subtraction to find the answer. Then, compute the answer using this definition.

a. 7 - 3
b. 6 - 0
c. 4 - 4

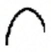
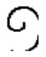
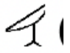
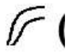
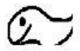
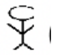
Parts b and c of Exercise 4 illustrate two familiar properties of subtraction.

The first property states that for any whole number m, $m–0 = m$. Using our knowledge of Set Theory, choose the first set A to have m elements and then choose a second set B that has zero elements – there is only one set you can choose and that is the null or empty set, { }. Then, A – { } = A. Therefore, it is a fact that m – 0 = 0.

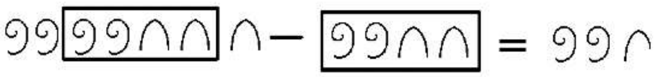
The second property states that for any whole number m, m – m = 0. This must be true because for any set A, whether it has m elements or any other number of elements, we know from Set Theory that A – A = { }.

We will use the Take-Away approach to perform subtraction problems in Egyptian now.

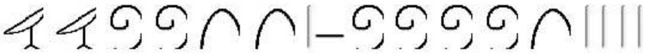
You are reminded of the symbols and their Hindu-Arabic equivalents below:

| (1)	 (10)	 (100)	 (1,000)	 (10,000)	 (100,000)	 (1,000,000)

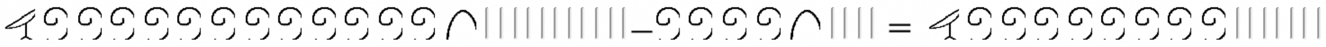
To use the Take-Away approach, we want to see the subtrahend as a subset of the minuend and then remove the subtrahend from the minuend. The symbols that remain in the minuend is the difference. In this first example, notice how the following subtraction is performed. In this case, you can see the subtrahend as a subset of the minuend. A box is put around what is to be taken away and the final answer is clear.



Sometimes, exchanges have to be made in the minuend before the subtraction can be done. For instance, consider the following subtraction:

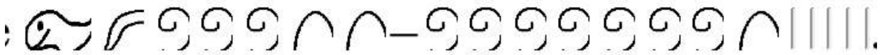


The first step would be to make some exchanges in the minuend. One lotus flower must be exchanged for ten scrolls and one heel bone must be exchanged for ten staffs. After doing that, the subtraction can be performed as shown below:

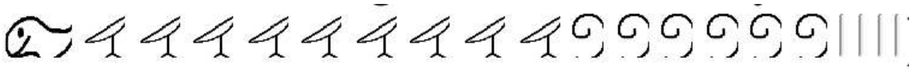


In the above example, put a box around the subset that is being removed from the minuend.

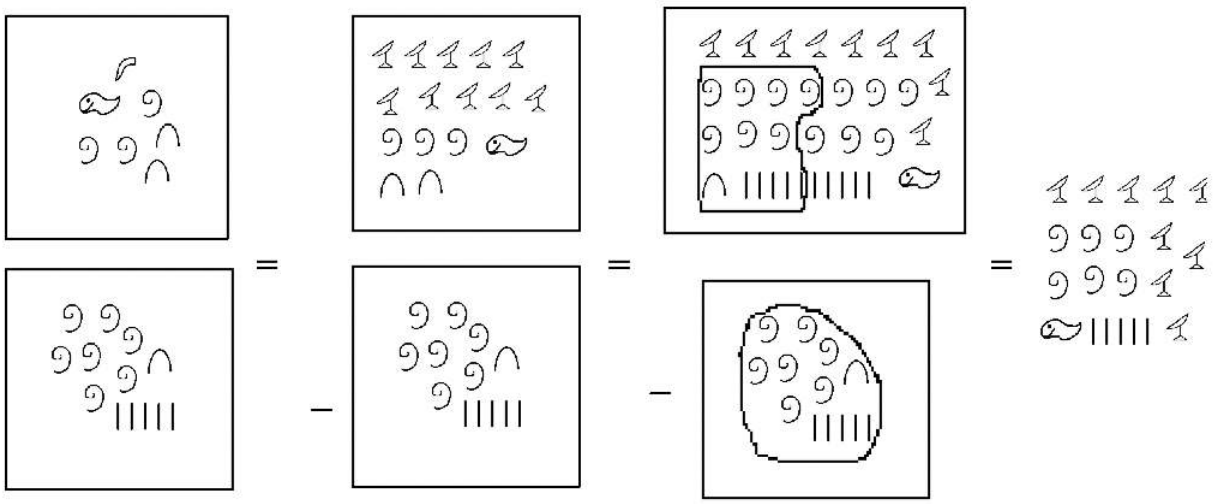
The next subtraction will be



. This time the subtraction will be shown as a vertical problem with the minuend and subtrahend shown enclosed in a box. In order to do the subtraction, first the pointed finger in the minuend must be exchanged for ten lotus flowers. Then, one lotus flower must be exchanged for ten scrolls and one heel bone must be exchanged for ten strokes. Finally, the subset (the subtrahend) is taken away from the minuend. I've traced out what is being removed. The symbols remaining in the minuend is the answer (

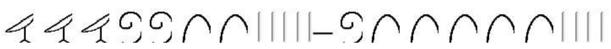
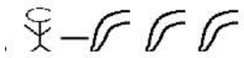


) as shown in the illustration below.



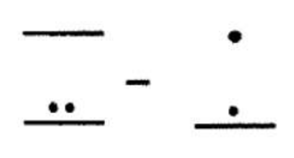
? Exercise 5

Perform the following subtraction problems in Egyptian, showing all steps. You can model the problem however you like as long as the steps are clear.

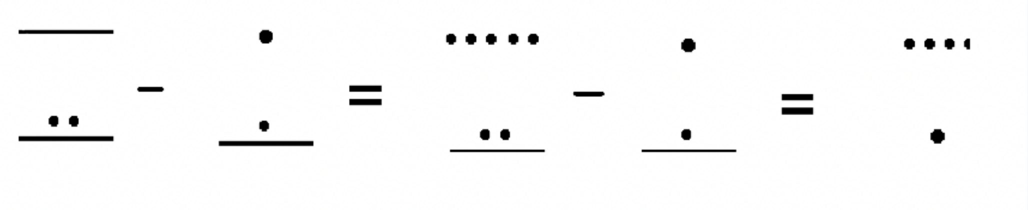
a.	b.
	

Let's use the take-away approach to subtract Mayan numerals. It's similar to addition in that you remove a subset *at each level* – exchanges or trades might have to be done before being able to take away at any given level. Pay close attention to which level you are on when making exchanges. A dot at one level can be traded in for a group of 20 at the next level down except from level three to level two – *one dot at the third level is replaced by a group of 18 at the second level*. Study the following examples. See if you can figure out which exchanges are being made. I will indicate exchanges which are being made from one level to the other with a downward arrow.

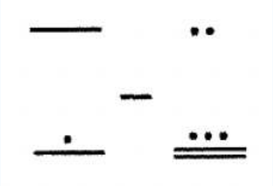
Example 1



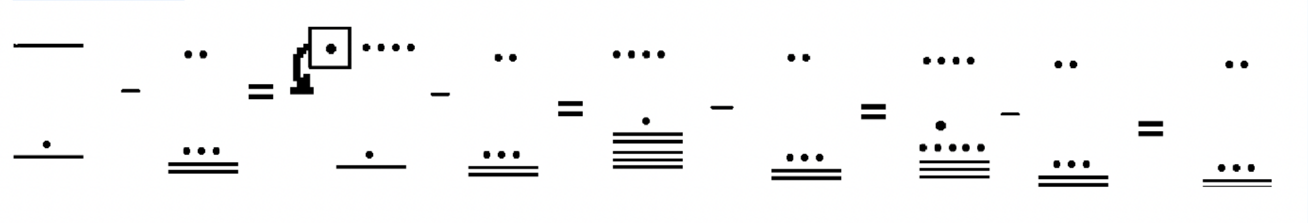
Solution



Example 2



Solution



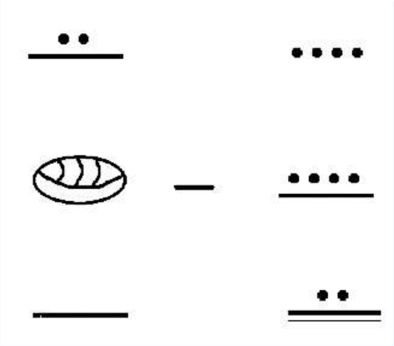
In Examples 1 and 2, note that as exchanges are being made in the minuend (so that you can use the take-away approach), the subtrahend just keeps getting repeated. Alternately, you can first show the exchanges being made for the minuend, make a break (like a dotted line as shown below) and then write the subtraction problem out and subtract. Here is another way, you might show the steps for Example 2.

Exercise 6

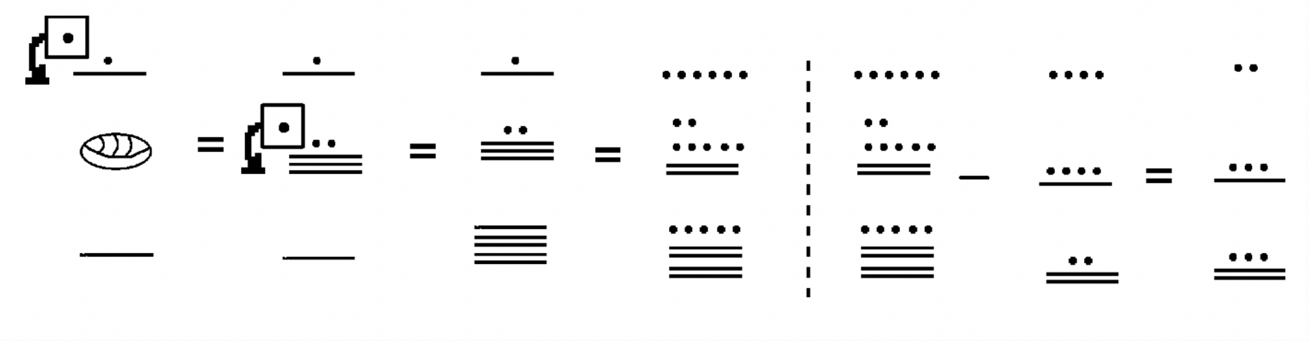
Perform the following subtraction problem in Mayan and show all of the steps.

Here are some more examples.

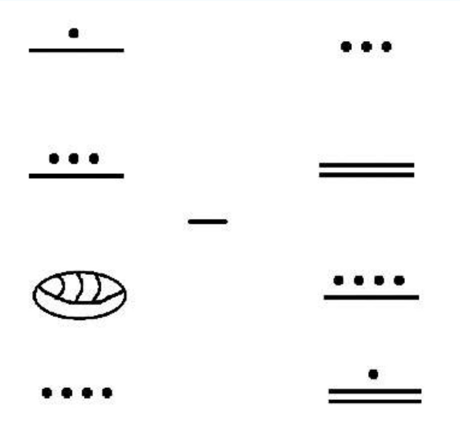
Example 3



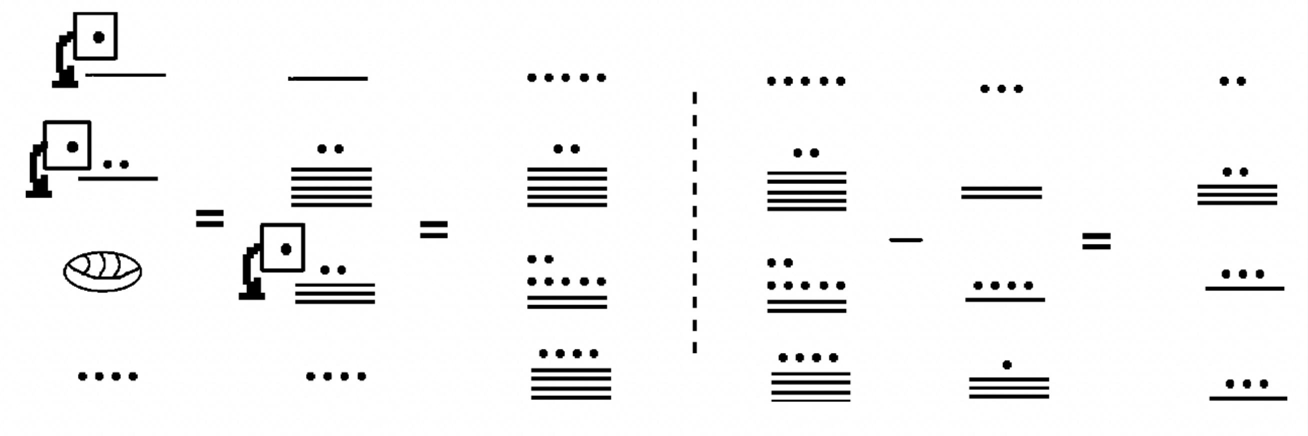
Solution



Example 4



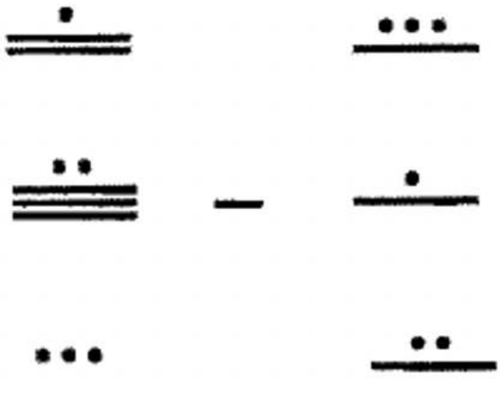
Solution



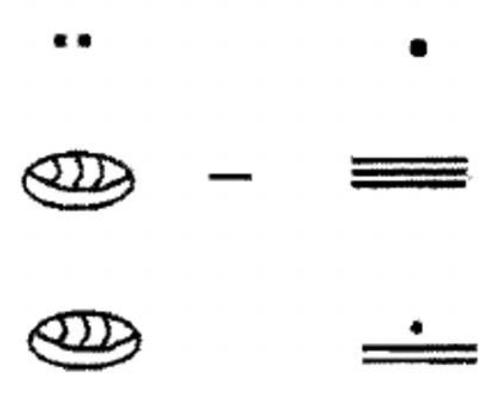
It takes a lot of concentration and effort to work these correctly. Perform each step carefully because it is easy to make mistakes. Each of the previous four examples can be checked by adding the answer to the subtrahend and seeing if the sum is the minuend. The other way to check is convert the minuend and subtrahend to Hindu-Arabic, subtract in Hindu-Arabic, then convert the difference to Mayan, making sure the answer is the same as the one you came up with when you did it in Mayan. Here is the check for Example 4: 46,084 –25,396 = 20,688. I made mistakes on this problem the first time through, but found my mistakes by checking and starting over. You should definitely check your answers. Try doing examples 3 and 4 above on your own before going on to the next exercise.

? Exercise 7

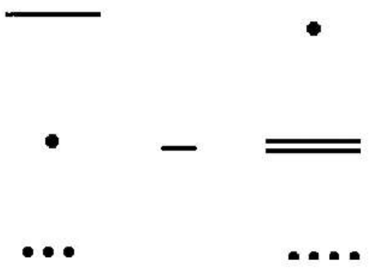
Perform the following subtraction problems in Mayan. Show all steps and check!



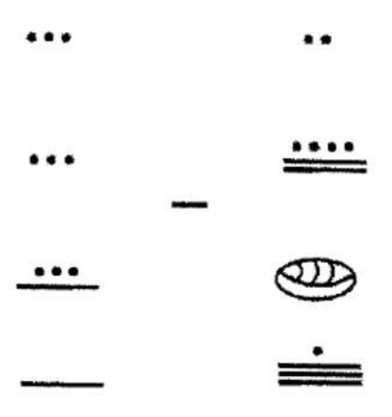
a.



b.

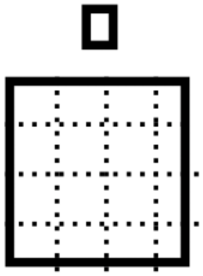
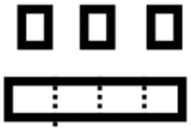


c.

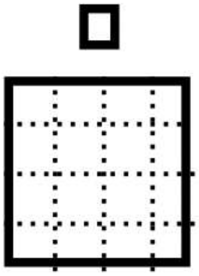
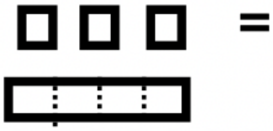
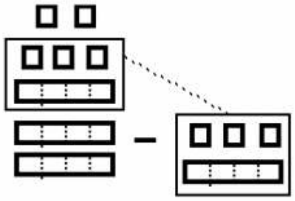
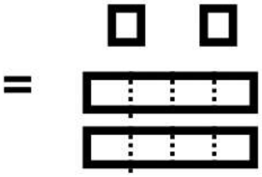


d.

Take out a set of Base Four blocks. We are going to do the subtraction problem 17 – 7 in Base Four. 101_{four} is the Base Four numeral for 17 and 13_{four} is the Base Four numeral for 7. We will make a pile, called Pile A, using the Base Four blocks to represent 17 and another pile, called Pile B, to represent 7. Your piles should look something like what you see to the right.

 Pile A	 Pile B

Below is an illustration of the subtraction problem being performed. In order to use the take-away approach, Pile B must be a subset of Pile A. That exact subset must be removed from Pile A. First, the flat in Pile A must be exchanged for 3 longs and 4 units so you can actually see Pile B inside Pile A. There is a box around the subset (Pile B) to be removed from Pile A note how it matches Pile B. After taking those blocks away, 2 longs and 2 units remain. Therefore, the completed subtraction problem is $101_{\text{four}} - 13_{\text{four}} = 22_{\text{four}}$.

 Pile A	 Pile B	 Pile A - Pile B	

Get out and use your base blocks to do the following exercises! Follow directions.

Exercise 8

Count out 17 unit blocks and make exchanges with base five blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base five, write 17 as a base five numeral on the space provided for Pile A below. Now, count out 9 more units and make exchanges with base five blocks. Put them in a pile, called Pile B. Since you've made exchanges in base five, write 9 as a base five numeral on the space provided for Pile B below. If necessary, make exchanges within Pile A so that an exact subset of Pile B is seen in Pile A. Take away those blocks in Pile A which represent the blocks in Pile B. The blocks left in Pile A are those that are left after subtracting the blocks that were in Pile B. If necessary, make any exchanges with base five blocks and then write this number as a base five numeral on the third blank provided below.

Below is the subtraction problem you have just performed in Base Five :

_____________ + ______________ = ______________________

Pile A Pile B Difference of Pile A and B

Convert the difference (answer) to Base Ten. It should be 8 since 17 – 9 = 8. Is it?

Illustrate the subtraction problem using blocks below. Show the steps.

Exercise 9

Count out 17 unit blocks and make exchanges with base eight blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base eight , write 17 as a base eight numeral on the space provided for Pile A below. Now, count out 7 more units and make exchanges with base eight blocks. Put them in a pile, called Pile B. Since you've made exchanges in base eight , write 7 as a base eight numeral on the space provided for Pile B below. If necessary, make exchanges within Pile A so that an exact subset of Pile B is seen in Pile A. Take away those blocks in Pile A which represent the blocks in Pile B. The blocks left in Pile A are those that are left after subtracting the blocks that were in Pile B. If necessary, make any exchanges with base eight blocks and then write this number as a base eight numeral on the third blank provided below.

Below is the subtraction problem you have just performed in Base Eight:

_____________ + ______________ = ______________________

Pile A Pile B Difference of Pile A and B

Convert the difference (answer) to Base Ten. It should be 10 since 17 – 7 = 10. Is it? Illustrate the subtraction problem using blocks below. Show the steps.

Exercise 10

Count out 21 unit blocks and make exchanges with base three blocks. Put them in a pile, called Pile A, and SAVE THIS PILE. Since you've made exchanges in base three , write 21 as a base three numeral on the space provided for Pile A below. Now, count out 7 more units and make exchanges with base three blocks. Put them in a pile, called Pile B. Since you've made exchanges in base three , write 7 as a base three numeral on the space provided for Pile B below. If necessary, make exchanges within Pile A so that an exact subset of Pile B is seen in Pile A. Take away those blocks in Pile A which represent the blocks in Pile B. The blocks left in Pile A are those that are left after subtracting the blocks that were in Pile B. If necessary, make any exchanges with base three blocks and then write this number as a base three numeral on the third blank provided below.

Below is the subtraction problem you have just performed in Base Three :

Pile A Pile B Difference of Pile A and B

Convert the difference (answer) to Base Ten. It should be 14 since 21 –7 = 14. Is it? Illustrate the subtraction problem using blocks below. Show the steps.

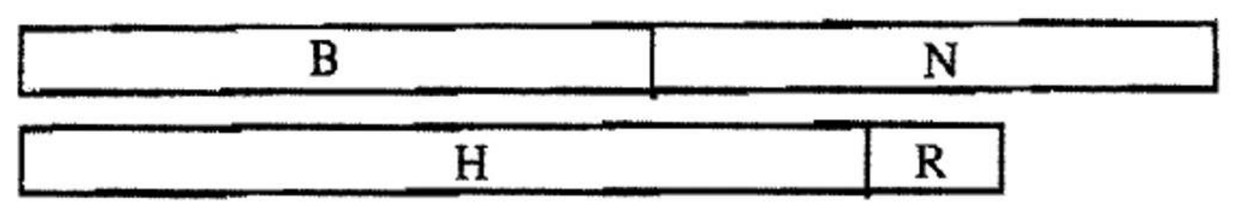
Exercise 11

Write the three subtraction problems from exercises 8, 9 and 10 in a vertical format like the one in Base Four (my example) shown to the right. Study this Base Four problem as well as the three problems you write down. Try to figure out a way to do the subtraction problems using paper and pencil instead of by using the Base Blocks. In other words, try to come up with your own algorithm (method) for doing subtraction in other bases. Explain your method and show a few examples. Later, you will be learning algorithms for subtraction.

$$\begin{array}{r} 101_{\text{four}} \\ -13_{\text{four}} \\ \hline 22_{\text{four}} \end{array}$$

Exercise 12

Using your C-Strips, make two trains as indicated. Let B + N be the first train, t_1, and let H + R be the second train, t_2. Place t_1 adjacent to t 2 as shown below. Find a train (write it as a single C-strip) that, when added to t 2 , will form a train equal in length to t_1. This train is called the *difference* of the two trains, t_1 and t_2 and is denoted by $t_1 – t_2$. What is $t_1 – t_2$?



Exercise 13

Use C-Strips to find the following differences. Draw a diagram of your work.

a. H – L	b. S – D	c. B – W
d. (B + K) – N	e. (O + Y) – P	

Exercise 14

Use the C-Strips to verify and illustrate each of the following statements:

a. (S + Y) – Y = S	b. (H – P) + P = H	c. (B + R) – B = R
d. H – (L + D) = H – L –D	e. (B + Y) – (K + L) = (B – K) + (Y – L)	

In Exercise 12, a new approach was used to defining difference. There are two distinct ways of defining subtraction – the *take-away approach* and the *missing addend approach*. We can use the model for how difference was defined using trains and now apply it to the definition of subtraction for whole numbers. Below is a definition of subtraction using the model for how difference was defined using trains in Exercise 12.

Definition of Subtraction *(Missing Addend Approach)*: Let a and b be any two whole numbers. **a – b** is the whole number **c** such that **a = b + c**. In other words, if **c** is added to the subtrahend, **b**, the sum is the minuend, **a**. The answer, **c**, is called the *missing addend.*

? Exercise 15

Is there a whole-number answer for every whole-number subtraction problem? In other words, is subtraction of whole numbers closed? Explain your answer and provide a counterexample if it is not closed

? Exercise 16

Determine which sets, if any, are closed under subtraction. Provide a counterexample if a set is not closed.

a. {0}	b. {0, 2, 4, 6, ...}

If you think of subtraction in terms of the missing addend approach, then we say that the statements a – b = c and a = b + c are equivalent to each other. Consider the statement, 8 – 2 = 6. It is equivalent to the statement 8 = 2 + 6. We also know that 2 + 6 = 6 + 2 because of the commutative property of addition. Therefore, 8 = 6 + 2, which in turn is equivalent to the statement 8 – 6 = 2. This gives us four facts about how to relate the numbers 2, 6 and 8 using addition and subtraction

8 – 2 = 6
8 = 2 + 6
8 – 6 = 2
8 = 6 + 2

Some teachers relate subtraction and addition by using the idea of "fact families" such as the four facts above, which is one fact family.

It is important to note that each addition statement gives us two subtraction statements, which is how many people learn their subtraction facts. Because of this relationship between addition and subtraction, once a child learns basic addition facts, the subtraction facts naturally follow.

? Exercise 17

Write down two subtraction statements that are equivalent to each addition statement.

You may wonder why I wrote the addition statements in Exercise 14 with the plus sign on the left of the equals sign, unlike the way I happened to write the two addition facts in the fact family relating 2, 6 and 8 at the top of the page. It's because 8 + 4 = 12 is the same fact as 12 = 8 + 4 due to the *Symmetric Property of Equality* which is defined below.

The **Symmetric Property of Equality** states that for any equation, if a = b, then it is also true that b = a

Therefore, it is perfectly okay to write the four facts above as

2 + 6 = 8
6 + 2 = 8
8 – 2 = 6
8 – 6 = 2

Use the symmetric property of equality to rewrite each equation.

? Exercise 18

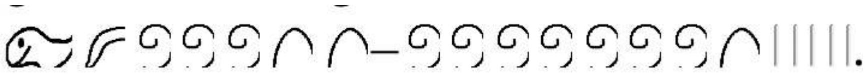
a. 7 – 3 = 4 : _____	b. 9 – 1 = 8 : _____

Exercise 19

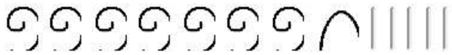
Is there a symmetric property of inequality? Explain.

Note: Inequality refers to less than (<) and/or greater than (>).

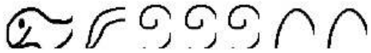
Let's do a subtraction problem using the missing addend approach in Egyptian. The set-up is exactly the same as when employing the take-away approach, but the way in which you go about finding the answer is different. The first example we looked at in Egyptian was



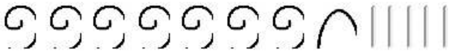
We ask ourselves "What must be added to



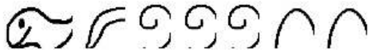
to get



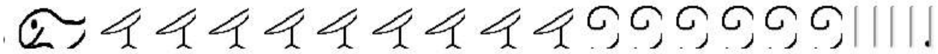
?" Another way to express this is to write:



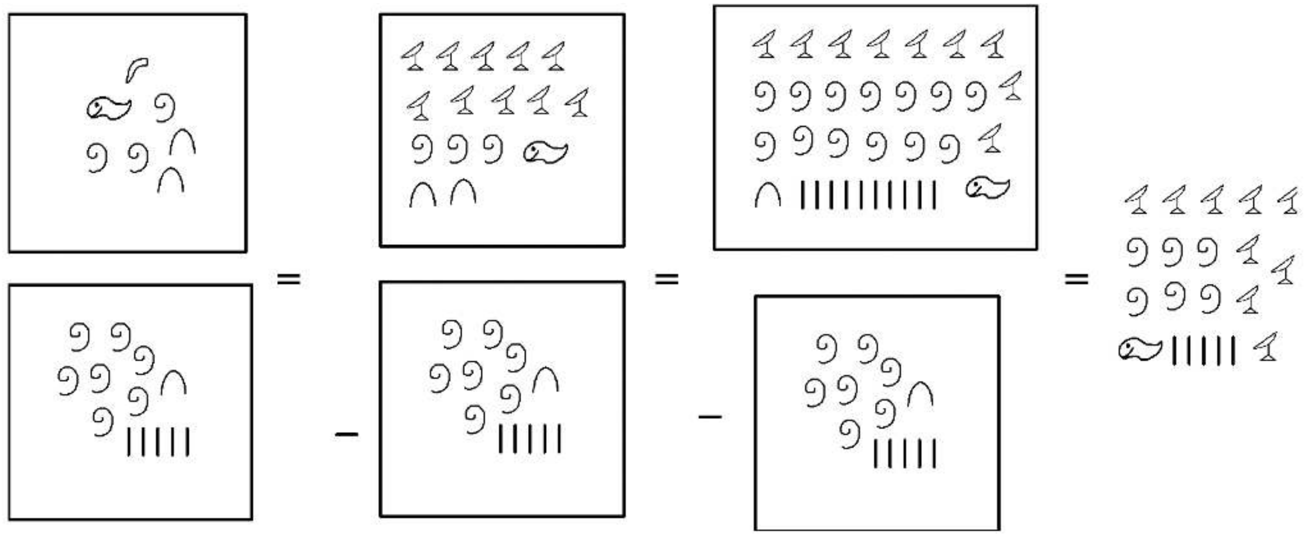
+ ????? =



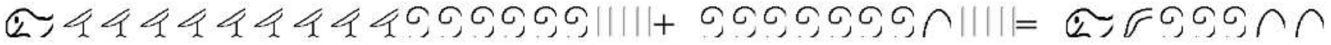
. We begin by converting in the minuend until it contains the subtrahend. We did these exact steps when using the take-away approach. But to get the final answer, we note what must be added to the subtrahend to get the minuend. The answer is



. The illustration is shown below.



To check, see if the missing addend (the answer) + the subtrahend = the minuend. In other words, is the following addition statement true?



Another way to check is to convert to Hindu-Arabic.

The same approach could be used to subtract in Mayan. The steps used to get ready to find the missing addend are exactly the same as when employing the take-away approach. But to compute the answer, at each level you decide which symbols must be added to the subtrahend to get what is in the minuend.

Although it may be hard to distinguish between the two approaches to subtraction, it's the way that you *think* about the problem that makes it different. For instance, one child learning about subtraction might think about the problem 8 – 3 by thinking "If I have 8 pennies in one hand and take away 3 of them to put in my pocket, how many are left in my hand?" This is the take-away approach. The child might compute the answer by counting backwards –7, 6, 5 or by actually using pennies or manipulatives where 8 pennies are put in one hand, 3 are taken away and what's left is counted to get the answer. Another child might think of the same subtraction problem in this way –"If I have 3 pennies, how many more pennies do I need so that I'll have 8 pennies?" This is the missing addend approach.

Exercise 20

Make up two word problems that would require the subtraction problem 8 – 3 to be computed. The first should use the take-away approach and the second should use the missing addend approach. Explain and show how you would solve each word problem using the given approach.

a. Take-Away Approach

b. Missing Addend Approach

Without using manipulatives, each subtraction problem can be worded as a missing addend addition problem. In other words, to find the answer to 8 – 3, you figure out what goes in the blank for 3 + ____ = 8. Since 3 + 5 = 8, then 8 – 3 = 5.

Exercise 21

Perform the following subtraction problems by employing the missing addend approach. Fill in the blanks

a. 9 – 7 = ____ because ______________

b. 13 – 6 = ____ because ______________

c. 88 – 8 = ____ because ______________

d. 70 – 14 = ____ because ______________

Use Base Blocks to compute the following subtraction problems using the missing addend approach. Begin by forming two piles, one for the minuend (Pile A) and one for the subtrahend (Pile B). Then figure out what must go in a third pile (Pile X) so that B + X = A. Show how you found the answer.

Exercise 22

$231_{\text{five}} - 140_{\text{five}}$

Exercise 23

$100_{\text{two}} - 11_{\text{two}}$

? Exercise 24

$135_{\text{twelve}} - T6_{\text{twelve}}$

The missing addend approach is sometimes called the *Additive (or Austrian) Algorithm.*

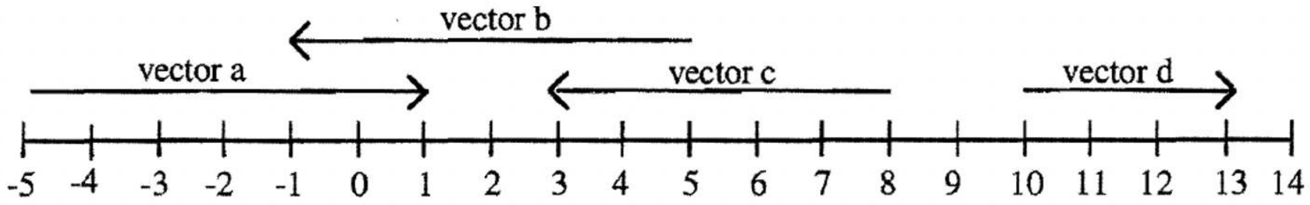
This approach is very useful for doing subtraction on the number line.

Let's review the definition of subtraction using the missing addend approach. It says the answer to a – b = c where c is the number that must be added to b to get c, or b + c = a.

Think about this using a number line. How would we find the answer to 7 – 3? We need to find a number to add on to 3 that gives us an answer of 7. In other words, what would go in the blank 3 + ____ = 7? I know you know the answer, but how could we get the answer by using a number line?

First, we have to define a new term: **vector**. A vector is a directed line segment. Basically, a directed vector looks like an arrow. It has a certain length and points in a certain direction. Since we'll be using horizontal number lines, we'll be using horizontal vectors. An arrow pointing to the right will denote a positive number and an arrow pointing to the left will denote a negative number.

Below is a number line, with some vectors shown above. Vector a is 6 units long and the arrow points to the right. Therefore, it represents the number +6. Vector b is also 6 units long, but it points to the left. Therefore, it represents the number -6.

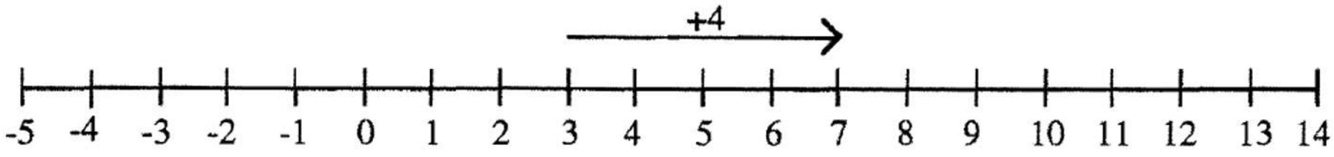


? Exercise 25

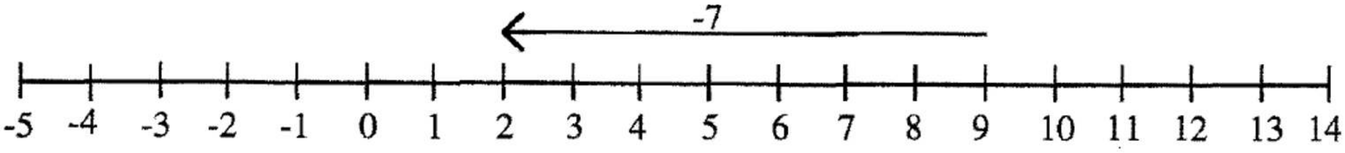
a. What number does vector c represent? __________

b. What number does vector d represent? __________

Now, we're ready to use vectors in conjunction with the missing addend approach to compute a subtraction problem. Let's go back to the problem 7 – 3. The missing addend approach states the answer to the subtraction problem is the number that when added to the number after the subtraction sign gives the number that is before the subtraction sign. Since 3 is the number after the subtraction sign, we have to find a number to add to 3 that will give an answer of 7. Using a number line, this means if you start at 3, how can you get to 7? In other words, draw a vector that starts at 3 and ends up at 7. Make sure the arrow points to 7. The number that the vector represents is the answer to the problem. The illustration is shown below. The vector has length 4 and points to the right, so the answer is 4, which is written above the vector to indicate the answer.



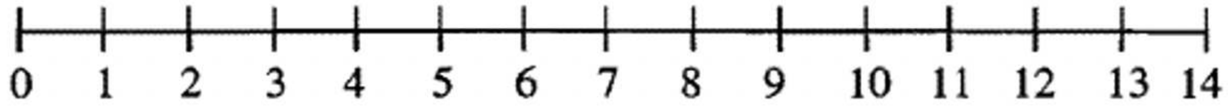
Let's try the problem 2 - 9. Then, 9 + ____ = 2. On the number line, draw a vector that starts at 9 and ends at 2. Then, see what number the vector represents to find the answer. The vector shown below gives the answer of -7.



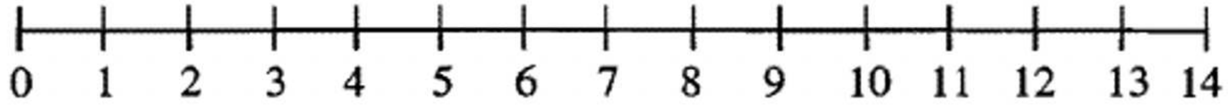
Exercise 26

Find the answer to the following subtraction problems by using directed vectors on the number line in conjunction with the missing addend approach. Draw the vectors.

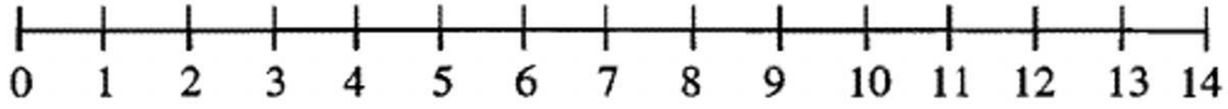
a. 10 - 4 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

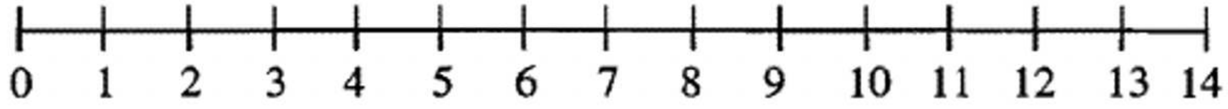
b. 12 - 5 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

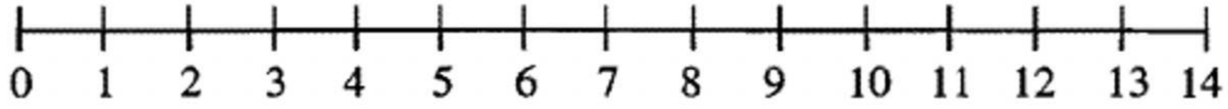
c. 8 - 3 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

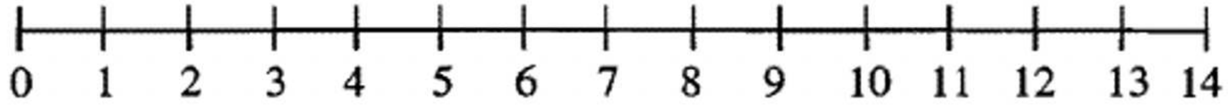
d. 9 - 7 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

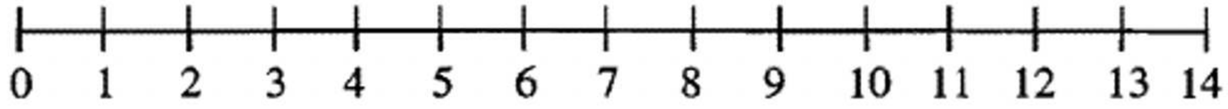
e. 3 - 10 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

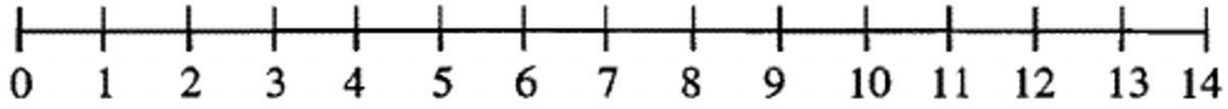
f. 5 - 8 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

g. 1 - 13 =

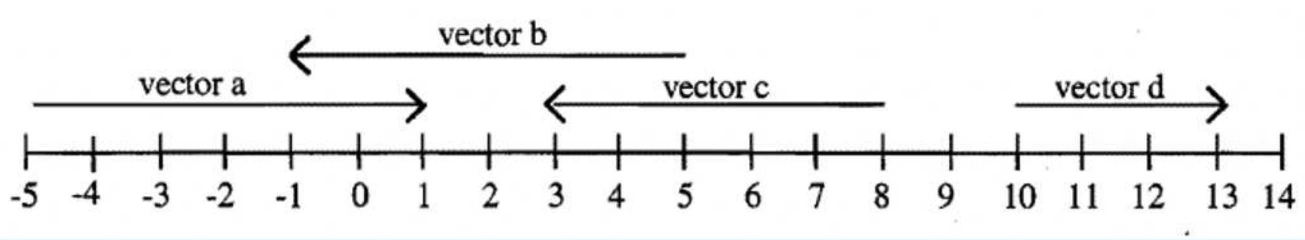

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

h. 9 - 9 =


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

Exercise 27

Looking at the vectors a – d below, figure out what subtraction problem was being performed, and then state the answer. For instance, vector b came from doing the problem -1 – 5. So, -1 – 5 = -6 (since vector b is 6 units long and goes left).


vector b
vector a
vector c
vector d
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

Vector a: ____	Vector b: ____	Vector c: ____

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3.5: Subtraction Algorithms

You will need: ***Base Blocks*** *(Material Cards 4-15)*

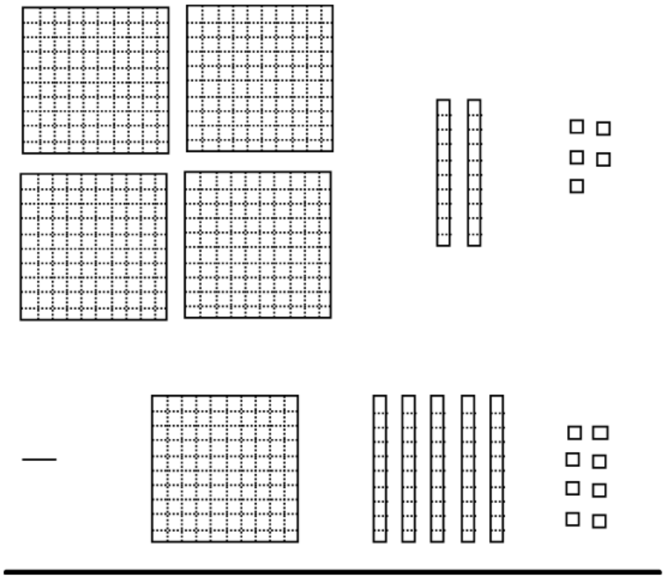
The most common subtraction algorithm is the ***Right to Left Standard Subtraction Algoithm***, which is where you start in the ones column and subtract, then move to the left and subtract at each column. The problem, of course, is when the top digit is less than the bottom digit and you have to regroup. Get out your Base Ten blocks now to see what is really going on.

Consider the subtraction problem 425 –158. From our knowledge of place value, we know that 425 is is 4 × 100 + 2 × 10 + 5 (or 400 + 20 + 5) and 158 is 1 × 100 + 5 × 10 + 8 (or 100 + 50 + 8). Using Base Ten blocks, 425 would be represented with 4 flats, 2 longs and 5 units whereas 158 would be represented with 1 flat, 5 longs and 8 units. So, the subtraction problem can be thought of in the following way:

Using Base Blocks

$$\begin{array}{r} (4\text{ flat(s) } +2\text{ long(s) } +5\text{ unit(s) }) \\ \underline{-(1\text{ flat(s) } +5\text{ long(s) } +8\text{ unit(s) })} \end{array}$$

Using blocks, the subtraction problem is shown below:

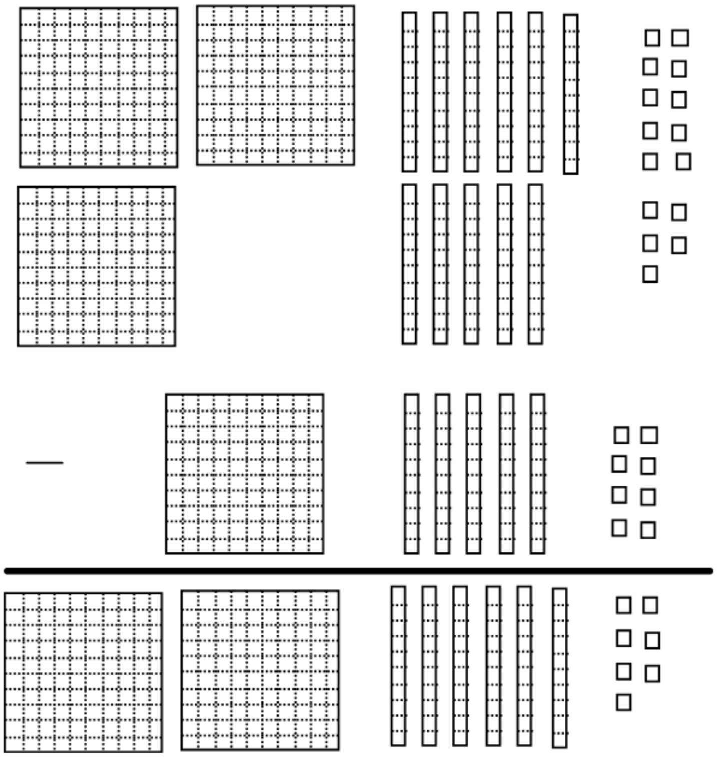


We can't subtract 8 units from 5 units, so a long is exchanged for ten units, which gives 4 flats, 1 long and 15 units in the minuend. Now, we also can't subtract 5 longs from 1 long, so 1 flat is exchanged for ten longs, which gives 3 flats, 11 longs and 15 units in the minuend. Now we can subtract at each column, which is shown on the next page. You can use either the take away approach or the missing addends approach to subtract. **Use your base blocks to work through these problems.**

Using Base Blocks

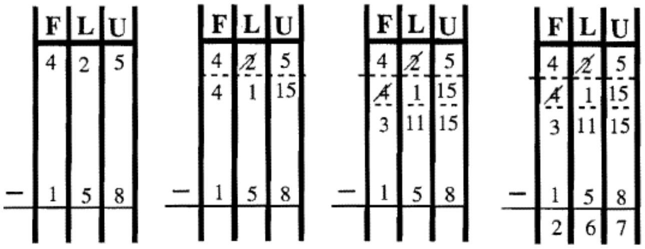
$$\begin{array}{l} (3\text{ flat(s) } +11\text{ long(s) } +15\text{ unit(s)}) \\ -(1\text{ flat(s) } +5\text{ long(s) } +8\text{ unit(s)}) \\ \hdashline 2\text{ flat(s) } +6\text{ long(s) } +7\text{ unit(s)}) = \underline{267} \end{array}$$

The subtraction problem using blocks is shown below:



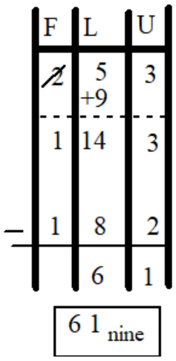
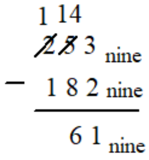
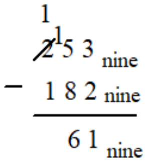
Since we are working in Base Ten, the answer (difference) is 267. So, 425 –158 = 267.

The problem can be worked using a chart where enough space is left between the minuend and subtrahend so exchanges can be made in the minuend. Notice the steps shown if you work this problem using a Base chart as shown below.

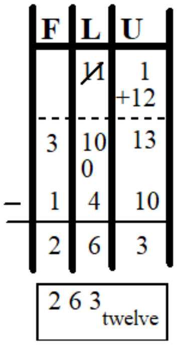
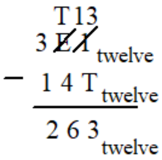
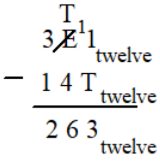


Now, we'll perform this basic algorithm in different bases. Pay attention to the base when you subtract! For instance, in the first example below you have to exchange a flat for 9 more longs (since it is Base Nine). One way to show this is to actually add 9 (the base) to the amount of longs already there, so I cross off the 5, and add 9 to get 14. If you write it that way, remember the number you are writing is in Base Ten and not in the Base you are working in. Writing it this way is probably easiest when you first start to subtract. A more elegant way is to write it the same way you do when working in Base Ten, which is by putting a 1 in front of the 5 in the longs column so it looks like 15 (it doesn't mean fifteen), which in Base Nine represents 14 longs (5 longs + 9 more from the exchange)! Both ways of writing it are shown. You might want to use your Base Blocks to help visualize the actual exchanges being made. Study the following examples. Check each answer by adding the answer to the subtrahend and see that you get the minuend!

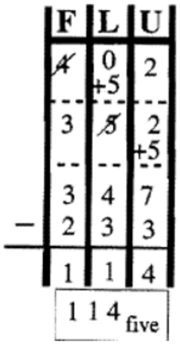
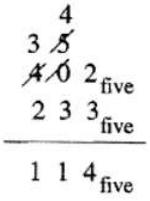
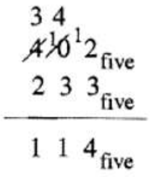
✓ Example 1

253_{nine} -182_{nine}			

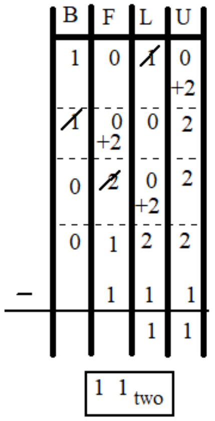
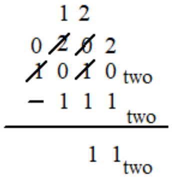
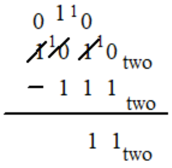
Example 2

$3E1_{twelve}$ $-14T_{twelve}$			

Example 3

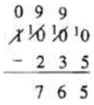
402_{five} -233_{five}			

Example 4

$\begin{array}{r} 1010_{\text{two}} \\ \underline{-111_{\text{two}}} \end{array}$			

Check: $\begin{array}{r} 11_{\text{two}} \\ \underline{+111_{\text{two}}} \\ 1010_{\text{two}} \end{array}$

In Examples 3 and 4, when I needed to make an exchange and there was a zero in the place value to the left, the exchange had to be made at a higher place value first. Then, exchanges are made down the line until you can subtract. This case happens in Base Ten all the time when you subtract a number from 100, 1000, 10000, etc. Most people just start crossing off each zero and putting 9's above them out of habit, without having any idea why they are doing it. To subtract 235 from 1000, first you should cross off the 1, put a zero above it and put a 1 in front of the zero in the hundred's place (which is trading in 1000 for 10 hundreds), then you should cross off that 10 in the hundred's place, put a 9 above it and put a 1 in front of the zero in the ten's place (which is exchanging 10 hundreds for 9 hundreds and 10 tens), then you should cross off that 10, put a 9 above it and a 1 in front of the zero in the one's place (which is exchanging 10 tens for 9 tens and 10 ones). Then, you can subtract as shown in the example to the right. It's important to really understand what's going on here because children usually have a very hard time when there is a zero to cross off.



Exercise 1

Perform the following subtraction problems. You can use blocks or charts to help visualize the problem, but you eventually want to work toward being able to do them without the manipulatives. Underneath each problem, check by adding the difference (answer) to the subtrahend (number being subtracted) and see if the sum is the minuend (the number on top). Use any addition algorithm you prefer.

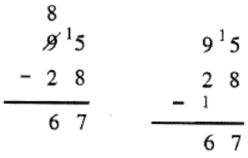
a. $\begin{array}{r} 452_{\text{thirteen}} \\ \underline{-13E_{\text{thirteen}}} \end{array}$	b. $\begin{array}{r} 831_{\text{nine}} \\ \underline{-670_{\text{nine}}} \end{array}$	c. $\begin{array}{r} 2004_{\text{six}} \\ \underline{-335_{\text{six}}} \end{array}$	d. $\begin{array}{r} 421_{\text{five}} \\ \underline{-232_{\text{five}}} \end{array}$	e. $\begin{array}{r} 100_{\text{three}} \\ \underline{-11_{\text{three}}} \end{array}$

Check each problem here:

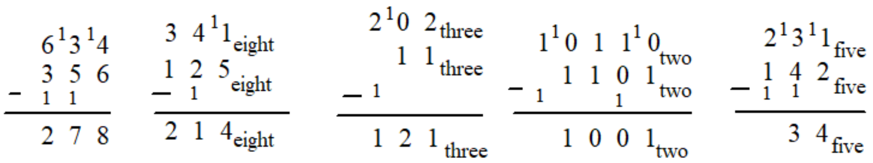
a.	b.	c.	d.	e.

The traditional, standard subtraction algorithm is not necessarily the best one and there are many other ones you'll be learning in the rest of this Exercise Set.

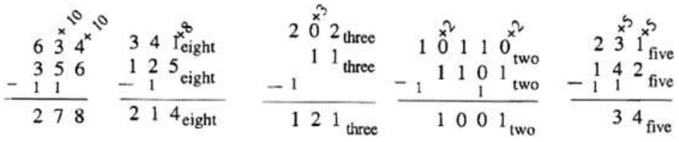
The next algorithm was taught to many people before the "new math" and so I call it the **Oldtimer's Algorithm**. It's similar to the Standard Algorithm in that you start on the right and move to the left. But you don't regroup (formally known as borrowing). First look again at an example of using the Standard Algorithm to subtract 95 – 28. Since, you can't subtract 8 from 5, you regroup from the 9 by crossing out the 9 and writing 8 so you can put a 1 in front of the 2. Now, in the unit's column, 15 – 8 = 7 and in the ten's column, 8 – 2 = 6. The answer is 67. This new method starts out the same way. You can't subtract 8 from 5, but instead of regrouping from the 9, you *add a 1 to the number under the 9*, and put a 1 in front of the 5. Notice that in the unit's column, 15 – 8 = 7 and in the ten's column, instead of 8 –2, you have 9 –3 which is the same answer of 6. The advantage of this method is if you put a 1 at the top of one column, you compensate by adding a 1 to the bottom of the next column. You don't have to worry about crossing things off and making a mess and you don't have to think about the digit in the column to the left until you go to that column to subtract – no trying to regroup and getting hung up there! Look at the two methods side by side. If you use the new method, you have to leave space between the subtrahend and subtraction bar in case you need to put a 1 there.


8
9 ¹5
– 2 8
6 7
9 ¹5
2 8
– 1
6 7

What I really like about the Oldtimer's method is that it is much easier to check. Just add up each column. There is no carrying : 7 + 8 = 15 and 6 + 1 + 2 = 9. It's a breeze. To check the Standard Algorithm, well it's something of a mess and you would need to write it down somewhere else and check it. Study the following examples.


6 ¹3 ¹4
3 5 6
– 1 1
2 7 8
3 4 ¹1 eight
1 2 5 eight
– 1
2 1 4 eight
2 ¹0 2 three
1 1 three
– 1
1 2 1 three
1 ¹0 1 1 ¹0 two
1 1 0 1 two
– 1 1
1 0 0 1 two
2 ¹3 ¹1 five
1 4 2 five
– 1 1
3 4 five

You still have to be very careful to pay close attention to the base. When you put a 1 in front of a digit, remember that it stands for the number that is the base. Another way to indicate the same thing is to write out what it stands for as shown below. The same five examples are illustrated again doing it this way. It's not as elegant but if it helps drive home what is really going on, do it this way.

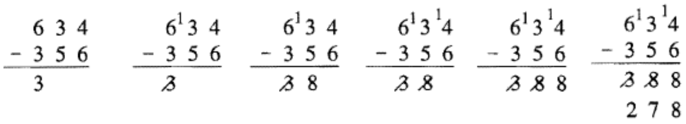

×10
×10
6 3 4
3 5 6
– 1 1
2 7 8
×8
3 4 1 eight
1 2 5 eight
– 1
2 1 4 eight
×3
2 0 2 three
1 1 three
– 1
1 2 1 three
×2
×2
1 0 1 1 0 two
1 1 0 1 two
– 1 1
1 0 0 1 two
×5
×5
2 3 1 five
1 4 2 five
– 1 1
3 4 five

Exercise 2

Subtract the following using the Oldtimer's Algorithm. Check your answers by adding up.

a. 5036 -3418	b. 314_{four} -221_{four}	c. $5T2_{\text{twelve}}$ $-23E_{\text{twelve}}$	d. 413_{six} -324_{six}	e. 1100_{two} -101_{two}
f. 537_{nine} -273_{nine}	g. 312_{five} -224_{five}	h. 502_{seven} -235_{seven}	i. 200_{three} -121_{three}	j. 523_{eight} -265_{eight}

Now that you have mastered that algorithm, here is another one for you. This is a **Left to Right Algorithm** where no regrouping takes place until the end of the problem – you regroup from the answer! Here's how it works. Start with the left-most column and subtract. Move to the next column. Subtract if possible. If you need to regroup, put a 1 in front of the top digit as usual, and put a slash through the previous digit in the answer. You'll take away from the answer in the end. Continue on to the next column until you've done the unit's column. Then go back and subtract one from all digits with a slash through them. Study the following example which is shown step by step.


6 3 4
– 3 5 6
3
6 ¹3 4
– 3 5 6
3
6 ¹3 4
– 3 5 6
3 8
6 ¹3 ¹4
– 3 5 6
3 8
6 ¹3 ¹4
– 3 5 6
3 8 8
6 ¹3 ¹4
– 3 5 6
3 8 8
2 7 8

Here are some examples in other bases. If you prefer, instead of putting the 1 in front of the digit when you regroup, you can put, for example, +4 if it's in Base Four, like was shown for the previous algorithm on the bottom of the last page.

$$\begin{array}{r} 4\ 3^{1}2_{\text{six}} \\ -\ 2\ 1\ 5_{\text{six}} \\ \hline 2\ \cancel{2}\ 3 \\ 2\ 1\ 3_{\text{six}} \end{array} \qquad \begin{array}{r} 1^{1}0\ 1^{1}0_{\text{two}} \\ -\ 1\ 0\ 1_{\text{two}} \\ \hline \cancel{1}\ 1\ \cancel{1}\ 1 \\ 1\ 0\ 1 \end{array} \qquad \begin{array}{r} 4^{1}6^{1}2_{\text{twelve}} \\ -\ 1\ T\ 3_{\text{twelve}} \\ \hline \cancel{3}\ \cancel{8}\ E \\ 2\ 7\ E \end{array} \qquad \begin{array}{r} 2^{1}1\ 2^{1}1_{\text{three}} \\ -\ 1\ 2\ 1\ 2_{\text{three}} \\ \hline \cancel{1}\ 2\ \cancel{1}\ 2 \\ 2\ 0\ 2 \end{array}$$

Exercise 3

Check the above examples by doing the following addition problems.

213_{six} $+215_{\text{six}}$	101_{two} $+101_{\text{two}}$	$27E_{\text{twelve}}$ $+1T3_{\text{twelve}}$	202_{three} $+1212_{\text{three}}$

Exercise 4

Subtract the following using the Left to Right Algorithm..

a. 8042 -1325	b. 4132_{five} -2413_{five}	c. 7210_{eight} -5564_{eight}	d. $T37_{\text{thirteen}}$ -569_{thirteen}	e. 1101_{two} -11_{two}

Exercise 5

Check the answers to Exercise 4 by doing the corresponding addition problem

In this algorithm, there is one more little detail. If you cross out a zero, you must continue crossing off the digit to the left of the zero. If there is more than one zero in a row as you go left, cross off each zero until you get to a nonzero digit. Then, subtract 1 from each number crossed off as before. Pay attention to the base: In the second example below, the number before 40five is 34five. In the third example, the number before 100two is 11two. Study the following examples.

$$\begin{array}{r} 6\ 7^{1}2\ 4 \\ -\ 2\ 7\ 6\ 3 \\ \hline \cancel{4}\ \cancel{0}\ 6\ 1 \\ 3\ 9\ 6\ 1 \end{array} \qquad \begin{array}{r} 3^{1}2\ 4^{1}1_{\text{five}} \\ -\ 1\ 3\ 4\ 2_{\text{five}} \\ \hline \cancel{2}\ \cancel{4}\ \cancel{0}\ 4 \\ 1\ 3\ 4\ 4_{\text{five}} \end{array} \qquad \begin{array}{r} 1^{1}0\ 1\ 1^{1}0_{\text{two}} \\ -\ 1\ 1\ 1\ 1_{\text{two}} \\ \hline \cancel{1}\ \cancel{1}\ \cancel{0}\ \cancel{0}\ 1 \\ 1\ 1\ 1_{\text{two}} \end{array} \qquad \begin{array}{r} 4\ 3^{1}2_{\text{six}} \\ -\ 2\ 3\ 5_{\text{six}} \\ \hline \cancel{2}\ \cancel{0}\ 3 \\ 1\ 5\ 3_{\text{six}} \end{array}$$

Exercise 6

Subtract the following using the Left to Right Algorithm.

a. 8042 -1345	b. 4132_{five} -2433_{five}	c. 7210_{eight} -5214_{eight}	d. $T37_{\text{thirteen}}$ -539_{thirteen}	e. 1100_{two} -101_{two}

Exercise 7

Check the answers to Exercise 6 by doing the corresponding addition problem.

Algebra will help you understand the next algorithm:

$M – S = M – S + x – x = M + x – S – x = (M + x) – (S + x)$

$M – S = M – S + x – x = M – x – S + x = (M –x) – (S – x)$

Basically, this states that to subtract two numbers, M (minuend) and S (subtrahend), you get the same answer if you first add (or subtract) the same number to both M and S before subtracting. The Oldtimer's Algorithm works because of this fact. For instance, when you put a 1 in front of the minuend's unit's digit and a 1 below the subtrahend's ten's digit, you are adding both a 10 to the minuend and to the subtrahend; then, you subtract.

This algorithm is particularly useful if the minuend has a string of zeros at the end. Subtract 1 from the minuend and subtrahend first and then do the subtraction. Look at how this works:

$\begin{array}{r} 100000 \\ \underline{-65378} \\ 34622 \end{array} = \begin{array}{r} 99999 \\ \underline{-65377} \\ 34622 \end{array}$		$\begin{array}{r} 3000 \\ \underline{-1264} \\ 1736 \end{array} = \begin{array}{r} 2999 \\ \underline{-1263} \\ 1736 \end{array}$

Well, what do you think? Isn't the second subtraction shown in each pair easier to do than the first one? I just subtracted 1 from the minuend and subtrahend first!

? Exercise 8

Make up two of your own subtraction problems using this method to subtract.

This algorithm also works well if the subtrahend is close to a power of the base, like 100, 1000, etc. For instance, for the subtraction problem 4503 – 997, add 3 to the minuend and subtrahend so that the problem becomes 4506 – 1000 = 3506. In the Base Four problem, $2302_{\text{four}} - 333_{\text{four}}$, add 1 to both to get $2303_{\text{four}} - 1000_{\text{four}} = 1303_{\text{four}}$

? Exercise 9

Make up two of your own subtraction problems using this method to subtract. Use a base other than 10.

The above method is sometimes called the **Complementary Method**. There is yet a more specific approach called the **Complementary Algorithm**. It relies on the idea of complements that we defined in an earlier exercise set. In Base Ten, pairs of complements were 1 & 9, 2 & 8, 3 & 7, 4 & 6 and 5 & 5. For the complementary method, you find a very specific complement of the subtrahend and add it to both the minuend and subtrahend before subtracting. The complement, in this case, is a 1 followed only by zeros such that the number of zeros is the same number of digits in the minuend. First, you'll have to come up with an easy way to find the complement of a given number.

Below are examples showing how to find the complement (C) of a given number (A), given you want A + C to add up to a number B.

✓ Example 1

A = 74 and B = 1000.

Solution

To find C, add 6 to 74 to get 80, then 20 to get 100 and 900 to get 1000. C = 926.

✓ Example 2

A = 24_{six} and B = 1000_{six}.

Solution

To find C, add 2 units to get 3 longs, then 3 longs to get 1 flat and 5 flats to get 1 block. Then C = 532_{six}.

There are other ways to figure out the complement. For instance, in Base Ten, if A is 28403 and you want a C such that A + C = 1000000, you can find the answer by writing the actual complement of 3, which is 7, for the unit's digit. Move to the left. You want a digit for each place value that when added to the digit there gives a sum of 9. Do this until the number has as many digits as there are zeros in 1000000. In this case, the answer is 971597. It's harder to explain than do! If you have your own way of figuring out the complement, that's great! Don't keep it a secret.

? Exercise 10

Find the complement (C) of the given number (A) such that A + C = B.

a. A = 538 and B = 10000

b. A = 212_{four} and B = 1000_{four}

c. A = 1011_{two} and B = 10000_{two}

Here is how this complementary method works. Consider the problem 452 – 74. We need to find the number that when added to 74 is 1000. How do you choose what it should add up to, as in 1000? If the *first* number (minuend) is a 2 digit numeral, you want a 1 with 2 zeroes (100). If the first number (minuend) is a 3 digit numeral, you want a 1 with 3 zeroes (1000), and so on. This step is the same in all bases. Back to the problem: 452 – 74: 926 is what should be added to 74 to get 1000 (from Example 1). Add 926 to the minuend and subtrahend to get the new subtraction problem which is: 1378 – 1000 = 378. *The answer is the new minuend without the first digit: 378.* Pretty wild! Look at $402_{\text{six}}–24_{\text{six}}$. We need to find the number that when added to 24_{six} is 1000_{six}., which is 532_{six} from Example 2. Add 532_{six} to the minuend and subtrahend to get the new subtraction problem which is $1304–1000_{\text{six}} = 304_{\text{six}}$. Again, the answer is the new minuend without the first digit: 304_{six}. So here's the trick – *Just add the complement to the minuend, take off the first digit and that's the answer! There is no subtracting.*

✓ Example 3

Subtract using the complementary method: 912 – 573

Solution

427 is the number that when added to 573 equals 1000. Add to the minuend: 427 + 912 = 1339. Take off the first digit and the answer is 339.

✓ Example 4

Subtract $301_{\text{four}}–212_{\text{four}}$ using the complementary algorithm.

Solution

From 10.b., the complement is 122_{four}. $301_{\text{four}} + 122_{\text{four}} = 1023_{\text{four}}$.
So the answer is 23_{four}.

? Exercise 11

Subtract $1100_{\text{two}}–1011_{\text{two}}$ using the complementary algorithm and your answer from 10.c.

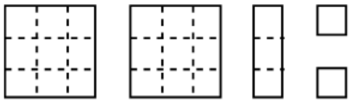
? Exercise 12

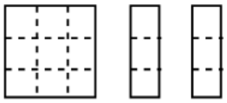
Perform the following subtraction problems using the complementary method.

a. $\begin{array}{r}5034\\ \underline{-357}\end{array}$	b. $\begin{array}{r}420_{\text{five}}\\ \underline{-231_{\text{five}}}\end{array}$

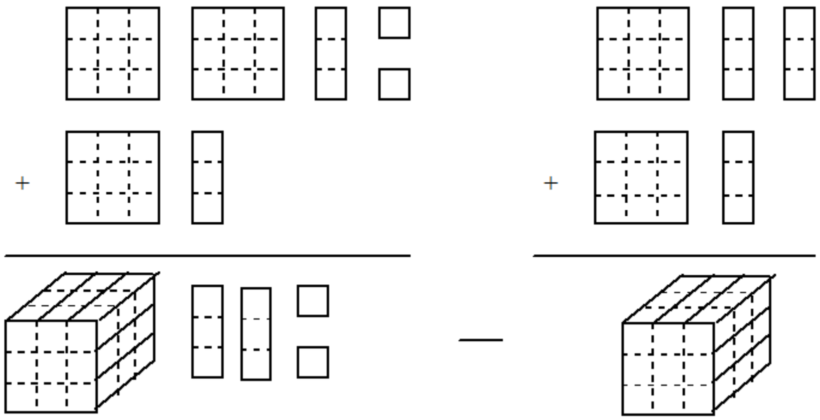
The complementary method is actually easier to do with the blocks –once you get the hang of it. Take out your Base Three blocks to do the following example and problems.

To compute $212_{\text{three}}–120_{\text{three}}$, represent 212_{three} and 120_{three} with the blocks as shown below.



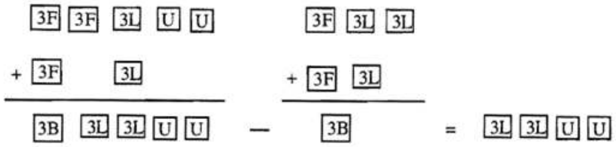


The minuend is shown on the left. Since its biggest "block" is a flat, the next "place value" up a block. So the problem is to figure out what needs to be added to the subtrahend (shown on the right) to make a block. It would be 1 more long and 1 more flat. In other words, if you added a long and flat to the subtrahend, you would have a block, right? Add that same amount to the minuend, shown on the left above. It should look like this:



If you subtract now, the answer is the new minuend with the largest block removed. Therefore, the answer is 2 longs and 2 units, or 22_{three} The crucial step is to figure out what needs to be added to the subtrahend to make the block. Then, also add it to the minuend, remove the largest block from the new minuend and that's the answer! Ta da!

Since drawing the blocks out is sometimes cumbersome, another way to pictorially show what is going on is to use abbreviated pictures for the Base Three blocks, flats, longs and units (3B, 3F, 3L, U) as shown below.



Exercise 13

On the next page, use the appropriate base blocks to subtract $120_{\text{three}} - 12_{\text{three}}$ and $213_{\text{four}} - 133_{\text{four}}$ **using the complementary method**. Explain how to do it and show full or abbreviated pictures with the base blocks.

a. Use Base Three Blocks and complementary method to subtract $120_{\text{three}} - 12_{\text{three}}$.

b. Use Base Four Blocks and complementary method to subtract $213_{\text{four}} - 133_{\text{four}}$.

Exercise 14

Explain how each student is doing the subtraction problem 634 – 152.

a. Mary does the subtraction in her head by thinking out loud "534, 484, 482". What is her reasoning?

b. Pedro does the subtraction by thinking out loud " 634 and 48 is 682, take away 200 is 482". What is his reasoning?

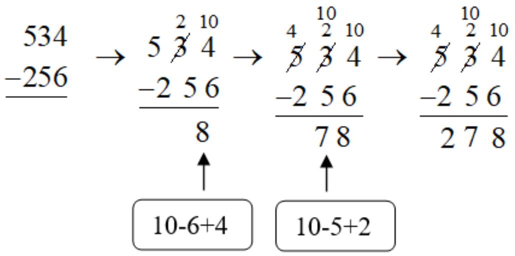
Another method to subtract is called the **Subtract from the Base Algorithm**. This one is especially nice for working in other bases. In this method you never need to subtract from a number larger than the base.

Standard algorithm:

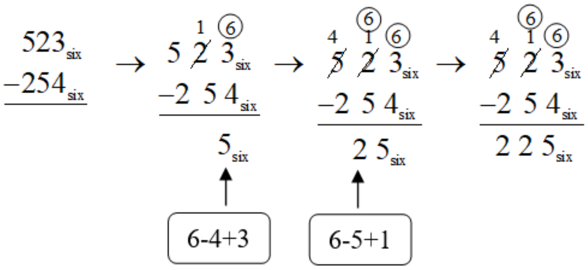
$$\begin{array}{r}534\\ -256\\ \hline\end{array}\text{ make exchanges }\begin{array}{ccc}\overset{4}{\cancel{5}} & \overset{12}{\cancel{3}} & \overset{14}{4}\\ -2 & 5 & 6\\ \hline 2 & 7 & 8\end{array}$$

Using the Subtract from the Base method if you have a place where the bottom number is bigger than the top number, regroup as in the traditional method then subtract the bottom number from the base and add the result to the top number. The advantage of this method is you never subtract a number from any number larger than the base. This is a good method for people who need to use fingers to subtract.

Subtract from the base algorithm:



When using this method in other bases circle the base number when regrouping. For example when working in base six, 6 does not exist and writing 10 can be confusing even though it is correct.



Exercise 15

Subtract the following using the Subtract from the Base Algorithm. These are the same problems from #6. Don’t forget to circle the base number when regrouping.

a. $\begin{array}{r}8042\\ \underline{-1345}\end{array}$	b. $\begin{array}{r}4132_{\text{five}}\\ \underline{-2433_{\text{five}}}\end{array}$	c. $\begin{array}{r}7210_{\text{eight}}\\ \underline{-5214_{\text{eight}}}\end{array}$	d. $\begin{array}{r}\text{T}37_{\text{thirteen}}\\ \underline{-539_{\text{thirteen}}}\end{array}$	e. $\begin{array}{r}1100_{\text{two}}\\ \underline{-101_{\text{two}}}\end{array}$

Do you know of another algorithm for addition or subtraction? If so, post it with an explanation on the Forum for extra credit.

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3.6: Homework

HW 1

Use the set theory definition of addition or subtraction to add or subtract each of the following. Define appropriate sets and show work. Refer back to pages 2 and 3 for addition and pages 41 and 42 for subtraction if you need help getting started.

a. 5 + 3	b. 5 - 4

HW 2

Verify each of the following:

a. 79 > 66 ____	b. 34 < 88 ____

HW 3

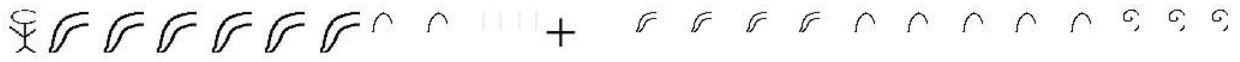
State whether each set is closed under addition and/or subtraction of whole numbers. Provide an example if it is closed, or provide a counterexample if it is not closed.

a. {0}	b. {0, 2, 4, ...}

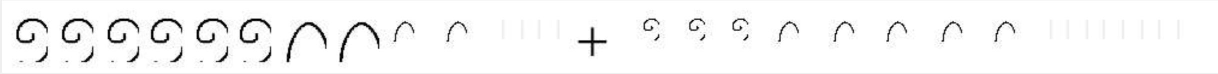
HW 4

Add the following in the system in which the numbers are given. Show work.

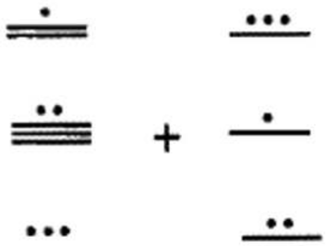
a.



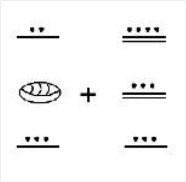
b.



c.



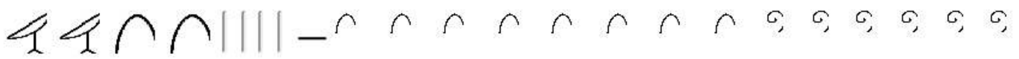
d.



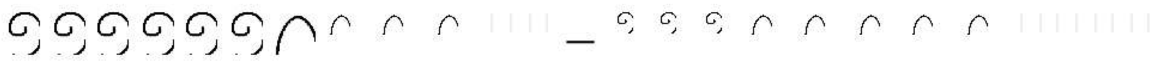
? HW 5

Subtract the following in the system in which the numbers are given. Show work.

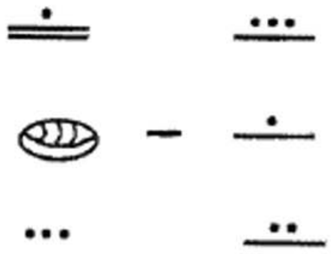
a.



b.



c.



For each pair of numbers, add them together. Then subtract the smaller number from the larger number. Show work and check your answer.

? HW 6

5432_{six} and 2453_{six}

a.	b.

? HW 7

$5T4_{\text{twelve}}$ and 387_{twelve}

a.	b.

? HW 8

Add 5643 + 7885 using each algorithm from this module. State the method for each.

? HW 9

Subtract 3586 from 8204 each algorithm from this module. State the method for each.

? HW 10

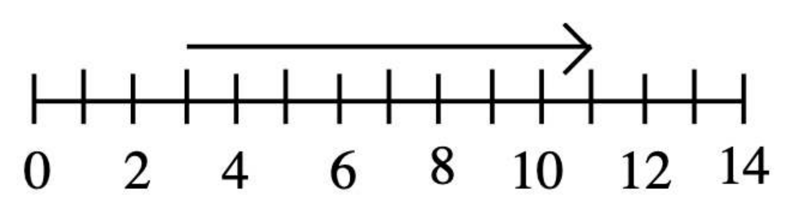
Write the addition table for Base Four.

HW 11

Someone does the following computation in a certain base, but is making a mistake. What base were they working in and what is the mistake? 132 + 243 = 320. What is the correct answer? Be sure to show work.

HW 12

Someone used the directed vector to do a subtraction problem. What was the subtraction problem used?



HW 13

Find the answer to the following subtraction problems by using directed vectors on the number line in conjunction with the missing addend approach. Draw the vectors.

a. 10 - 4	b. 6 - 11

HW 14

In what number base or bases between 2 and 13 are each of the following calculations valid?

a. $\begin{array}{r} 57 \\ +64 \\ \hline 132 \end{array}$	b. $\begin{array}{r} 341 \\ +542 \\ \hline 865 \end{array}$	c. $\begin{array}{r} 101 \\ +111 \\ \hline 212 \end{array}$	d. $\begin{array}{r} \mathrm{T}4 \\ +58 \\ \hline 12\mathrm{W} \end{array}$

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CHAPTER OVERVIEW

4: Multiplication of Understanding Elemementary Mathmatics

The teacher said to the student, "Multiply 3×7*."*

The student replied, "But I don't know how to multiply."

Then the teacher asked, "Do you know how to add?"

The student replied, "Sure, that's easy."

Then the teacher said, "Well, then, you can multiply. Just add three sevens together!"

The student marveled at this notion. The student simply added 7+7+7, and answered, "Is 3×7 *equal to 21?"*

The teacher said , "Yes! I knew you could multiply. I'm very proud of you."

The student was proud too...and very happy.

- 4.1: Definition and Properties
- 4.2: Multiplication Algorithms
- 4.3: Homework

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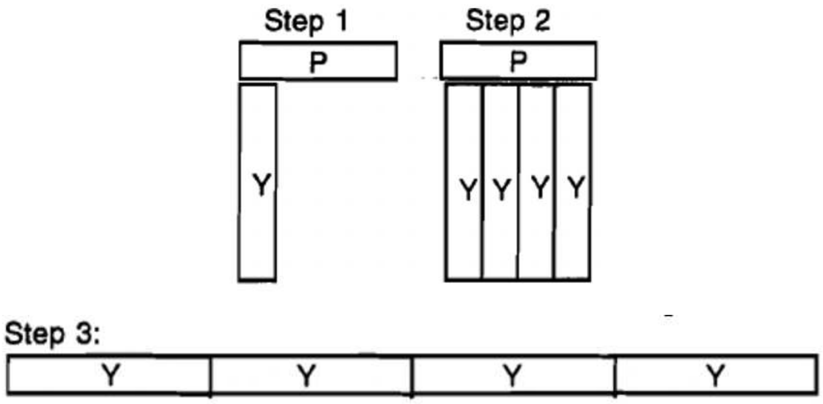
4.1: Definition and Properties

This exercise set presents a different, challenging way for you to look at multiplication. You will be using the Centimeter Strips (C-strips) to explore and discover the commutative, associative and distributive properties of multiplication. Below is a reminder of the different C-Strips we'll be working with along with the abbreviation used for each highlighted in **bold**:

1 cm by 1 cm	**W**HITE	1 cm by 2 cm	**R**ED
1 cm by 3 cm	**L**IGHT GREEN	1 cm by 4 cm	**P**URPLE
1 cm by 5 cm	YELLOW	1 cm by 6 cm	**D**ARK GREEN
1 cm by 7 cm	BLAC**K**	1 cm by 8 cm	BROW**N**
1 cm by 9 cm	**B**LUE	1 cm by 10 cm	**O**RANGE
1 cm by 11 cm	**S**ILVER	1 cm by 12 cm	HOT PINK

We will begin by defining a way to "multiply" two C-strips together. Consider multiplying the Purple C-strip by the Yellow C-Strip. Using abbreviations, we will write $P \times Y$, which we will read aloud as "P cross Y". The first step in this "multiplication" is to place the first strip (P) down in a horizontal position and then to place the second strip (Y) underneath the first strip so that it is perpendicular to the first strip. The second step is to fill in more yellow C-strips alongside the one already there so that you form a rectangle of yellow strips where the length is the yellow strip and the width is the purple strip. Last, remove the horizontal strip (purple in this case) and form a train of the vertical strip(s) (yellow in this case). This train is $P \times Y$ and is shown below. See the illustrations of the three steps.



We say $P \times Y$ is the train consisting of 4 yellow C-strips. *It is important to notice that the answer is a train that is formed with only yellow strips, which is the* ***second*** *letter in the multiplication.*

? Exercise 1

a. Make the train $P \times Y$ as explained above. **Save it until you finish part d.**

b. Find $Y \times P$ by following the same procedure used to find $P \times Y$, by doing the first ***two*** steps and showing them below.

Step 1:	*Step 2:*

c. This train you eventually end up with should consist of only purple C-strips. How many purple strips are there? Draw a picture of the train $Y \times P$.

d. Place the two trains, $P \times Y$ and $Y \times P$, side by side and compare the lengths of the two trains. What do you notice?

To make notation simple, let's agree to say that **two trains are equal if they have the same length**. In other words, from what you discovered in Exercise 1, $P \times Y = Y \times P$.

For the following exercises, describe the trains formed by doing the following multiplications with C-strips. You should physically form each train by going through the same steps used to multiply $P \times Y$ on the previous page. For part a and b, fill in the first blank with a numeral (how many) and the second blank with a color. Save each train in order to answer part c. Use your observation to write the equations asked for in Exercise 1d. For instance, since you discovered $P \times Y = Y \times P$, this translates to $4 \times 5 = 5 \times 4$, since P is 4 cm long and Y is 5 cm long. Use this example to fill in blanks for exercises 2, 3 and 4.

✓ Example a

Find $P \times Y$. This train consists of

✓ Example b

Find $Y \times P$. This train consists of

? Exercise 2

a. Find $R \times D$. This train consists of _____ C-strips

b. Find $D \times R$. This train consists of ______ C-strips

c. $R \times D$ is made up of ____ white strips and $D \times R$ is made up of _____ white strips.

d. What do you notice when you compare the lengths of the trains $R \times D$ and $D \times R$?

e. From your work in parts a - d, write an equation including C-strips R and D; then translate the equation to one with numbers.

? Exercise 3

a. Find $K \times L$. This train consists of _____ C-strips

b. Find $L \times K$. This train consists of _____ C-strips

c. $K \times L$ is made up of ____ white strips and $L \times K$ is made up of _____ white strips.

d. What do you notice when you compare the lengths of the trains $K \times L$ and $L \times K$?

e. From your work in parts a - d, write an equation including C-strips K and L; then translate the equation to one with numbers.

? Exercise 4

a. Find $H \times W$. This train consists of _____ C-strips

b. Find $W \times H$. This train consists of _____ C-strips

c. $H \times W$ is made up of ____ white strips and $W \times H$ is made up of ____ white strips.

d. What do you notice when you compare the lengths of the trains $H \times W$ and $W \times H$?

e. From your work in parts a - d, write an equation including C-strips H and W; then translate the equation to one with numbers.

The above problems illustrate the *commutative property of multiplication* for trains.

The commutative property of multiplication for trains states that if s and t are any two trains, then $s \times t = t \times s$

Note: A train can consist of one or more C-strips. In other words, a single white (or any other color) C-strip is actually a train. There doesn't have to be a caboose and engine!

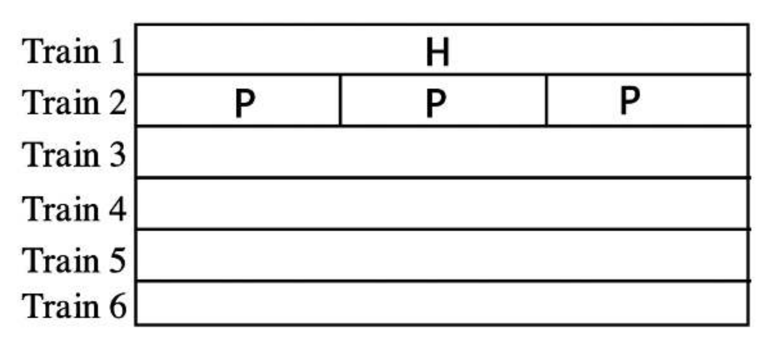
Exercise 5

Fill in the blanks to each of the following problems. To do this, take the color c-strip shown after the multiplication sign and make a train out of that color that is equal in length to the c-strip on the right side of the equal sign. Second, take the train and form it into a rectangle. Then find a c-strip that fits across the width of the rectangle. Write the abbreviation for that c-strip in the blank.

a. ____ $\times R = H$	b. ____ $\times P = P$	c. ____ $\times L = D$
d. ____ $\times R = N$	e. ____ $\times L = B$	f. ____ $\times W = K$
g. ____ $\times R = D$	h. ____ $\times D = H$	i. ____ $\times Y = O$

Exercise 6

a. Take out the Hot Pink (H) C-strip. Use your C-strips (one color at a time) to see if you can form a train made up of c-strips of the same color equal in length to the hot pink c-strip in other words, see if you can do it with all whites (always possible for any train), or all reds, or all light greens, etc. You should be able to make six different trains each train is made up of a single color. Draw a picture of each of these trains under the Hot Pink one shown. I've drawn two trains for you already one is simply the hot pink strip (a train consisting of just one c-strip), and a second is made up of three purple c-strips.



b. Your work from exercise 5 should help complete this exercise. Take each train, one at a time, and put each strip in a vertical position, but side by side to form a rectangle. Then, find a c-strip that fits across the width (or top) of the rectangle. This should tell you from which multiplication problem each train was formed. Write an equation using c-strips and then translate to an equation with numbers. For instance, for train 2, first I would make a rectangle out of the three purple c-strips. Next, I would try to find a c-strip to fit across the top, which would be light green. Therefore, train 2 was formed from the multiplication $L \times P$ remember that since the train is formed with purple c-strips, P is the *second* letter in the multiplication. So, the equation in c-strips is $L \times P = H$, and the numerical equivalent is $3 \times 4 = 12$. Follow this same procedure for the other four trains you made in part a.

Train 1 illustrates the multiplication $W \times H = H$, or $1 \times 12 = 12$. (Note that if the hot pink strip is vertical, the white c-strip fits across the top.)

Train 2 illustrates the multiplication $L \times P$ = H, or $3 \times 4 = 12$

Train 3 illustrates the multiplication ____ × ____ = H, or ____ × ____ = 12

Train 4 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 5 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 6 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

? Exercise 7

For train 2 in exercise 6, I made a train of purple c-strips equal in length to the hot pink c-strip. Now, make a train of purple c-strips equal in length to the brown (N) c-strip. Take the train of the purple c-strips and make it into a rectangle.

Which color c-strip measures across the width of this rectangle? _______

Put the answer (abbreviation for the c-strip) in the blank to solve the multiplication: ____ × P = N.

Now, we'll explore more about the equations formed in exercise 5. Consider part a. You should have written $D \times R = H$. This is analogous to the statement $6 \times 2 = 12$. But, if you actually look at the original train that was formed to find the answer, you would see that there were 6 red c-strips that made a train equal in length to the hot pink c-strip. This gives rise to one of the whole number definitions of multiplication. *Multiplication* is really just the same thing as ***Repeated Addition.***

Definition: Multiplication of Whole Numbers using the repeated addition approach

If a and b are whole numbers, then a × b = b + b+ b+ b + ... + b, where there are a addends of b in this sum.

I like to think of using the definition like this:

3 × 4 means 3 fours added together or 3 × 4 = 4 + 4 + 4 = 12

4 × 3 means 4 threes added together, or 4 × 3 = 3 + 3 + 3 + 3 = 12

5 × 2 means 5 twos added together, or 5 × 2 = 2 + 2 + 2 + 2 + 2 = 10

2 × 5 means 2 fives added together, or 2 × 5 = 5 + 5 = 10

Note that although 3 × 4 gives the same answer as 4 × 3, it does not *mean* the same thing! One means 4 + 4 + 4 and the other means 3 + 3 + 3 + 3. Before we delve any further into this discussion, I want you to use the repeated addition definition to compute the following:

? Exercise 8

Use the repeated addition definition of multiplication to compute the following. First, write out the meaning of the multiplication, and then compute the answer.

a. 6 × 3 =

b. 2 × 9 =

c. 9 × 2 =

Now, it may seem obvious to you that 3 × 4 is the same answer as 4 × 3. That's because you've no doubt been using the commutative property of addition for years. Whether or not you really thought about why it was true is another thing. But, although 3 × 4 gives the same answer as 4 × 3, it does not mean the same thing. One means 4 + 4 + 4 and the other means 3 + 3 + 3 + 3. To a child just learning about addition and multiplication, there is no reason for them to conclude that one sum (4 + 4 + 4) happens to give the same answer as the other sum (3 + 3 + 3 + 3). For instance, if I was asked to compute the following two sums, it wouldn't be evident to me at all that the answer to each would be end up being the same:

17 + 17 + 17 + 17 and 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4

Actually, I find it quite amazing that they are equal. But, indeed they are, because the first sum comes from the multiplication problem 4 × 17 (since there are 4 seventeens added together), and the second sum comes from the multiplication problem 17 × 4 (since there are 17 fours added together). Later, when we discuss a more geometrical definition for multiplication, it becomes obvious why the commutative property of multiplication is true.

The commutative property of multiplication for whole numbers states that if a and b are *any* two numbers, then

$$a \times b = b \times a.$$

So, the very exciting thing about the commutative property of multiplication is that, given a problem like 100 × 7, where it would take a long time to write out and compute the answer using the strict definition of multiplication by adding 7 over and over again, we can use our knowledge of the commutative property, that is 100 × 7 = 7 × 100, and compute the second multiplication instead. Wouldn't you agree that 100 + 100 + 100 + 100 + 100 + 100 + 100 is quicker and easier to compute than adding together 100 sevens? Thank goodness for the commutative property of multiplication!!!

Before going on to another property, let's practice verifying that the commutative property of multiplication is true for some numbers by using the repeated addition definition of multiplication. The commutative property requires two numbers (they don't have to be two different numbers) and there are two sides of an equation to verify. The difference between the expressions a × b and b × a is that the order of the numbers is different. Remember that the commutative property means there has been an order change. To verify the commutative property of multiplication, you need to use the repeated addition definition of multiplication and add from left to right, one number at a time. The next example shows how to verify the commutative property of multiplication, given a set of two numbers.

✓ Example

Verify the commutative property of multiplication is true for the given set of numbers. You'll have to remember how to add in base six to do part b.

a. {5, 3}

Solution

I need to show $5 \times 3 = 3 \times 5$. Since $5 \times 3 = 3 + 3 + 3 + 3 + 3 = 6 + 3 + 3 + 3 = 9 + 3 + 3 = 12 + 3 = 15$ and since $3 \times 5 = 5 + 5 + 5 = 10 + 5 = 15$, then $5 \times 3 = 3 \times 5$, because both equal 15.

b. $\{3_{\text{six}}, 4_{\text{six}}\}$

Solution

I need to show $3_{\text{six}} \times 4_{\text{six}} = 4_{\text{six}} \times 3_{\text{six}}$ Since $3_{\text{six}} \times 4_{\text{six}} = 4_{\text{six}} + 4_{\text{six}} + 4_{\text{six}} = 12_{\text{six}} + 4_{\text{six}} = 20_{\text{six}}$ and since $4_{\text{six}} \times 3_{\text{six}} = 3_{\text{six}} + 3_{\text{six}} + 3_{\text{six}} + 3_{\text{six}} = 10_{\text{six}} + 3_{\text{six}} + 3_{\text{six}} = 13_{\text{six}} + 3_{\text{six}} = 20_{\text{six}}$, then $3_{\text{six}} \times 4_{\text{six}} = 4_{\text{six}} \times 3_{\text{six}}$, because both equal 20_{six}.

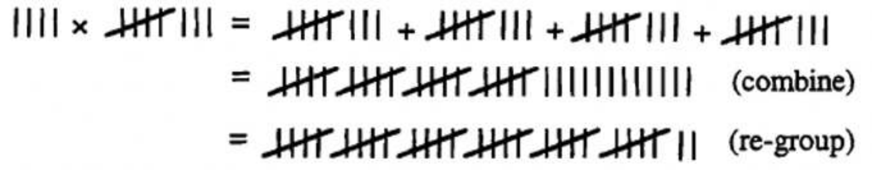
? Exercise 9

Verify the commutative property of multiplication by writing an equation that must be true and using the repeated addition definition of multiplication is true using the given set of numbers. Simplify each expression separately; then show they are equal.

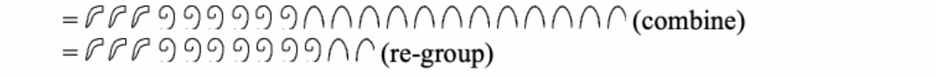
a. {4, 3}
b. $\{2_{\text{five}}, 3_{\text{five}}\}$
c. Make up a set in a base other than 10 and verify the commutative property of multiplication.

We can also use the **repeated addition definition** of multiplication to multiply in other numeration systems. Study the following examples and then work through the problems in exercise 10.

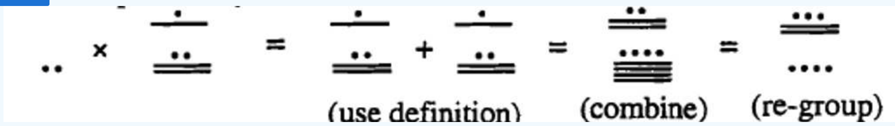
✓ Example in Tally



Example in Egyptian



Example in Mayan

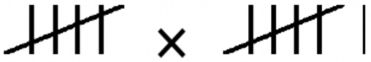


Notice that all of these examples are done completely in the numeration system given. All of the steps are shown in that numeration system. This is how you should do the problems in exercise 10. If you want, you can check the answer after you complete the problem by converting to base ten. For instance, the Mayan example in base ten is 2 × 132. So the answer should be 264. If you convert the final answer to base ten, it is also 264.

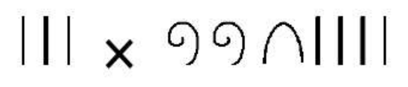
Exercise 10

Use the repeated addition definition of multiplication to compute the following. Make sure you write out the meaning of the multiplication **in the system given** (see examples on previous page), showing all of the work and exchanges (if necessary), to compute the answer. Do not do the problem in base ten, although you can use base ten to check the answer when you are done.

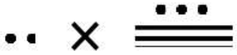
a.



b.



c.



Exercise 11

Next, we will form the trains for $(R \times L) \times P$ and for $R \times (L \times P)$. We can only work on one at a time. In both cases, *you must do what's inside the parentheses first.*

a. First, we'll do $(R \times L) \times P$. Begin by forming the train $R \times L$ to use in part b.

b. Next, we need to form the train $(R \times L) \times P$. The first part, $R \times L$, which you formed in part a, should be in a horizontal position, and the purple c-strip goes underneath it, perpendicular to the train $R \times L$. Note that although $R \times L$ is made up of more than one c-strip, it still forms one train across the top. In the space to the right, draw a picture of what this looks like so far.

c. Do the multiplication by filling in the rest of the purple c-strips in the vertical position under the train $R \times L$; then form a train out of the rectangle of purple c-strips. **Keep this train until you have finished all of exercise 11.** How many *purple c-strips* are in the train formed by doing the multiplication $(R \times L) \times P$?

d. Now, we'll do $R \times (L \times P)$. Begin by forming the train $L \times P$ to use in part e.

e. Next, form the train $R \times (L \times P)$. Put the first part, R, in the horizontal position. The second part, $L \times P$, formed in part d, should go in the vertical position as one long train, underneath and perpendicular to the train R. Don't separate the purple strips in $L \times P$ line up as one long vertical train (made up of more than one purple C-strip) under the red c-strip. To the right, draw a picture of what this looks like so far.

f. Do the multiplication by filling in the rest of the purple c-strips to form the rectangle; then make a train out of the rectangle of purple c-strips. How many purple c-strips are in the train formed by doing the multiplication $R \times (L \times P)$?

g. Compare the length of train $(R \times L) \times P$, from part c with length of train $R \times (L \times P)$, from part f. What do you notice?

h. From your observation, write an equation using the c-strips:

i. Translate the equation from part h to one with numbers:

j. Write the left side of the equation you wrote down in part i (which has numbers, not c-strips), and simplify by following the order of operations do what is in parentheses first. Then, do the same for the right side of the equation.

Left side:

Right side:

k. Are the two expressions you simplified in part (j) equal? If so, you have verified the equation from part (i) is true

? Exercise 12

Next, we will form the trains for $(Y \times R) \times L$ and for $Y \times (R \times L)$.

a. First, we'll do $(Y \times R) \times L$. Begin by forming the train $Y \times R$.

b. Next, we need to form the train $(Y \times R) \times L$. The first part, $Y \times R$, which you formed in part a, should be in a horizontal position, and the light green c-strip should go underneath it, perpendicular to the train $Y \times R$. Note that although $Y \times R$ is made up of more than one c-strip, it still forms one train that is across the top. That's okay. Below, draw a picture of what this looks like so far.

c. Do the multiplication by filling in the rest of the light green c-strips and then making a train out of the rectangle of light green c-strips. **Keep this train until you have finished all of exercise 12.** How many light green c-strips are in the train formed by doing the multiplication $(Y \times R) \times L$?

d. Now, we'll do $Y \times (R \times L)$. Begin by forming the train $R \times L$.

e. Next, form the train $Y \times (R \times L)$. Put the first part, Y, in the horizontal position. The second part, $R \times L$, formed in part d, should go in the vertical position as one long train, underneath and perpendicular to the train Y. Don't separate the light green strips in $R \times L$ line up as one long vertical train (made up of more than one light green C-strip) under the yellow c-strip. To the right, draw a picture of what this looks like so far.

f. Do the multiplication by filling in the rest of the light green c-strips to form the rectangle, and then making a train out of the rectangle of light green c-strips. How many light green c-strips are in the train formed by the multiplication $Y \times (R \times L)$?

g. Compare the length of train $(Y \times R) \times L$, from part c with the length of train $Y \times (R \times L)$, from part f. What do you notice?

h. From your observation, write an equation using the c-strips:

i. Translate the equation from part h to one with numbers:

j. Write the left side of the equation you wrote down in part i (which has numbers, not c-strips), and simplify by following the order of operations do what is in parentheses first. Then, do the same for the right side of the equation.

Left side:

Right side:

k. Are the two expressions you simplified in part (j) equal? If so, you have verified the equation from part (i) is true.

Exercises 11 and 12 illustrate the *associative property of multiplication* for trains.

The *associative property of multiplication* for trains states that if r, s and t are *any* three trains, then $(r \times s) \times t = r \times (s \times t)$.

Before going on to any more properties, we'll practice verifying that the associative property of multiplication is true for some numbers in base ten later on, we'll do it in other bases, too. The associative property requires three numbers (they don't have to be three different numbers) and there are different expressions on each side of the equation. The difference between the left and right sides is that the parentheses are around a different pair of numbers. Remember that the associative property means there is a parentheses change, whereas the commutative property means there has been an order change. To verify, you need to use the order of operations when simplifying each side. If each expression simplifies to same thing, the equation is true. In the order of operations, you always simplify what is in parentheses first. The following are some examples of how to verify the associative property of multiplication, given a set of three numbers.

The *associative property of multiplication* for whole numbers states that if a, b and c are *any* three numbers, then $(a \times b) \times c = a \times (b \times c)$.

Example 1

Verify the associative property of multiplication is true using the three numbers {4, 5, 7} in your example.

Note: Since the associative property states $(a \times b) \times c = a \times (b \times c)$, one of the numbers given is put in for a, the other for b and the other for c. The choice is yours. For example, we can show any of the following six equations are true:

a. $(4 \times 5) \times 7 = 4 \times (5 \times 7)$ or d. $(4 \times 7) \times 5 = 4 \times (7 \times 5)$
b. $(5 \times 4) \times 7 = 5 \times (4 \times 7)$ or e. $(7 \times 4) \times 5 = 7 \times (4 \times 5)$
c. $(7 \times 5) \times 4 = 7 \times (5 \times 4)$ or f. $(5 \times 7) \times 4 = 5 \times (7 \times 4)$.

If this was a question on a test, you would only have to verify one of the six equations shown there are always six possibilities one might choose, depending on which numbers you put in for a, b or c. I will show you how to do it if you decided to verify the equation (e). I'll also verify it for another equation (f). Then, for practice, you get to verify the other four equations.

One Solution to Example 1: I'll show: $(7 \times 4) \times 5 = 7 \times (4 \times 5)$

Left side: $(7 \times 4) \times 5 = 28 \times 5 = 140$ and Right side: $7 \times (4 \times 5) = 7 \times 20 = 140$

Since I used the order of operations to simplify each expression, and both expressions are equal, the associative property is true using the numbers 4, 5 and 7.

Here is a different way to verify it using a different equation.

Another Solution to Example 1: I'll show: $(5 \times 7) \times 4 = 5 \times (7 \times 4)$

Left side: $(5 \times 7) \times 4 = 35 \times 4 = 140$ and Right side: $5 \times (7 \times 4) = 5 \times 28 = 140$

Since I used the order of operations to simplify each expression, and both expressions are equal, the associative property is true using the numbers 4, 5 and 7.

Exercise 13

Verify the associative property of multiplication is true using the numbers 4, 5 and 7, by doing the order of operations on each side. Do this four times, using the equations a, b, c and then d, as shown on the previous page.

a. Verify: $(4 \times 5) \times 7 = 4 \times (5 \times 7)$
b. Verify: $(5 \times 4) \times 7 = 5 \times (4 \times 7)$
c. Verify: $(7 \times 5) \times 4 = 7 \times (5 \times 4)$
d. Verify: $(4 \times 7) \times 5 = 4 \times (7 \times 5)$

To verify the associative property of multiplication is true by example, first you must write an equation that must be true using any three numbers (or by using three particular numbers given to you.) Then, by doing the order of operations on each side (each individual expression), you must show each side of the equation gives the same result.

✓ Example 2

Verify the associative property is true using the numbers 10, 8 and 3.

a. I will show this equation is true: $(10 \times 8) \times 3 = 10 \times (8 \times 3)$

Since $(10 \times 8) \times 3 = 80 \times 3 = 240$ and $10 \times (8 \times 3) = 10 \times 24 = 240$, then the equation $(10 \times 8) \times 3 = 10 \times (8 \times 3)$ is true.

? Exercise 14

Verify the associative property is true using the numbers given.

a. {9, 7, 6}

b. any three numbers of your choice.

Time to work on another property...

STEP 1: Form the train $P \times (R + L)$. To do this, the purple strip should be place in a horizontal position, and the train R + L should be placed in a vertical position underneath the purple c-strip. Then, the rectangle needs to filled in. Lastly, make a long train out of the rectangle. Note that this train is made up of 8 c-strips (4 reds and 4 light greens). **Save this train**.

STEP 2: Form two trains: $P \times R$ and $P \times L$. Then add the two trains together (in other words, $P \times R + P \times L$) to form one long train. Note that this train is made up of 8 c-strips (4 reds and 4 light greens). Compare the length of the train formed in Step 1 with this train. Are they are the same length? Are they made up of the same exact c-strips?

We have just shown and verified that $P \times (R + L) = P \times R + P \times L$.

? Exercise 15

a. Form the train $L \times (P + Y)$ by first forming the train P + Y. Then do the multiplication $L \times (P + Y)$. Exactly what c-strips make up this train?

b. Form the train $L \times P$ and $L \times Y$. Now add them together to form one long train. Exactly what c-strips make up this train?

c. Compare the trains from part a and b, and write an equation (with c-strips).

d. Translate the equation from part c by writing an equation using numbers.

e. Verify the equation you wrote in part d by first simplifying the left hand side of the equation using the order of operations and then by simplifying the right hand side as well. If the answer to each expression is the same, you have verified the equation.

left side:

right side:

? Exercise 16

a. Form the train $R \times (W + D)$ by first forming the train W + D. Then do the multiplication $R \times (W + D)$. Exactly what c-strips make up this train?

b. Form the train $R \times W$ and $R \times D$. Now add them together to form one long train. Exactly what c-strips make up this train?

c. Compare the trains from part a and b, and write an equation (with c-strips).

d. Translate the equation from part c by writing an equation using numbers.

e. Verify the equation you wrote in part d by first simplifying the left hand side of the equation using the order of operations and then by simplifying the right hand side as well. If the answer to each expression is the same, you have verified the equation.

left side:

right side:

The above problems illustrate the distributive property of multiplication over addition for trains.

The distributive property of multiplication over addition for trains states that if r, s and t are *any* three trains (each train may be made up of one or more c-strips)**, then** $r\times(s+t)=(r\times s)+(r\times t)$.

Now, it's time to practice verifying the distributive property of multiplication over addition is true for some numbers in base ten — later on, we'll do it in other bases, too. The distributive property requires three numbers (they don't have to be three different numbers) and there are two sides of an equation to verify. Note the difference between the left and right sides. To verify, you need to use the order of operations when simplifying each side. In the order of operations, you always simplify what is in parentheses first. If each expression simplifies to same thing, the equation is true. The following are some examples of how to verify the distributive property of multiplication, given a set of three numbers.

The distributive property of multiplication over addition for whole numbers states that if a, b and c are *any* three numbers, then $a\times(b+c)=(a\times b)+(a\times c)$.

✓ Example 1

Verify the distributive property of multiplication is true using the three numbers {4, 5, 7} in your example.

Note: Since the distributive property states $a\times(b+c)=(a\times b)+(a\times c)$, one of the numbers given is put in for a, the other for b and the other for c. The choice is yours. For example, we can show any of the following six equations are true:

a. $4\times(5+7)=(4\times 5)+(4\times 7)$ or d. $5\times(7+4)=(5\times 7)+(5\times 4)$
b. $4\times(7+5)=(4\times 7)+(4\times 5)$ or e. $7\times(4+5)=(7\times 4)+(7\times 5)$
c. $5\times(4+7)=(5\times 4)+(5\times 7)$ or f. $7\times(5+4)=(7\times 5)+(7\times 4)$

If this was a question on a test, you would only have to verify one of the six equations shown. There are always six possibilities one might choose, depending on which numbers you put in for a, b or c. I will show you how to do it if you decided to verify the equation (c). I'll also verify it for another equation (f). Then, for practice, you get to verify the other four equations.

One Solution to Example 1: I'll show: $5\times(4+7)=(5\times 4)+(5\times 7)$

Left side: $5\times(4+7)=5\times(11)=55$ and Right side: $(5\times 4)+(5\times 7)=20+35=55$

Since I used the order of operations to simplify each expression, and both expressions are equal, the distributive property is true using the numbers 4, 5 and 7.

Here is a different way to verify it using a different equation.

Another Solution to Example 1: I'll show: $7\times(5+4)=(7\times 5)+(7\times 4)$

Left side: $7\times(5+4)=7\times 9=63$ and Right side: $(7\times 5)+(7\times 4)=35+28=63$

Since I used the order of operations to simplify each expression, and both expressions are equal, the distributive property is true using the numbers 4, 5 and 7.

? Exercise 17

Verify the distributive property of multiplication is true using the numbers 4, 5 and 7, by doing the order of operations on each side. Do this four times, using the equations a, b, d and then e, as shown on the previous page.

a. Verify: $4\times(5+7)=(4\times 5)+(4\times 7)$
b. Verify: $4\times(7+5)=(4\times 7)+(4\times 5)$
c. Verify: $5\times(7+4)=(5\times 7)+(5\times 4)$
d. Verify: $7\times(4+5)=(7\times 4)+(7\times 5)$

To verify the distributive property of multiplication is true by example, first you must write an equation that must be true using any three numbers (or by using three particular numbers given to you.) Then, by doing the order of operations on each side (each individual expression), you must show each side of the equation gives the same result.

✓ Example 2

Verify the distributive property is true using the numbers 10, 8 and 5.

a. I will show this equation is true: $10 \times (8+5) = (10 \times 8) + (10 \times 5)$

Solution

Since $10 \times (8+5) = 10 \times 13$ and $(10 \times 8) + (10 \times 5) = 80 + 50$, then the equation $10 \times (8+5) = (10 \times 8) + (10 \times 5)$ is true.

? Exercise 18

Verify the distributive property of multiplication over addition is true for the three numbers specified:

a. {7, 8, 12}
b. {28, 65, 35}
c. any three numbers of your choice.

? Exercise 19

Use the definition of multiplication for trains to compute the following. Then translate to make an equation using numbers.

a. $W \times R =$ _____ translates to _______________

b. $W \times B =$ _____ translates to _______________

c. $W \times D =$ _____ translates to _______________

d. $W \times S =$ _____ translates to _______________

e. $N \times W =$ _____ translates to _______________

f. $H \times W =$ _____ translates to _______________

Exercise 19 illustrates that **1 is the identity element for multiplication**. In other words, for any number m, $\mathbf{m} \times \mathbf{1} = \mathbf{m}$ **and** $\mathbf{1} \times \mathbf{m} = \mathbf{m}$.

? Exercise 20

The identity element for multiplication if we are referring to trains is what you fill in the blank to make the following statement true. Fill in the blanks to find the identity.

For any train t, $\mathbf{t}\times$ _____ $= \mathbf{t}$ **and** _____ $\times \mathbf{t} = \mathbf{t}$.

The distributive property of multiplication over addition is used quite often to simplify arithmetic problems. Consider the following examples:

Someone who needed to multiply 57×102 might find it easier to think of the problem as $57 \times (100+2) = (57 \times 100) + (57 \times 2) = 5700 + 114 = 5814$.

Someone who needed to compute $47 \times 38 + 47 \times 62$ might realize this is really just the same as $47 \times (38+62) = 47 \times 100 = 4700$.

This next example illustrates that multiplication also distributes over subtraction:

Someone who needed to multiply 38×99 might find it easier to think of the problem as $38 \times (100-1) = (38 \times 100) - (38 \times 1) = 3800 - 38 = 3762$.

Exercise 21

Use the distributive property of multiplication over addition (or subtraction) to rewrite the following problems. Then, simplify. For d, e and f, make up three of your own problems where it would be easier to use the distributive property first before simplifying. Show how to use the property and simplify.

a. 764×999

b. 324×102

c. $83 \times 74 + 83 \times 26$

d.

e.

f.

Exercise 22

So far, you have worked with three operations; addition, subtraction and multiplication. You have learned many properties. Take a little time to reflect on and review these by writing the properties or answering the questions as stated below:

a. State the associative property of multiplication and provide an example.

b. State the commutative property of addition and provide an example.

c. State the identity element for addition and what it means and provide an example.

d. State the commutative property of multiplication and provide an example.

e. State the distributive property of multiplication over addition and provide an example.

f. State the identity element for multiplication and provide an example.

g. Provide a counterexample to show that subtraction is not commutative

h. State the associative property of addition.

i. Provide a counterexample to show that subtraction is not associative.

There is another way to define multiplication using Set Theory. In order to do this, we have to remember how to take the **Cartesian product** of two sets, A and B.

The **Cartesian product** of set A with set B, which is written $A \times B$ and is read as "A cross B" is the set of *all possible* ordered pairs (a,b), where $a \times A$ and $b \times B$.

If you are finding the Cartesian product, the answer is a set that contains ordered pairs.

Example 1

If A = {x, y, z} and B = {a, b}, find $A \times B$.

Solution

$A \times B$ = {(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}

Example 2

If E = {1, 2} and F = {2, 3}, find $E \times F$.

Solution

$E \times F$ = {(1, 2), (1, 3), (2, 2), (2, 3)}

Exercise 23

Write the Cartesian product. Each answer is a set containing ordered pairs!

a. {3, 4} × {2, 6} = _______

b. {6, 7, 8, 9} × {5} = ________

c. {r, s, t} × { } = _________

d. {a} × {a} = _________

e. {x, y} × {x, y} = ________

f. {1, 3, 5} × {1, 3, 5} = ________

Set Theory Definition for Multiplying two whole numbers:

For any two sets A and B, $\mathbf{n(A) \cdot n(B) = n(A \times B)}$.

In other words, to multiply two numbers, a and b, write one set A which has a elements in it and another set B that has b elements in it. To find the product, write the Cartesian
product of A and B and count the elements in the set.
Note: There are no restrictions on what elements you choose for sets A and B. They can be disjoint or they can have elements in common. It’s up to you.

Example 1

Use the set theory definition of multiplication to show that $3 \cdot 2 = 6$

Solution

Let A = {v, w, x} and B = {1, 2}. Since n(A) = 3 and n(B) = 2, then

$$\begin{aligned} 3 \cdot 2 &= n(A) \cdot n(B) && \text{by substituting n(A)for 3 and n(B)for 2} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{(v,1),(v,2),(w,1),(w,2),(x,1),(x,2)\}) && \text{by computing } A \times B \\ &= 6 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $3 \cdot 2 = 6$.

Example 2

Use the set theory definition of multiplication to show that $1 \cdot 3 = 3$

Solution

Let A = {v} and B = {1, 2, 3}. Since n(A) = 1 and n(B) = 3, then

$$\begin{aligned} 1 \cdot 3 &= n(A) \cdot n(B) && \text{by substituting n(A)for 1 and n(B)for 3} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{(v,1),(v,2),(v,3)\}) && \text{by computing } A \times B \\ &= 3 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $1 \cdot 3 = 3$

Example 3

Use the set theory definition of multiplication to show that $2 \cdot 0 = 0$

Solution

Let A = {v, w} and B = { }. Since n(A) = 2 and n(B) = 0, then

$$\begin{aligned} 2 \cdot 0 &= n(A) \cdot n(B) && \text{by substituting n(A)for 2 and n(B)for 0} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ &= n(\{\ \}) && \text{by computing } A \times B \\ &= 0 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $2 \cdot 0 = 0$

Example 4

Use the set theory definition of multiplication to show that $4 \cdot 3 = 12$

Solution

Let A = {e, f, g, h} and B = {g, h, i }. Since n(A) = 4 and n(B) = 3, then

$$\begin{aligned} 4 \cdot 3 &= n(A) \cdot n(B) && \text{by substituting n(A)for 2 and n(B)for 0} \\ &= n(A \times B) && \text{by the set theory definition of multiplication} \\ , h), (e, i), (f, g), (f, h), (f, i), (g, g), (g, h), (g, i), (h, g), (h, h), (h, i)\}) && && \text{by computing } A \times B \\ &= 12 && \text{by counting the elements in } A \times B \end{aligned}$$

Therefore, $4 \cdot 3 = 12$

Exercise 24

Use the set theory definition of multiplication to verify each multiplication. Show and justify each step. Only the solution to 24a is shown in the solutions.

a. $5 \cdot 2 = 10$

b. $3 \cdot 1 = 10$

c. $0 \cdot 2$

Here's a more geometrical approach to defining multiplication of two whole numbers.

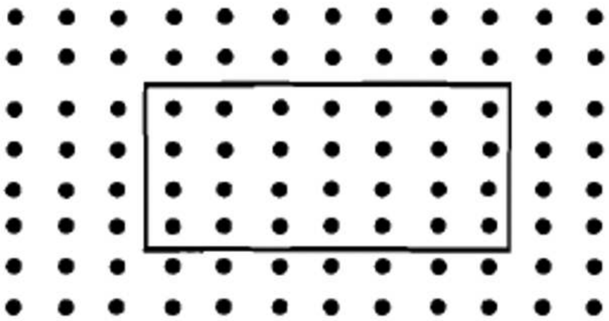
Definition

To find the product of any two whole numbers, m and n, make a rectangular array of objects having m rows and n columns. The product of m and n, written $m \times n$, equals the number of objects in the array.

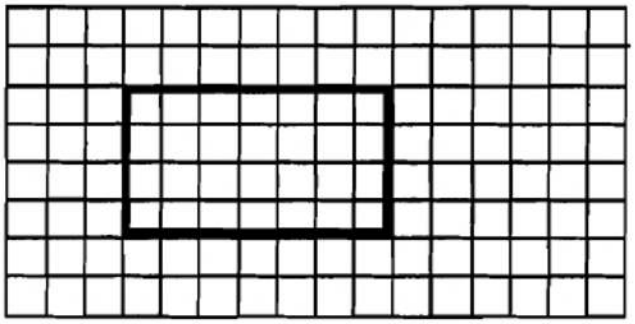
The cool thing about using this geometrical approach is that *any* objects can be used, and one only needs to be able to count to get the answer. You don't need to use addition in order to multiply! I personally advocate having lots of graph paper around for anyone just learning to multiply. To compute 4×7, you or the learner can mark off a rectangle on the graph paper that has 4 rows and 7 columns of boxes. The number of boxes in the rectangle is the answer to the problem. This is a useful tool for first learning a particular multiplication fact, or for anyone (adults, too) who forgot a fact and needs to figure it out again. The "objects" used on a graph paper are the individual boxes on the graph paper.

Here is how six different students might geometrically show how to multiply 4×7.

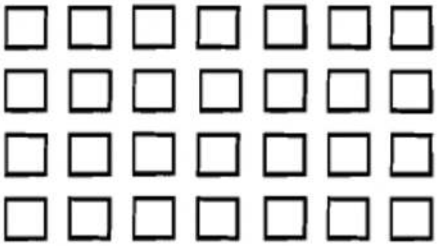
This student took some dotted paper and marked off a rectangular array containing 4 rows and 7 columns of dots.



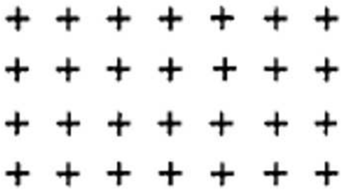
This student took a piece of graph paper and then marked off a rectangle having four rows and seven columns.



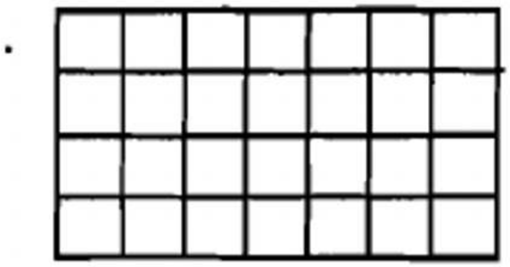
This student arranged four rows and seven columns of squares into a rectangular array.



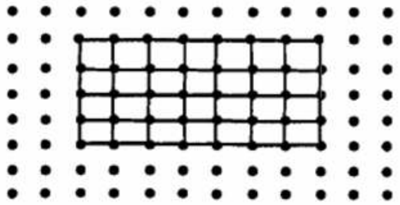
This student arranged four rows and seven columns of objects (addition signs) into a rectangular array.



This student drew a rectangular array having four rows and seven columns.



This student took some dotted paper and then drew a rectangular array having four rows and seven columns.

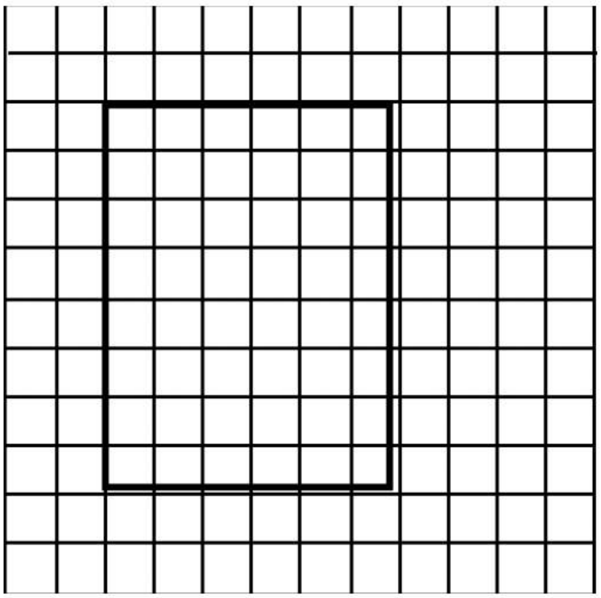


There is no point to making anyone stress out if they forgot, didn't memorize or couldn't recall the answer to 4×7. The point is to *KNOW WHAT MULTIPLICATION MEANS* so you **can** figure a problem out again if you have to. Of course, there is merit to using flashcards and eventually memorizing basic multiplication facts like your multiplication tables. If not, it takes longer to do problems, you aren't able to focus on more advanced issues, and you might not trust your own mind. You shouldn't have to think about multiplication every time you have to multiply two numbers together, but you should *be able* to think about it. Knowing ***how*** to get the answer is better than having memorized a bunch of facts if you don't know where they came from, what multiplication really means, or how to figure it out again in case you forget the answer to some fact. And, as far as I'm concerned, nothing's wrong with using your fingers, either! Let people do what is comfortable for them, if they understand and can get the right answer, that's GREAT!

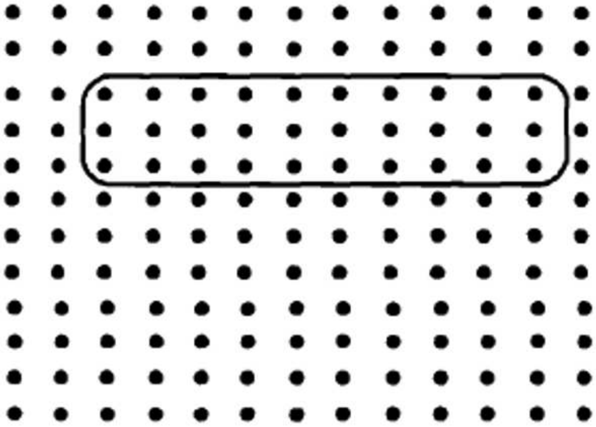
Exercise 25

Write the multiplication problem represented by each geometrical representation:

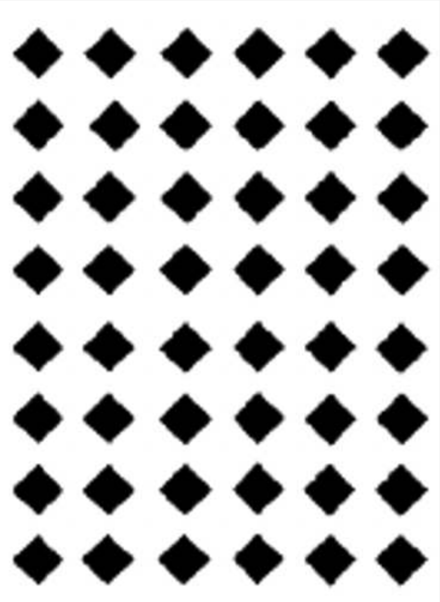
a. _____



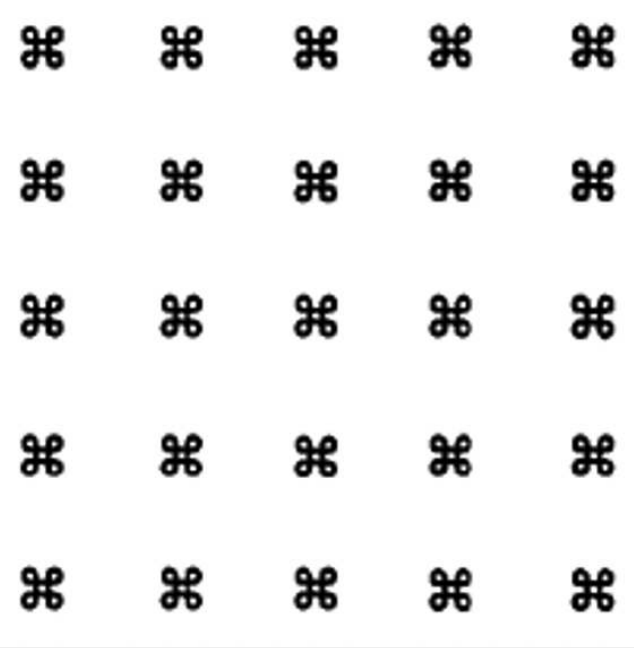
b. _____



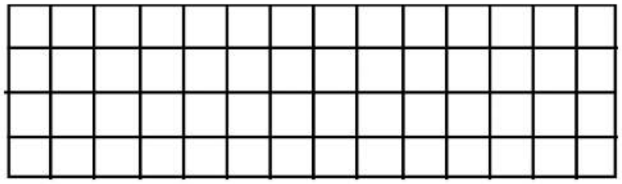
c. _____



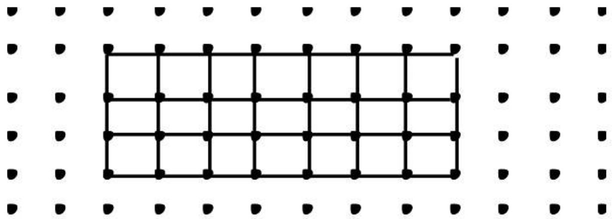
d. _____



e. _____



f. _____



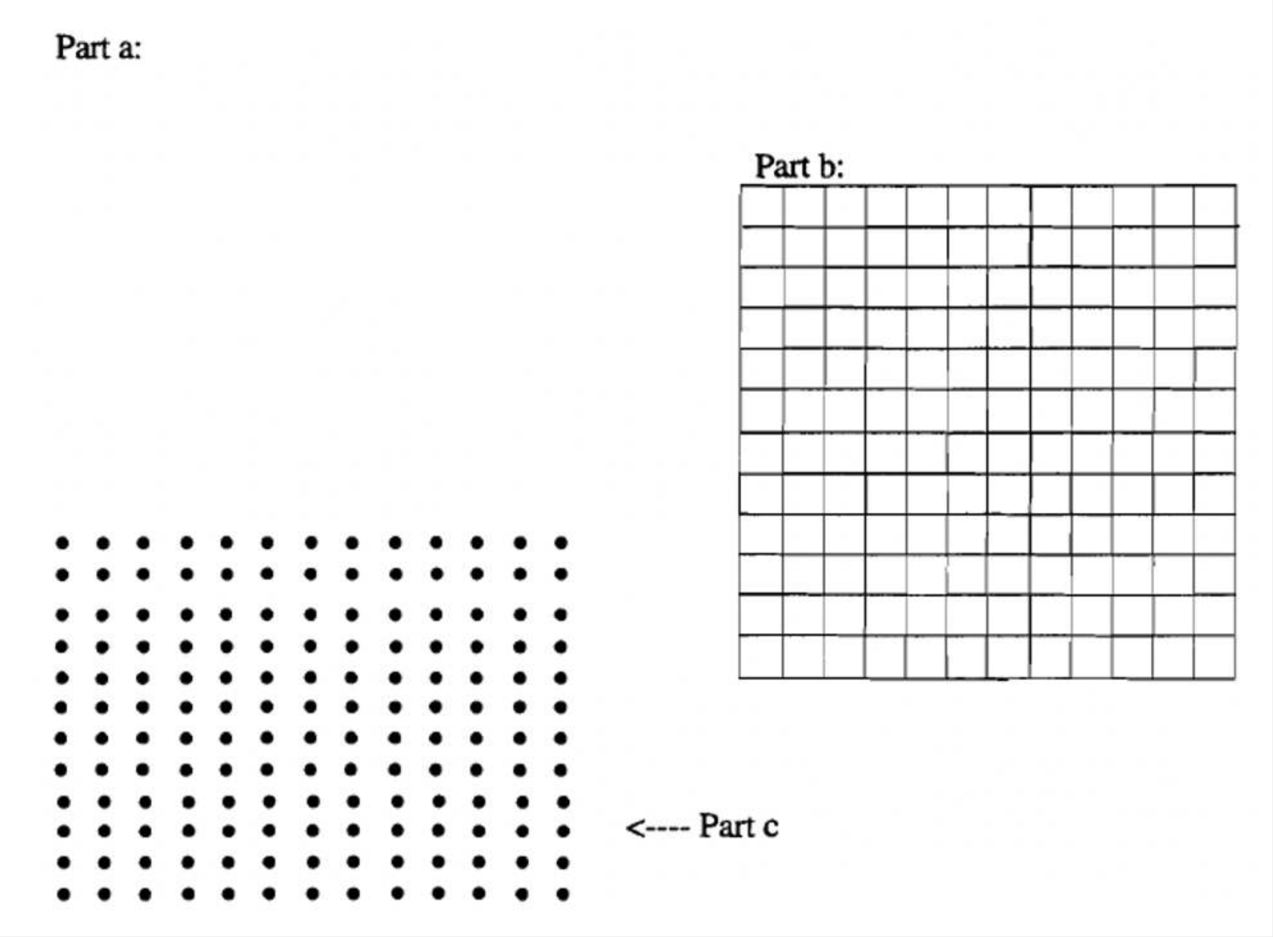
Exercise 26

Show two different geometrical representations for 4×3

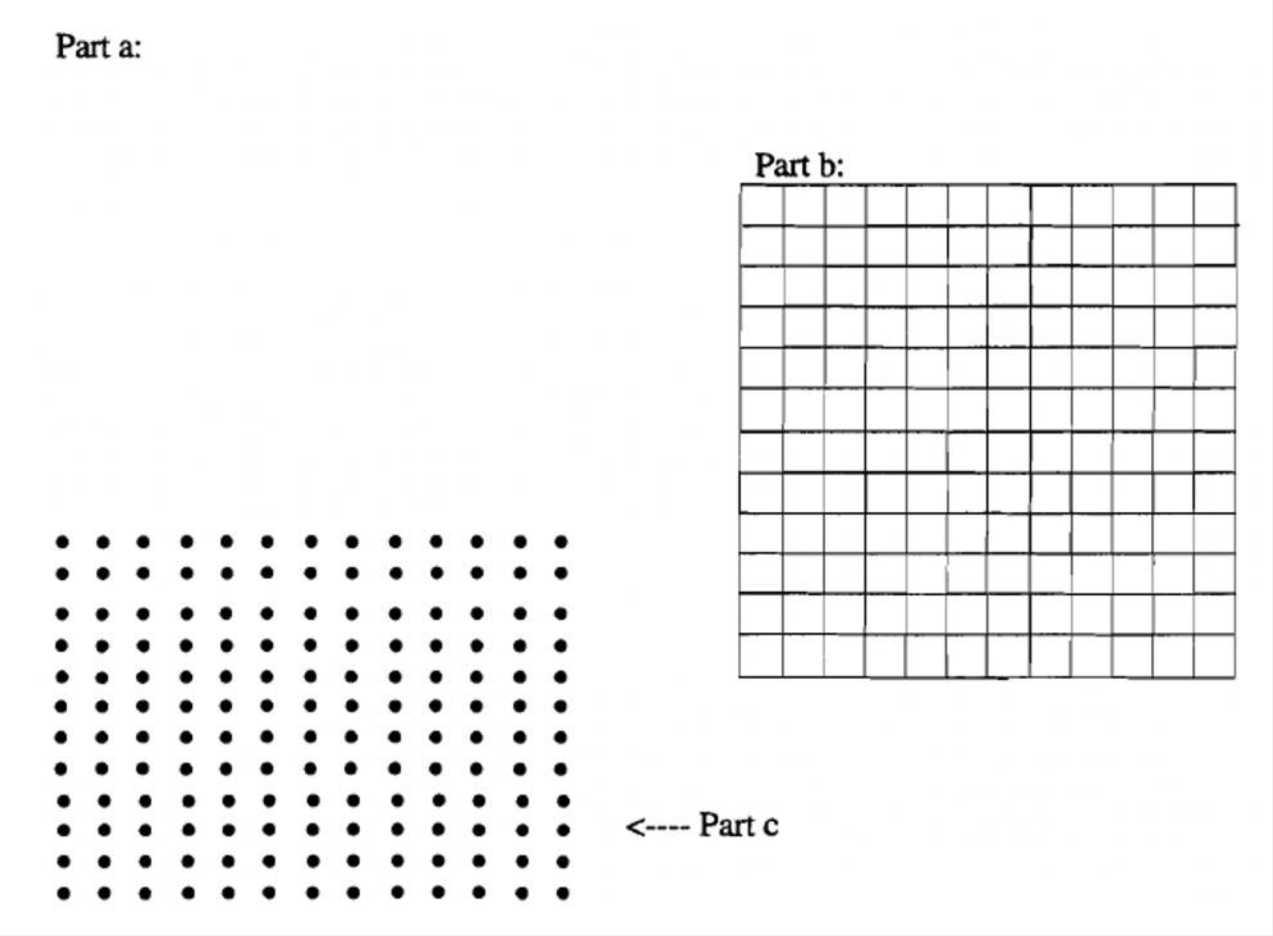
Exercise 27

For each multiplication, do the following to represent the multiplication: For part a, draw a rectangular array of objects, for part b use the graph paper, and for part c use the dotted paper. There are two ways you can choose to use the dotted paper.

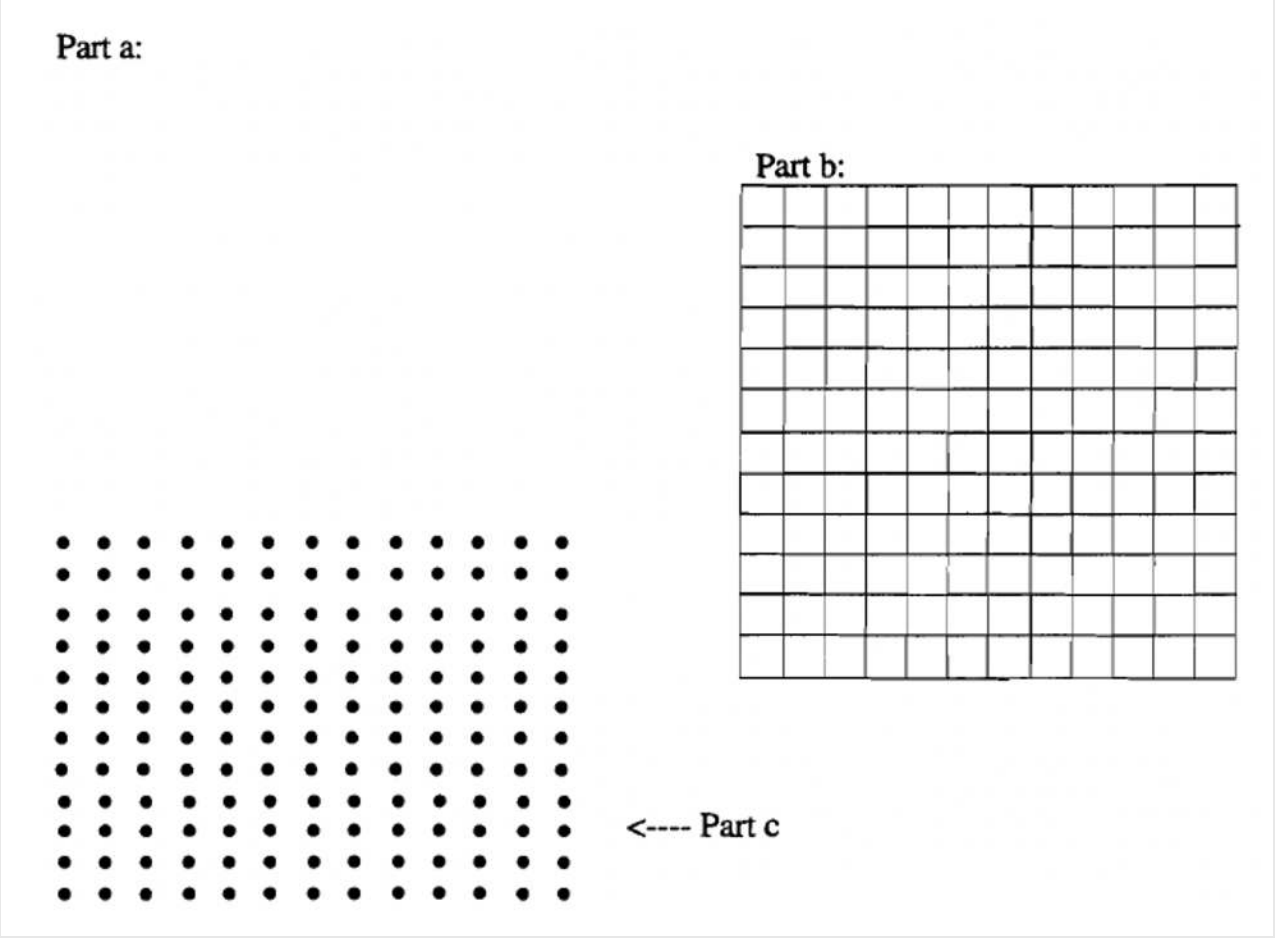
a. 2×6



b. 5×3


Part a:
Part b:
<---- Part c

c. 3×7



Exercise 28

a. Draw a rectangular array of boxes (a rectangle) on the graph paper provided to do the multiplication 4×6:

b. Turn your page sideways and look at the rectangle. Draw what the picture looks like:

c. What multiplication problem is now represented? ________________

Exercise 28 should convince you that multiplication is commutative. In fact, the commutative property of multiplication is easiest to see using the geometrical approach. A rectangular array of objects having m rows and n columns represents the multiplication $m \times n$. Once the rectangle is rotated 90° (or looked at sideways), there are now n rows and m columns, which represents the multiplication $n \times m$. But of course, it's really the same rectangle of objects, so the number of objects in $m \times n$ equals the number of objects in $n \times m$. Therefore, $m \times n = n \times m$. Okay, just for fun, one more time:

The commutative property of multiplication for whole numbers states that if a and b are *any* two numbers, then $a \times b = b \times a$.

Wow! Isn't this exciting? There are several ways to think about and define multiplication, and we can show the commutative property still holds each time. Does it get any better than this? Well,...YES...but that's another story.

It's a little bit trickier to show on a two-dimensional page that the associative property of multiplication holds. With your work with the C-strips, you know multiplication is associative. Here's one way to geometrically model the multiplication of three numbers.

Definition

To find the product of any three whole numbers, $(a \times b) \times c$, first make a rectangular array of cubes (dice, sugar cubes, etc.) having a rows and b columns. Then, fill in c layers of the bottom rectangle to obtain a rectangular box, where the base of the box has width a and length b, and the height of the box is c.

Basically, you need to make (or visualize) a 3-dimensional box to do multiplication on three numbers. As it turns out, no matter how you construct it, if at the end, one dimension is a, one is b and the other is c, you've got the same box, and it holds the same number of cubes in it. Therefore, multiplication is associative. We won't be going through the motions of showing it in 3-dimensions, but elementary school children should have the benefit of working through it with cubes. I hope you'll model it this way if you become an elementary school teacher.

Since multiplication is both commutative and associative, any time more than two numbers are being multiplied together, the parentheses do not have to be shown, and the numbers can be multiplied together in any order.

For instance, notice how differently one might compute $4 \times 10 \times 3 \times 5$

Student 1: This person just goes left to right. First, 4×10 is 40, then 40×3 is 120, then $120 \times 5 = 600$.

Student 2: This person first multiplies 4×5 to get 20, then multiplies 10×3 to get 30, then multiplies the two answers together, 20×30 to get 600.

Student 3: This person does 4×5 first to get 20, then multiplies by 3 to get 60, then multiplies by 10 to get 600.

Student 4: This person multiplies 3×5 first to get 15, then multiplies that by 4 to get 60, and then multiplies by 10 to get 600.

And there are many more possibilities. The point is that you have the option of multiplying in any order and any combination you want. The following show how this is very convenient for certain computations:

✓ Example

$5 \times 87 \times 2$: Well, it's easier to multiply the 5 and 2 first to get 10 and then multiply by 87 to get 870. Otherwise, you could do 5×87, which isn't so easy to do in your head.

✓ Example

$8 \times 22 \times 5$: Well, I'd do 8 times 5 first to get 40 and then multiply by 22 to get 880.

? Exercise 29

Make up a multiplication problem, where it would be easier to multiply by doing some rearranging first. Explain your steps and how to get the answer.

Exercise 30

We're going to show that $3 \times (4+2) = (3 \times 4) + (3 \times 2)$. Let's first work on the right side of the equation, which has two parts.

a. Draw a rectangular array of dots or stars to represent 3×4.

b. Now, draw a rectangular array of dots or stars to represent 3×2.

c. Since the right side of the equation says to join the two parts together (addition), then draw the array from part a below, and right next to it, draw the array from part b.

d. Now, let's do the left hand side of the equation, $3 \times (4+2)$. We first have to compute in the parentheses to get 3×6. So, make a rectangular array of dots or stars to represent 3×6.

e. Is the number of dots or starts in part c equal to the number of dots or starts in part d?

Exercise 31

What property did you just illustrate in exercise 30?

Well, I hope you realize you just showed that multiplication distributes over addition. In fact, it is also true that multiplication distributes over subtraction. In other words,

For any three whole numbers, a, b and c, where b > c, $a \times (b+c) = (a \times b) + (a \times c)$ and $a \times (b – c) = (a \times b) – (a \times c)$.

Exercise 32

Show that $2 \times (3+4) = (2 \times 3) + (2 \times 4)$ using a geometrical approach, like you did in exercise 30. Explain the steps.

Exercise 33

a. Form the train $P \times (Y – L)$ by first forming the train Y – L. Find out what C-strip Y – L is. Then do the multiplication $P \times (Y – L)$. Exactly how many white C-strips make up this train?

b. Form the train $P \times Y$ and $P \times L$. Now subtract the second train from the first train. Exactly how many white C-strips make up this train?

c. Compare the lengths of the trains from part a and b.

Exercise 33 showed that $P \times (Y – L) = P \times Y – P \times L$. This is one illustration that multiplication distributes over subtraction.

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4.2: Multiplication Algorithms

You will need: ***Base Blocks*** *(Material Cards 4-15)*

There are many algorithms for multiplication. Personally, I think the way most of us learned it in school is one of the hardest algorithms. It's an uphill battle getting new methods into the school system. But, it's worth pursuing. It's a good thing to know what multiplication *means*, and how to multiply any two numbers together. However, there is no reason that everyone should have to use the same method to obtain the answer. In this exercise set, you'll learn some new algorithms for multiplication. You'll also learn how to multiply in different bases.

Many people use repeated addition all the time to multiply. Consider the multiplication problem 26 times 17. Basically, this means add 26 seventeens added together. This could be done by adding them together one at a time. That would take a long time.

Another approach would be to add 10 seventeens together first to get 170, then add 10 more seventeens together to get 170 again, then add 5 more seventeens together which is 85, and then add one more seventeen. Adding 10 seventeens (170), 10 seventeens (170), 5 seventeens (85) and 1 seventeen (17) gives 170 + 170 + 85 + 17 = 442, which is exactly the answer to 26 times 17. That should be no surprise since we added 26 (10+10+5+1) seventeens together.

Someone else might add 10 seventeens (170), 10 seventeens (170), 3 seventeens (51) and 3 more seventeens (51) to get 170 + 170 + 51 + 51 = 340 + 102 =442.

There are several other ways people might break up the number 26 and still get the correct answer. This approach may be one you frequently or maybe this is the first time you've thought of it this way. In any case, it's different from just doing it the same old way.

Because of the commutative property, you could have done 17 times 26 instead. Maybe you'd add 10 twenty-sixes (260), then 5 more twenty-sixes (130) and 2 more twenty-sixes (52) to get 260 + 130 + 52 = 442.

One reason it's advantageous to do multiplication using repeated addition is that many problems can be done in your head this way. But, of course, every new method takes practice and getting used to, and it's common for people to revert back to using their old, familiar methods even if it takes longer and requires paper. That's okay, too. It is, however, very important that a teacher knows several ways so that he or she can provide more opportunities for students. What seems obvious to one student is foreign to another, so you give alternatives and present different ways of looking at things.

? Exercise 1

Use some form of repeated addition to do the following multiplications. Explain how you do each one.

a. 23×13

b. 14×14

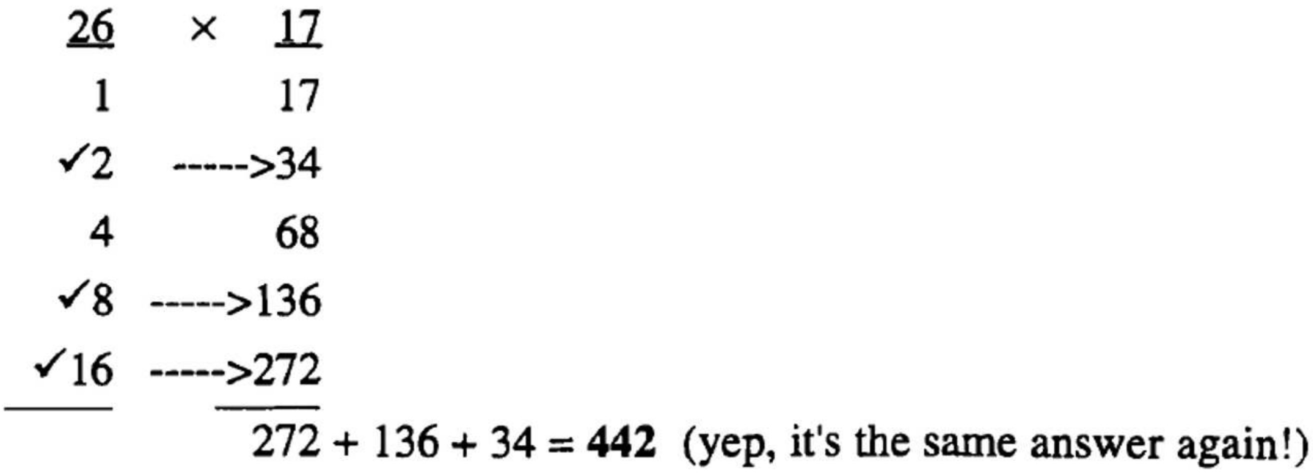
One new algorithm (for most of you reading this) is called **Duplation**. This is actually a very old algorithm used in ancient times by the Egyptians. It also uses the principle that multiplication is simply repeated addition. It is called duplation because it uses the idea of doubling numbers to get the answer. Let's look again at the first multiplication we did in this section. It was 26×17. Since 26 seventeens must be added together, the number 26 is broken down into powers of two. It's easy to get powers of two: simply start with 1, then double it to get 2, then double it to get 4, then double it to get 8, then double it to get 16, etc. To figure out how far you have to go in the doubling process, stop if doubling gives a number bigger than the number you are trying to break down (26, in this case). Now, all this sounds more complicated than it really is. Before explaining too much more, notice how you'll write it. Write 26 at the top and underline it. Underneath, start with 1, then, double continuously, stopping at 16 since if you doubled another time, you'd get a number bigger than 26. To the right of 26, write the number 17 and underline it.

Underneath it, first write 17, then double 17 to get 34, then double 34 to get 68, double 68 to get 136, etc., until you have the same amount of numbers under the 17 as you do under the 26.

26	×	17
1	17	
2	34	
4	68	
8	136	
16	272	
……	……	

Do you see the corresponding numbers? 1 corresponds with 17, because 17 is 1×17, 2 corresponds with 34, because 34 is 2×17, 4 corresponds with 68, because 68 is 4×17, 8 corresponds with 136, because 136 is 8×17, and 16 corresponds with 272, because 272 is 8×17. Now, you may need to think about this for a few minutes. The left side keeps track of how many of some number you are adding together. So, the 1×17 should be clear. So, 2×17 is simply 17 doubled. Now if 2×17 is 34, then 4×17 is twice as many as 2×17, so double 34 to get 68. This continues on and on. So, if 8×17 is 136, then if we double 136, we should have 16×17. Isn't it neat how we know that $16 \times 17 = 272$ and we just double a few numbers to get there?

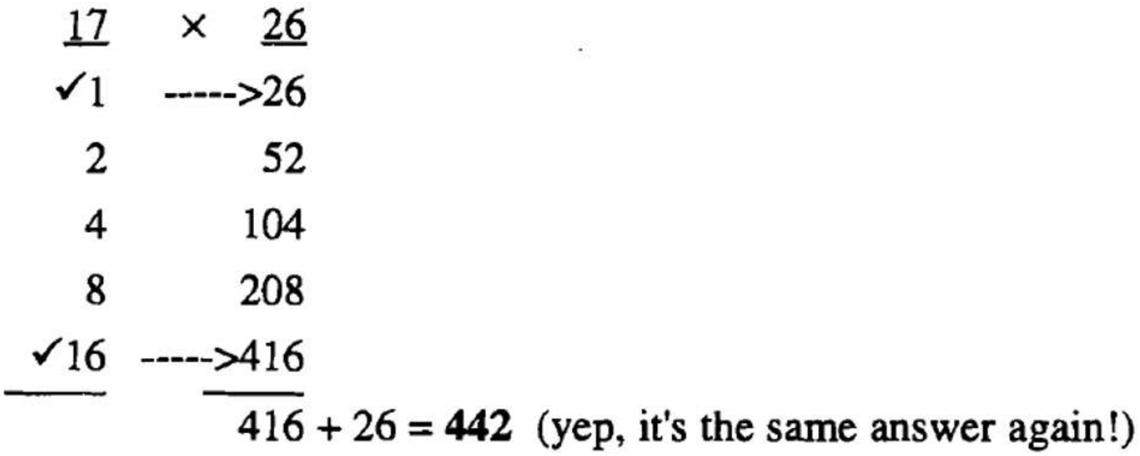
Okay, now we only need to add 26 seventeens together. Simply start at the bottom of the first column, and check off numbers that add up to 26 (this is like doing it in base two). After you check off the numbers in the left column, circle or point to their corresponding numbers in the right column. It's those numbers in the right column that you add together to get the answer. Watch how the rest of the problem is done:



26	×	17
1		17
✓2		----->34
4		68
✓8		----->136
✓16		----->272

$272 + 136 + 34 = \mathbf{442}$ (yep, it's the same answer again!)

Let's do the same problem over again, but use the commutative property of multiplication. In other words, use the duplation method to compute: 17×26. This time, in the left column, you check off numbers that add up to 17 it's coincidental that I had to double to 16 again and stop. In the right hand column, start with 26 and double away.



17	×	26
✓1		----->26
2		52
4		104
8		208
✓16		----->416

$416 + 26 = \mathbf{442}$ (yep, it's the same answer again!)

Here are two more examples for you to study. I'll do 15×35 and 35×15.

15 × 35
✓1 ----->35
✓2 ----->70
✓4 ----->140
✓8 ----->280
280+140+70+35=525

35 × 15
✓1 ----->15
✓2 ----->30
4 60
8 120
16 240
✓32 ----->480
480+30+15=525

Of course, because of the commutative property of multiplication, the answer is 525, no matter which way you do it. Are you ready to try a couple of duplation problems of your own? Well, ready or not, here they come!

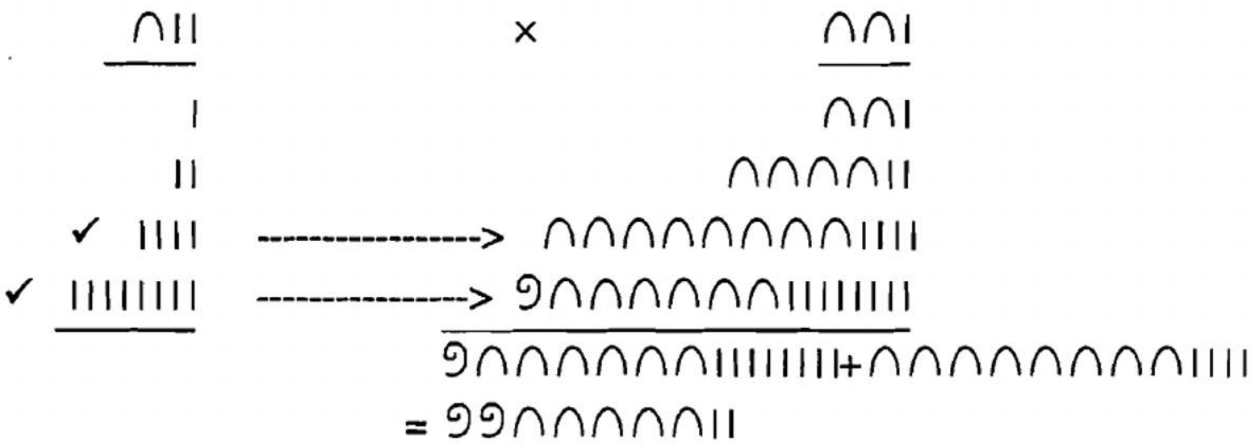
? Exercise 2

Multiply 13×29 using duplation. Then, multiply 29×13 using duplation. Show all of your work. Make it clear what is added together for each problem.

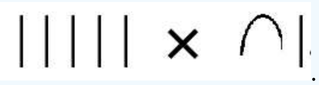
? Exercise 3

Use duplation to multiply 27 by 14, and then 14 by 27. Show your work.

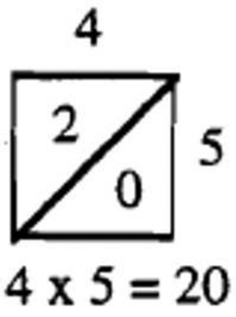
Here is an example of how the Egyptians multiplied using duplation.

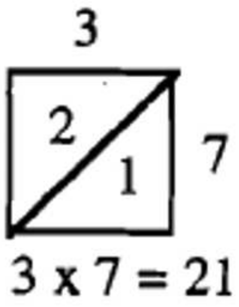


? Exercise 4

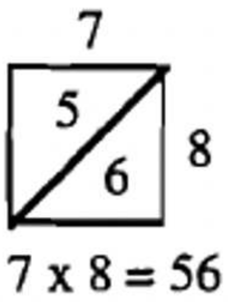
Use duplation to multiply .

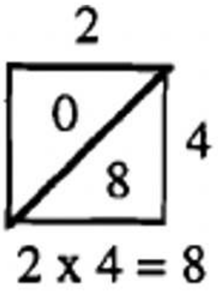
Do the problem entirely in Egyptian.



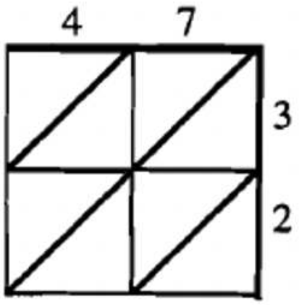


You are going to learn a cool and easy way to multiply numbers together. But first, I'll show you how to multiply 2 single digits together. Make a square with a diagonal in it. Put one number on top and the other on the right of the square. Multiply the two digits together and put the answer in the box with the diagonal. Look at the examples to the right.

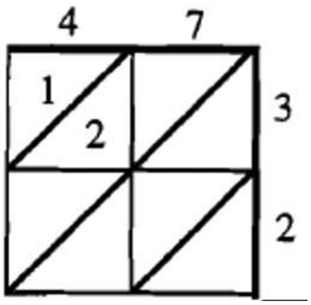




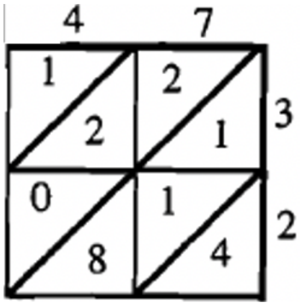
The method you are now going to learn is called the **LATTICE METHOD** and it could be used for multiplying 2 digit or 3 digit or even bigger numbers together. The example well do is 47×32.



STEP 1: To begin, put one of the numbers at the top (47) of a rectangle (that has a space for each digit in the number) and the other number along the side (32) of the same rectangle (that has a space for each digit). In this case, we need a rectangle that has two spaces across the top and 2 spaces along the side. Then, you make diagonals. Look to the right.

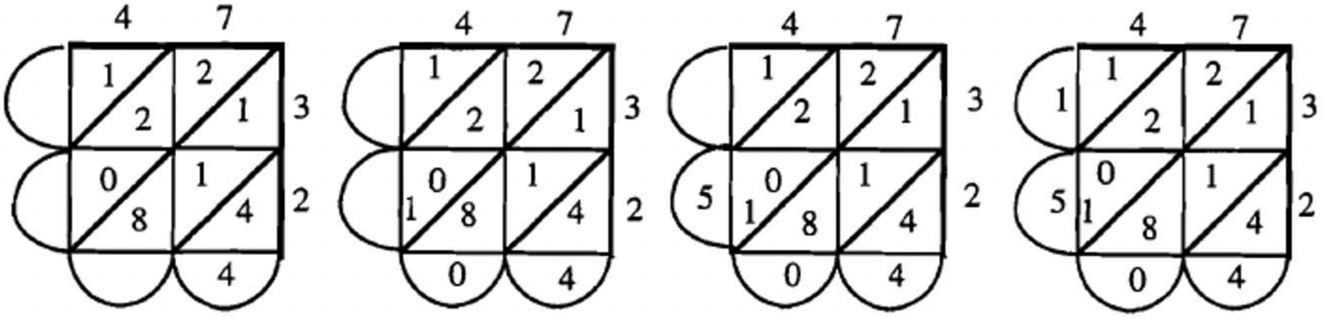


STEP 2: Multiply each of the digits on the top by each of the digits along the side, and put the answer where they would meet. For instance, look where the 12 goes when you multiply 4 times 3.



STEP 3: There are three other multiplications to do: 7 times 3, 4 times 2 and 7 times 2. Do you see how all the answers are filled in? It doesn't matter which ones you multiply and put in first.

STEP 4: There are four diagonals (made up of little triangles) slanting to the right inside the rectangle. Starting with the one in the right bottom corner, add the digits inside each diagonal and fill put the sum in the "cup" under its diagonal. In the first diagonal is 4, so 4 is put in the "cup" under its diagonal (bottom right of rectangle). In the next diagonal are three digits a 1, 1 and 8 which adds up to 10, so put a 0 in the cup under that diagonal (at the bottom to the left) and a 1 (for the carry) in one of the triangles to the left. The next diagonal has a 2, 2, 0 and 1, which adds up to 5, and goes in the "cup" under its diagonal (left, bottom). In the last diagonal, there is just a 1, so a 1 is put in the "cup"under its diagonal (left, top). Look at how I filled in the numbers one at a time.



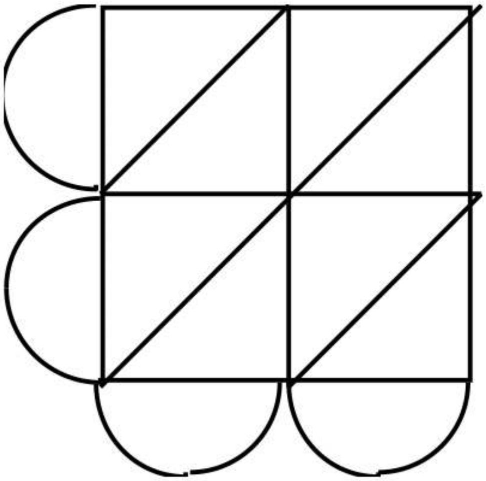
STEP 5: The numbers you filled in under each diagonal in the "cups" (outside the rectangle) form the answer. It is read by starting at the left top and reading down and then across the bottom. In this case, the answer is 1504. Isn't that cool? That's all there is to it!

Most of my students love this very old Lattice method of multiplying. It works as easily for small numbers as it does for large numbers. First, you simply do a series of simple single digit multiplications, and then you add, with carrying. Actually you could use the left to right (with underlining for carrying) method of addition for the last step. The main disadvantage is that you have to draw a lattice, which is a small price to pay to make multiplication easier and less complicated. Most likely, it takes less time than using the standard algorithm with carrying, because there's less chance of error and confusion with all that multiplying and carrying, etc. I propose all students learn multiplication this way first, and let them stay with it if they like it. Who needs to shorten it up and make it more complicated than it really is? Ah, well, I take it you get my point.

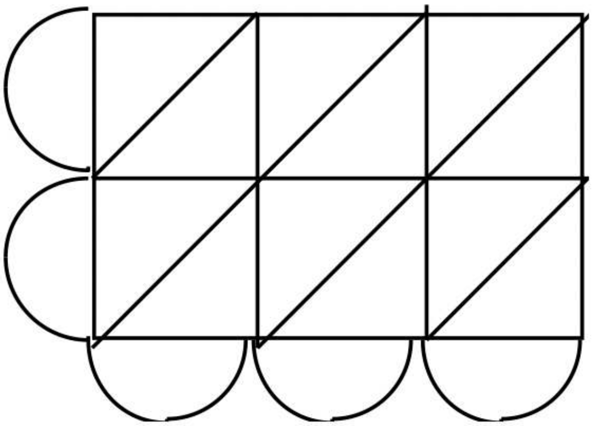
? Exercise 5

Use the lattices provided in a - d to multiply the given numbers. Then write the answer in the space provided. For e, draw your own lattice to compute the multiplication. By the way, you don't have to draw the little cups. That was my own little invention so it would be easy to identify the answer. There are two more problems (f and g) on the next page. Check your answers with a calculator

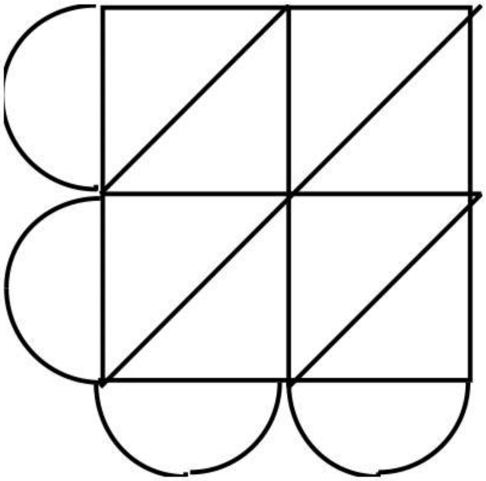
a. $63 \times 57 =$ _____



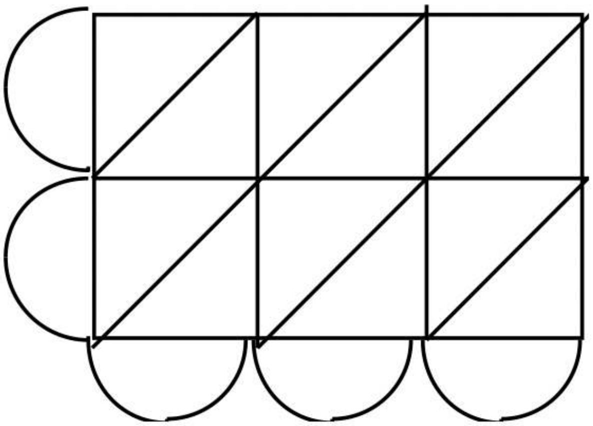
b. $342 \times 85 =$ _____



c. Make up a problem:



d. Make up a problem:



e. $58 \times 67 =$ _____ Draw your own lattice.

Exercise 6

a. Get out your Base Four Blocks. Take 41 units, and trade them in using your Base Four blocks. You should have some flats, longs and units which you can write as a numeral in Base Four.

Write the Base Four numeral here:

b. If you wanted to multiply 2 times that number, you would need to make two piles representing that number, join the two equal piles together (since multiplication is repeated addition) and then take the new pile and make trade-ins. Make two equal piles with the blocks, where each pile is the same as the one you formed from trade-ins in part a.Draw a picture of what the two piles look like.

c. Join the two piles together and make trade-ins to get a new base four numeral. Write this Base Four numeral here:

d. Fill in the first blank with what you got for part a and the second blank with what you got for part c. You just did a multiplication problem in Base Four using repeated addition!

$2_{\text{four}} \times \_\_\_\_\_\__{\text{four}} = \_\_\_\_\_\__{\text{four}}$

Exercise 7

a. Write the Base Four numeral from 6a here:

b. If you wanted to multiply 3 times that number, you would need to make three piles representing that number, join the three equal piles together (since multiplication is repeated addition) and then take the new pile and make trade ins. Make three equal piles with the blocks, where each pile is the same as the one you formed from trade-ins in part a. Draw a picture of what the three piles look like.

c. Join the three piles together and make trade-ins to get a new base four numeral. What Base Four numeral did you get?

d. Fill in the first blank with what you got for part a and the second blank with what you got for part c. You just did another multiplication problem in Base Four using repeated addition!

$3_{\text{four}} \times \underline{\hspace{2cm}}_{\text{four}} = \underline{\hspace{1.5cm}}_{\text{four}}$

Exercise 8

Using the same procedure as in 6 and 7, take out your base three blocks and do the following multiplication: $2_{\text{three}} \times 221_{\text{three}}$. Explain your work, and show pictures.

Exercise 9

If you wanted to do $10_{\text{three}} \times 221_{\text{three}}$, you would need to trade the long (10_{three}) in for three units, make three piles using Base three blocks representing 221_{three}, and do the same thing you did in exercise 7. Explain how you do this and show pictures.

This method of repeated addition with the blocks would be too cumbersome if we were working with higher numbers. For instance, if you were trying to use this method to multiply $321_{\text{four}} \times 312_{\text{four}}$, the first number when broken down into single units would consist of more than 50 units, and that would be way too many piles to make! Instead, we'll think about and figure out what it means to multiply using larger blocks than units. For instance, we'll figure out what a flat times a long is, or what a long times a flat is in any base! We'll be using these abbreviations: U for unit, L for long, F for flat, B for block, LB for long block, FB for flat block, and BB for block block.

Exercise 10

a. Get out your base three blocks. You're going to find the product of a long and a long similar to how you did products with the C-strips in Exercise Set 1. $L \times L$ is obtained by putting one long in the horizontal position and the second long in the vertical position, underneath and perpendicular to the first long. Then form a rectangle of longs underneath. Take the rectangle (don't include the top horizontal long) and trade up in base three, if possible. What is the product of long and a long in base three?

b. Get out your base four blocks. Find the product of long and a long in base four using the method explained in part a. What is the product of long and a long in base four?

c. Get out your base five blocks. Find the product of long and a long in base five using the method explained in part a. What is the product of long and a long in base five?

d. In any base, what is the product of a long and a long? $L \times L =$

Exercise 11

a. Get out your base three blocks. You're going to find the product of a flat and a long now. First, we need to convert the flat to longs so that we can make a horizontal train across the top of the T. Do this. Then, put a long in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up in base three, if possible. What is the product of flat and a long in base three?

b. Get out your base four blocks. Find the product of flat and a long in base four using the method explained in part a. What is the product of flat and a long in base four?

c. Get out your base six blocks. Find the product of flat and a long in base six using the method explained in part a. What is the product of flat and a long in base six?

d. In any base, what is the product of a flat and a long? $F \times L =$

? Exercise 12

a. Get out your base three blocks. You're going to find the product of a block and a long now. First, we need to convert the block to flats, and then the flat to longs so that we can make a horizontal train across the top of the T. Do this. Then, put a long in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up in base three, if possible. What is the product of block and a long in base three?

b. Get out your base four blocks. Find the product of block and a long in base four using the method explained in part a. What is the product of block and a long in base four?

c. Get out your base two blocks. Find the product of block and a long in base two using the method explained in part a. What is the product of block and a long in base two?

d. In any base, what is the product of a block and a long? $B \times L =$

? Exercise 13

Summarize the results from part d of exercises 10, 11 and 12.

a. $L \times L =$ _______

b. $L \times F =$ _______

c. $L \times B =$ ________

Use the commutative property of multiplication to compute the following:

d. $F \times L =$ ________

e. $B \times L =$ ________

? Exercise 14

In your own words, what happens when something is multiplied by a long?

Did you notice that when something is multiplied by a long, it bumps up to the next place value. This is analogous to multiplying by 10 in base 10, since 10 is a long in base ten. For instance, in base 10, 347 represents 3 hundreds, 4 tens and 7 ones; when you multiply 347×10, you get 3470, 3 thousands, 4 hundreds and 7 tens, the place value for each digit moved up one place value.

? Exercise 15

Consider multiplying $312_{\text{four}} \times 10_{\text{four}}$. This means you are multiplying a number in base four by a long in base four. Get out your base four blocks and represent the number 312_{four}.

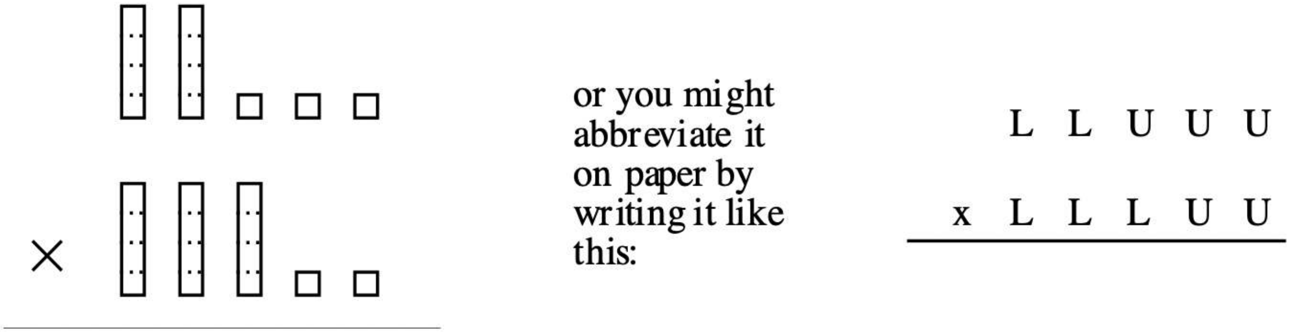
a. You have ____ flats, _____ longs and _____ units.

b. If you multiply each of these by the long, now how many of each type of place value block do you have in base four?

To multiply a two-digit numeral, X, in a base by some number, Y, in that base, the idea of repeated addition can be used. For instance, if we represent the number with base blocks, we'll want to create an "answer pile" where we put the blocks representing the product as we multiply by the two-digit numeral. Every unit in X multiplied by Y gives us one set of base blocks contained in

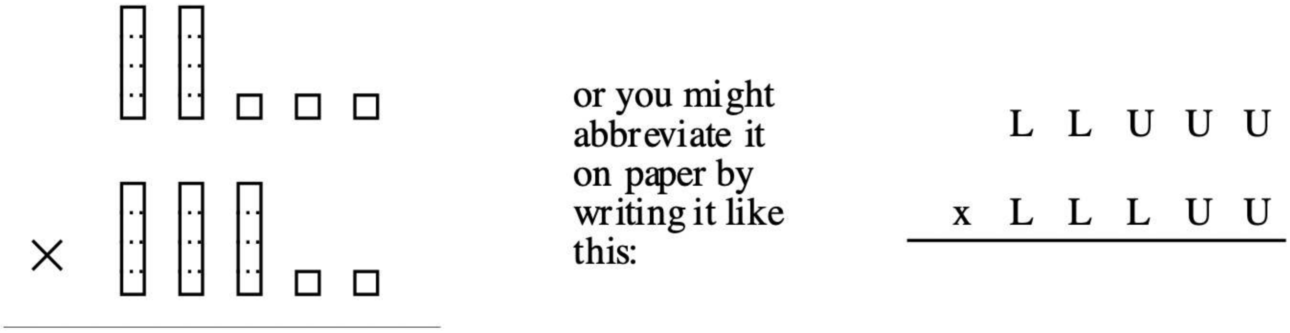
Y in the answer pile. Every long in X multiplied by Y gives us the next place value up for each of the base blocks contained in Y. This is much more complicated to explain than to do with the blocks. So, let's do an example.

Get out your Base Four Blocks. We'll multiply $23_{\text{four}} \times 32_{\text{four}}$. With your blocks, represent this as a multiplication problem. It should look something like this:

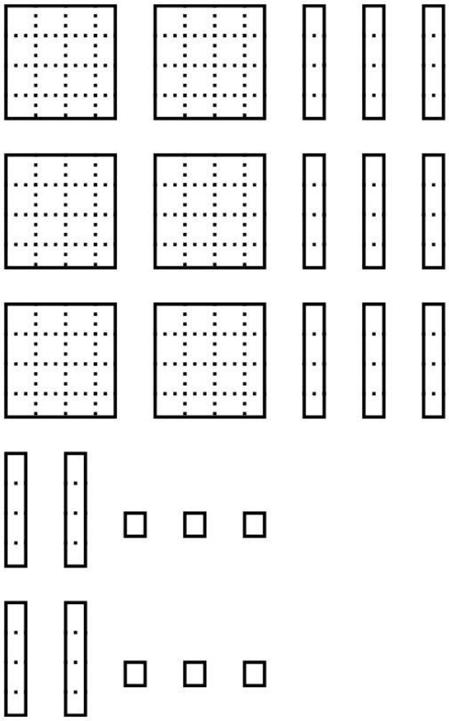

×
or you might abbreviate it on paper by writing it like this:
L L U U U
x L L L U U

Recall that each long in the bottom multiplied by the top makes each top piece move up to the next higher place value block. So, when the first bottom long is multiplied by the top number, 2 flats and 3 longs go into the answer pile. When the second bottom long is multiplied by the top number, another 2 flats and 3 longs go into the answer pile. When the third bottom long is multiplied by the top number, yet another 2 flats and 3 longs go into the answer pile. When I do this with the blocks, I usually move each part of the multiplier (bottom number) I've done so far to the right or out of the way. In any case, in your answer pile, you should now have 6 flats and 9 longs in the answer pile. We still have to multiply by each of the two units. Each unit multiplied by the top number gives a set of the top number; it's like multiplying by 1. So, when the first bottom unit is multiplied by the top number, 2 longs and 3 units go into the answer pile. When the second bottom unit is multiplied by the top number, another 2 longs and 3 units go into the answer pile. Create a place for the answer pile and do this with your blocks. Try it yourself, then check on the next page.

I'll show the answer pile below: each row shows what I got as I multiplied each bottom piece (3 longs and 2 units) by the top. By the way, you could have multiplied by the units first and then by the longs I'll do it that way for the next example. It all just eventually ends up in the answer pile and all exchanges have to made in the end, so you can write it in Base Four. This is $23_{\text{four}} \times 32_{\text{four}}$:


×
or you might abbreviate it on paper by writing it like this:
L L U U U
x L L L U U

THE Answer PILE IS SHOWN BELOW:

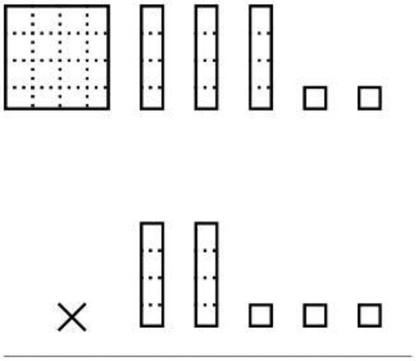


or you might abbreviate it on paper by writing it like this:

F F L L L
F F L L L
F F L L L
L L U U U
L L U U U

Now, make all your trade-ins with these pieces. You should end up with 2 blocks, 1 flat, 2 longs and 2 units, which is the number 2122_{four}, written in base four.

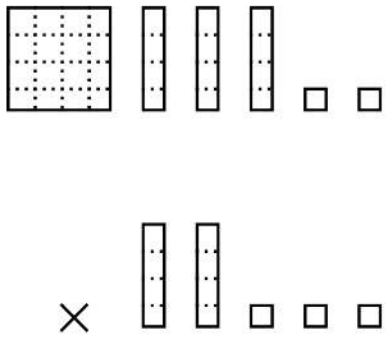
Next, we'll multiply $132_{\text{four}} \times 23_{\text{four}}$. With your blocks, represent this as a multiplication problem. It should look something like this:



or you might abbreviate it on paper by writing it like this:

F L L L U U
x L L U U U

Each unit multiplied by the top number gives a set of the top number it's like multiplying by 1. So, when the first bottom unit is multiplied by the top number, a flat, 3 longs and 2 units go into the answer pile. When the second bottom unit is multiplied by the top number, another flat, 3 longs and 2 units go into the answer pile. When the third bottom unit is multiplied by the top number, yet another flat, 3 longs and 2 units go into the answer pile. Create a place for the answer pile and do this with your blocks. When I do this with the blocks, I usually move each part of the multiplier I've done so far to the right or out of the way. In any case, in your answer pile, you should now have 3 flats, 9 longs and 6 units in the answer pile. We still have to multiply by each of the two longs. Recall that each long multiplied by the top makes each top piece move up to the next higher place value block. So, when the first bottom long is multiplied by the top number, a block, 3 flats and 2 longs go into the answer pile. When the second bottom long is multiplied by the top number, another block, 3 flats and 2 longs go into the answer pile. I'll show the answer pile below: each row shows what I got as I multiplied each bottom piece (3 units and 2 longs) by the top. By the way, you could have multiplied by the longs first and then by the units. It all just eventually ends up in the answer pile and all exchanges have to made in the end, so you can write it in Base Four. This is $132_{\text{four}} \times 23_{\text{four}}$:

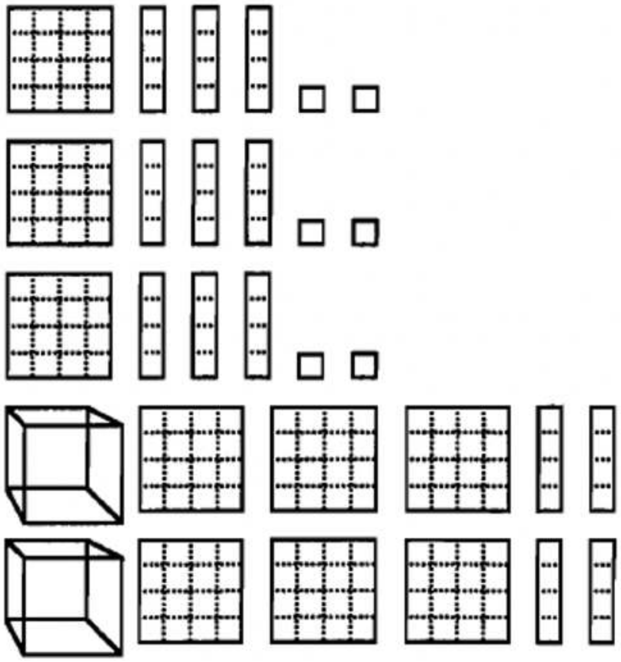


or you might abbreviate it on paper by writing it like this:

F L L L U U

x L L U U U

THE Answer PILE IS SHOWN BELOW:



or you might abbreviate it on paper by writing it like this:

F L L L U U

F L L L U U

F L L L U U

B F F F L L

B F F F L L

Now, make all your trade-ins with these pieces. You should end up with 1 long block, 1 block, 2 longs and 2 units, which is the number 11022_{four}, written in base four. Make sure you do and understand these last two problems before going on.

? Exercise 16

Multiply each of the following using base blocks. Then explain and draw the steps in the space below as done in the previous two examples.

a. Use Base Four blocks to multiply $12_{\text{four}} \times 31_{\text{four}}$

b. Use Base Three blocks to multiply $112_{\text{three}} \times 21_{\text{three}}$

? Exercise 17

a. Get out your base three blocks. You're going to find the product of a flat and a flat now. We need to convert the first flat to longs, make a train, and put this horizontal train across the top of the T. Do this. Then, convert the second flat to longs, make a train, put this train in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of a bunch of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up as much as possible in base three, if possible. What is the product of flat and a flat in base three?

b. Here's another way to find the product of a flat and a flat in base three. For each unit in the first flat, put a flat in an answer pile. Take all those flats in the answer pile and trade them in to see what you get. What is the product of flat and a flat in base three?

c. Get out your base two blocks. Find the product of flat and a flat in base two using the method explained in part a or b. What is the product of flat and a flat in base two?

d. Complete the following: In any base, $F \times F =$ _____

Another way to think about what happens when you multiply by a flat is to think of a flat as a long times a long. Let's say you are multiplying something by a flat. First, multiply it by a long, which means it moves up one place value, and then multiply that answer by a long once more, which means it moves up one more place value. So, anything multiplied by a long moves up one place value, and anything multiplied by a flat moves up two place values. Multiplying by a flat is analogous to multiplying by 100 in base ten. 100b in any base b represents a flat. Think about what happens when a block is multiplied by anything.

? Exercise 18

Complete the following:

a. $U \times U =$ ____	f. $F \times U =$ _______
b. $U \times L =$ _____	g. $B \times U =$ _______
c. $U \times F =$ _____	h. $L \times L =$ _______
d. $U \times B =$ _____	i. $L \times F =$ _______
e. $L \times U =$ _____	j. $L \times B =$ _______

? Exercise 19

Use base six blocks to multiply the following. $105_{\text{six}} \times 123_{\text{six}}$. If you have trouble, go on to the next page, where it is explained and then try it again. Show the steps and diagram.

If you had trouble with #19, below is an abbreviated picture of of the steps. The multiplication shown on the left is $105_{\text{six}} \times 123_{\text{six}}$ and the multiplication shown on the right is $123_{\text{six}} \times 105_{\text{six}}$ In both cases, I started multiplying with the biggest block at the bottom and moved left to right. In both cases, the final result is the same, and after all exchanges have been made, there is 1 long block, 3 blocks, 4 flats and 3 units, which is 13403_{six}, written in base six. It takes a lot of practice to really do this with the blocks. But it's fun once you get it!

F U U U U U	F L L U U U
x F L L U U U	x F U U U U U
LB F F F F F	LB B B F F F
B L L L L L	F L L U U U
B L L L L L	F L L U U U
F U U U U U	F L L U U U
F U U U U U	F L L U U U
F U U U U U	F L L U U U

? Exercise 20

Use base three blocks to multiply $102_{\text{three}} \times 120_{\text{three}}$. Show steps like done in the example shown above.

We can shorten up this process of multiplying with the blocks even further. In exercise 18, you showed the products obtained when multiplying two base blocks together. The interesting thing is that a flat times a long is a block, no matter what base you are in. You can actually make up a multiplication chart for using blocks.

? Exercise 21

Complete the rest of the multiplication chart below.

×	**U**	**L**	**F**	**B**
U				
L	L		B	
F				FB
B		LB		

In our first example with the blocks, we computed $23_{\text{four}} \times 32_{\text{four}}$. Since we are multiplying 2 longs and 3 units by 3 longs and 2 units, the multiplication can be written as $(2L + 3U) \times (3L + 2U)$. Applying the distributive property (or FOIL), we get

$(2L + 3U) \times (3L + 2U)$

= $2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$

= $\mathbf{(2 \times 3)(L \times L) + (2 \times 2)(L \times U) + (3 \times 3)(L \times U) + (3 \times 2)(U \times U)}$

= 6F + 4L + 9L + 6U

= 6F + 13L + 6U **

At this point exchanges have to be made. In Base Four,

6F = 1B + 2F, 13L = 3F + 1L and 6U = 1L + 2U

= 1B + 2F + 3F + 1L + 1L + 2U

= 1B + 5F + 2L + 2U

The 5 flats are now exchanged for a block and a flat

= 1B + 1B + 1F + 2L + 2U

= 2B + 1F + 2L + 2U

which is written as 2122_{four} in base four. This is the same answer we got before.

**If I were doing 23×32 in a different base, all of steps up to this point are exactly the same. For instance, on the next page, the same basic problem is done in base six. All of the steps are the same, but notice that once exchanges are made, they are done with base six blocks in mind.

Also, it isn't necessary to write out the step that is in bold type. That's an application of the commutative and associative properties, and I wanted to show that step for clarification.

Here is the same basic problem done in Base Six.

$23_{\text{six}} \times 32_{\text{six}} = (2L+3U)\times(3L+2U)$
$= 2L\times 3L + 2L\times 2U + 3U\times 3L + 3U\times 2U$
$= (2\times 3)(L\times L) + (2\times 2)(L\times U) + (3\times 3)(L\times U) + (3\times 2)(U\times U)$
$= 6F + 4L + 9L + 6U$
$= 6F + 13L + 6U$
$= 1B + 2F + 1L + 1L$
$= 1B + 2F + 2L$
$= 1220_{\text{six}}$

Here is the same basic problem done in Base Nine.

$23_{\text{nine}} \times 32_{\text{nine}} = (2L+3U)\times(3L+2U)$
$= 2L\times 3L + 2L\times U3U\times 3L + 3U\times 2U$
$= 6F + 4L + 9L + 6U$
$= 6F + 13L + 6U$
$= 6F + 1F + 4L + 6U$
$= 7F + 4L + 6U$
$= 746_{\text{nine}}$

? Exercise 22

Show the steps using the blocks as shown in the previous examples to multiply the following.

a. $23_{\text{five}} \times 32_{\text{five}}$

b. $42_{\text{eight}} \times 53_{\text{eight}}$

Here is an example of using this method for numerals with more than two digits each. You still use the distributive property. Each term in the first parentheses is multiplied by each term in the second parentheses.

$212_{\text{five}} \times 102_{\text{five}} = (2F+1L+2U)\times(1F+2U)$
$= 2F\times 1F + 2F\times 2U + 1L\times 1F + 1L\times 2U + 2U\times 1F + 2U\times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 2LB + 1B + 1B + 1F + 2L + 4U$
$= 2LB + 2B + 1F + 2L + 4U$
$= 22124_{\text{five}}$

$212_{\text{eight}} \times 102_{\text{eight}} = (2F+1L+2U)\times(1F+2U)$
$= 2F\times 1F + 2F\times 2U + 1L\times 1F + 1L\times 2U + 2U\times 1F + 2U\times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 21624_{\text{eight}}$

In this last example, we got away without having to make any exchanges. We aren't so lucky in this next example, which is in base three. Base Three is harder to work with since it only takes three of something before you have to make an exchange. Sometimes it seems that all you are doing is exchanging!

$212_{\text{three}} \times 102_{\text{three}} = (2F+1L+2U)\times(1F+2U)$
$= 2F\times 1F + 2F\times 2U + 1L\times 1F + 1L\times 2U + 2U\times 1F + 2U\times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 2LB + 1B + 2B + 2L + 1L + 1U$
$= 2LB + 3B + 3L + 1U$
$= 2LB + 1LB + 1F + 1U$
$= 3LB + 1F + 1U$
$= 1FB + 1F + 1U$
$= 100101_{\text{three}}$

? Exercise 23

Show the steps using the blocks as shown in the previous examples to multiply:

$212_{\text{four}} \times 102_{\text{four}}$

Exercise 24

Show the steps using the blocks as shown in the previous examples to multiply:

a. $361_{\text{nine}} \times 15_{\text{nine}}$

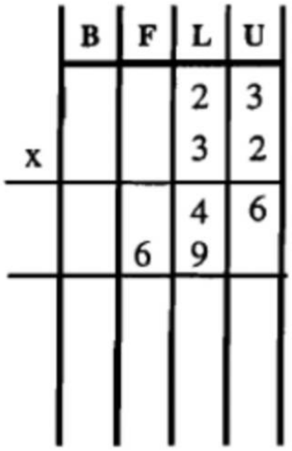
b. $111_{\text{two}} \times 11_{\text{two}}$

Another way to multiply with the blocks is by using a chart. This is similar to how we did it for addition. It's actually very similar to doing it using the regular multiplication algorithm, except we don't have to write the carry as we go. You simply multiply each digit of one numeral by each digit in the other numeral, and put it in the proper column. Then, you add the like terms, and still do exchanges. Here is an example writing it out with the distributive property. Below, contrast with the method using a chart.

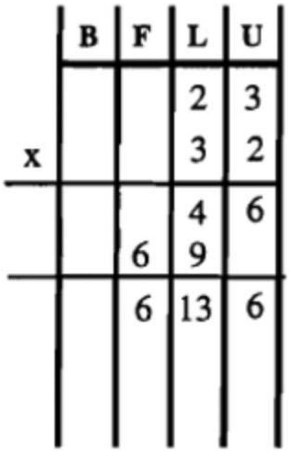
$$\begin{aligned}
&23_{\text{six}} \times 32_{\text{six}} = (2L+3U)\times(3L+2U)\\
&=2L\times 3L+2L\times 2U+3U\times 3L+3U\times 2U\\
&=6F+4L+9L+6U\\
&=6F+13L+6U\\
&=1B+2F+1L+1L\\
&=1B+2F+2L\\
&=1220_{\text{six}}
\end{aligned}$$

In step one, the individual multiplications are done. If you get a 2-digit numeral, just write it in. No need to carry. In step 2, the like terms are added. Then, exchanges are made. I did the exchanges by exchanging the units first. In step 3, When 6 units were exchanged, I had 0 units and 1 more long, so I crossed off the 13 and put 14 in the next column to the left. In step 4, I exchanged 14 longs for 2 flats and 2 longs, so I crossed off 6 in the flat column, and added 2 more to get 8, and crossed off 14 in the long column since there were only 2 left. In the last step, 8 flats are exchanged for 1 block and 2 flats. I've shown the individual steps, all the work is shown in Step 5. That's all the space it takes.

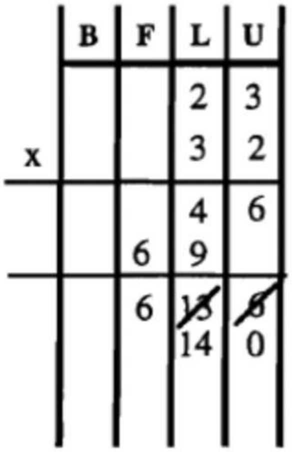
Step 1


B F L U
2 3
x 3 2
4 6
6 9

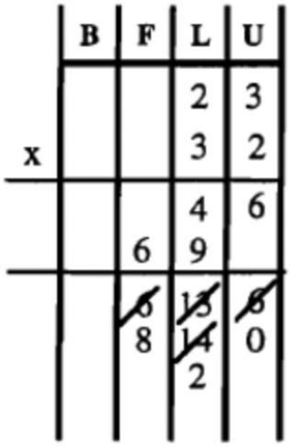
Step 2


B F L U
2 3
x 3 2
4 6
6 9
6 13 6

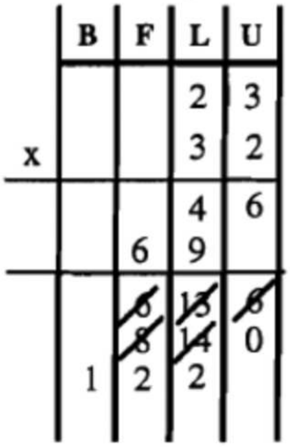
Step 3


B F L U
2 3
x 3 2
4 6
6 9
6 ~~13~~ ~~6~~
14 0

Step 4

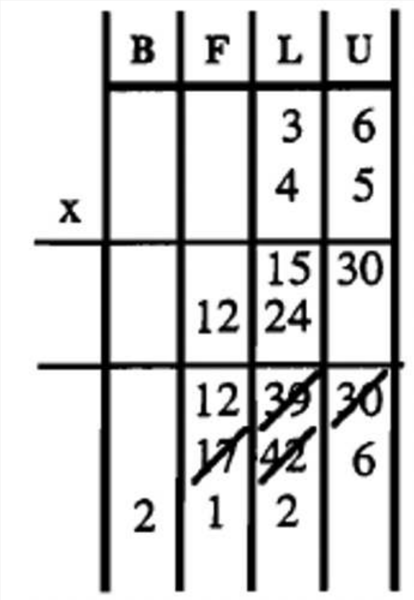

B F L U
2 3
x 3 2
4 6
6 9
~~6~~ ~~13~~ ~~6~~
8 ~~14~~ 0
2

Step 5

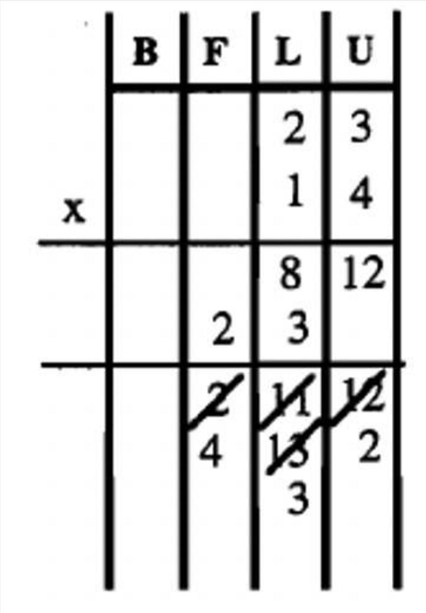

B F L U
2 3
x 3 2
4 6
6 9
~~6~~ ~~13~~ ~~6~~
~~8~~ ~~14~~ 0
1 2 2

Here are three more examples of using the charts to multiply with the blocks.

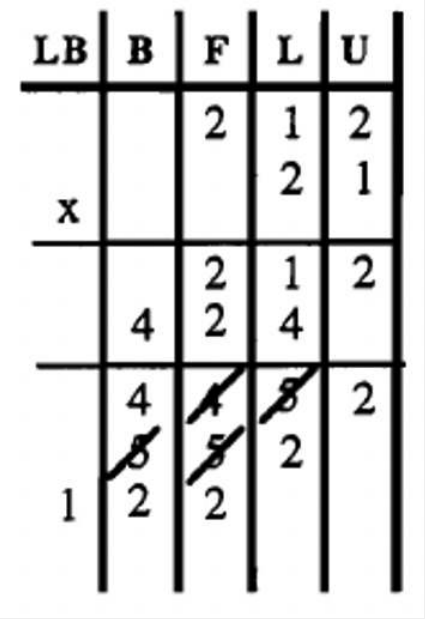
$36_{\text{eight}} \times 45_{\text{eight}}$	$23_{\text{five}} \times 14_{\text{five}}$	$212_{\text{three}} \times 22_{\text{three}}$


B F L U
3 6
x
4 5
15 30
12 24
12 39 30
17 42 6
2 1 2

= 2126_{eight}


B F L U
2 3
x
1 4
8 12
2 3
2 11 12
4 13 2
3

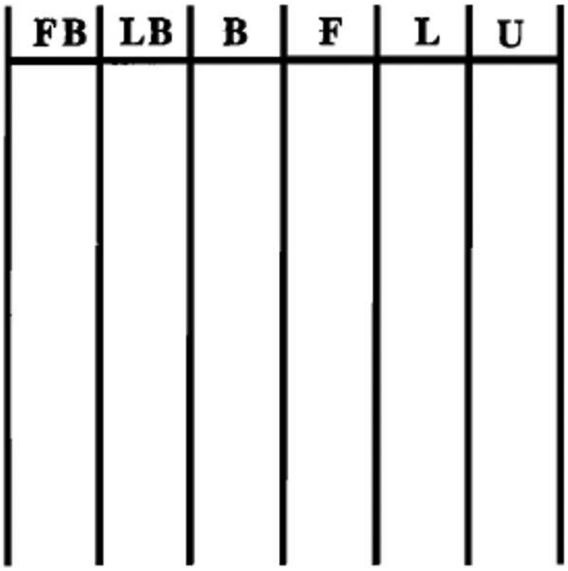
= 432_{five}


LB B F L U
2 1 2
x
2 1
2 1 2
4 2 4
4 4 8 2
8 5 2
1 2 2

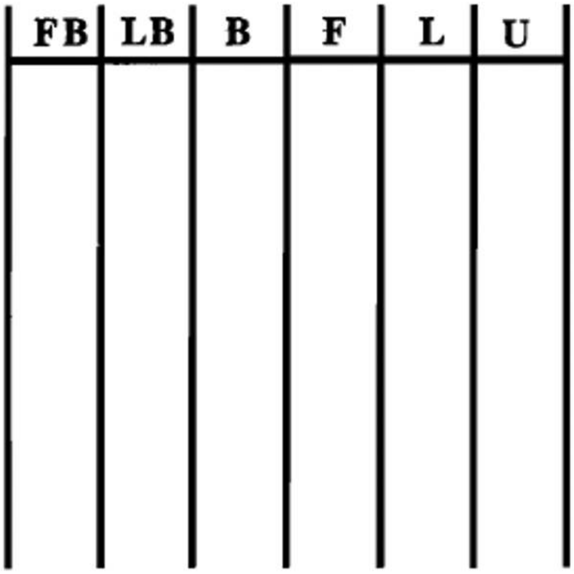
= 12222_{three}

? Exercise 25

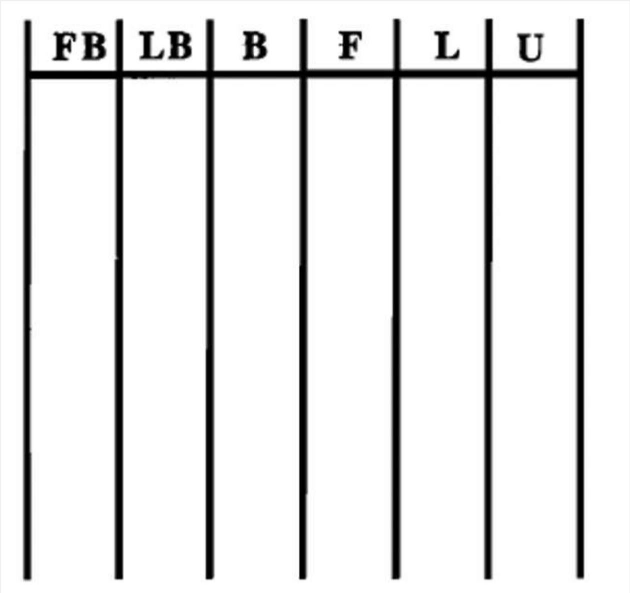
Use the base block charts, writing the numbers in terms of actual base blocks at first, as shown in the above examples, to compute the following. Show all your work and steps below.



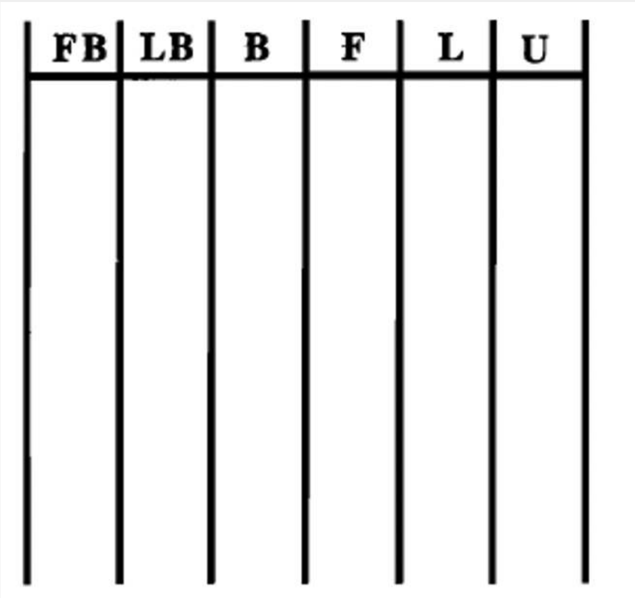
a. $34_{\text{six}} \times 25_{\text{six}} =$



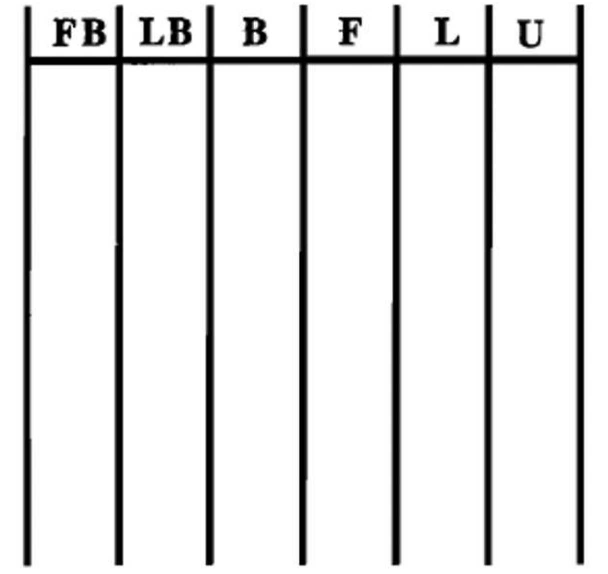
b. $32_{\text{eleven}} \times 4T_{\text{eleven}} =$



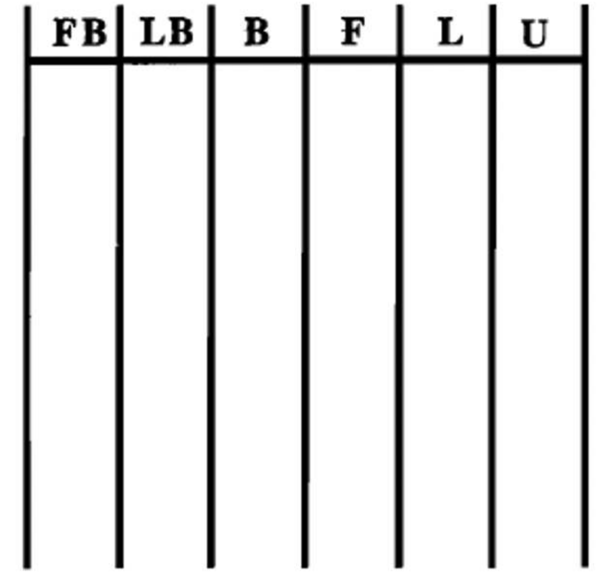
c. $111_{\text{two}} \times 101_{\text{two}} =$



d. $1E_{\text{twelve}} \times 53_{\text{twelve}} =$



e. $212_{\text{three}} \times 22_{\text{three}} =$



f. $23_{\text{four}} \times 32_{\text{four}} =$

Eventually, it would be nice to be able to do the multiplications without having to think in terms of the actual blocks. The traditional algorithm could be used. To do this, you have to make sure you immediately convert to the given base before writing the answer. For instance, $4_{\text{five}} \times 4_{\text{five}} = 31_{\text{five}}$ because 4 units times 4 units is 16 units, which in base five converts to 3 longs and 1 unit, which is written as 31_{five} in base five. A good start is to make up some multiplication tables in other bases. A base seven multiplication table is shown on the next page. Keep in mind that there are seven digits in base seven, so you'll have a table to fill in that is 7 rows by 7 columns. Because of the commutative property, there is symmetry in the table, which means many answers are duplicated. Furthermore, multiplying by 0 or 1 is trivial. At the top of the table, we'll indicate it is in base seven, and then leave off writing the base to the right and below each numeral.

Table 4.2.1: Base Seven Multiplication Table

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0

1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	31	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Make sure you don't read these as base ten numerals. 26 means "two six, base seven", which is 2 longs and 6 units, or 20 base ten units! An easy way to get started is to do the two rows and two columns where you multiply by 0 and 1, then, go down the diagonal, from upper left to bottom right. Then, fill in the numbers above the diagonal. Using the commutative property, fill in the bottom of the diagonal.

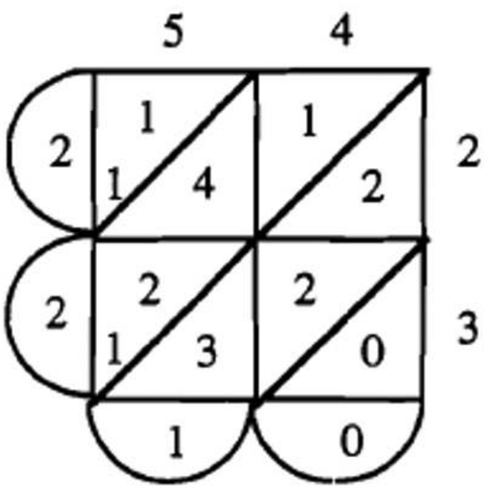
Exercise 26

Make up a multiplication table for each base indicated.

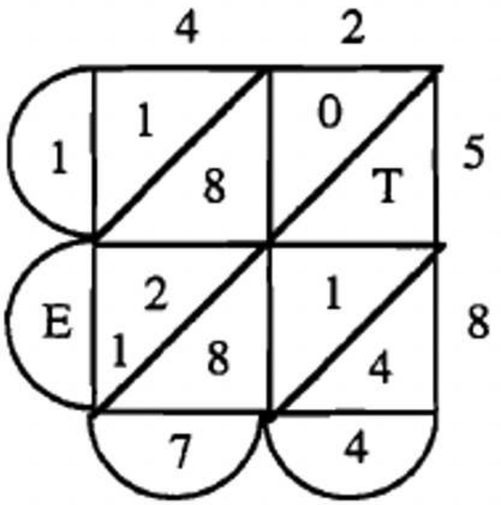
a. Base Two Multiplication Table	b. Base Three Multiplication Table
c. Base Six Multiplication Table	d. Base Eight Multiplication Table
e. Base Twelve Multiplication Table	

If you can quickly figure out the answer to multiplying two single digits in any given base, you can now use a lattice or the standard algorithm to do multiplication in different bases. Here are some examples of multiplying numbers in different bases using the lattice method.

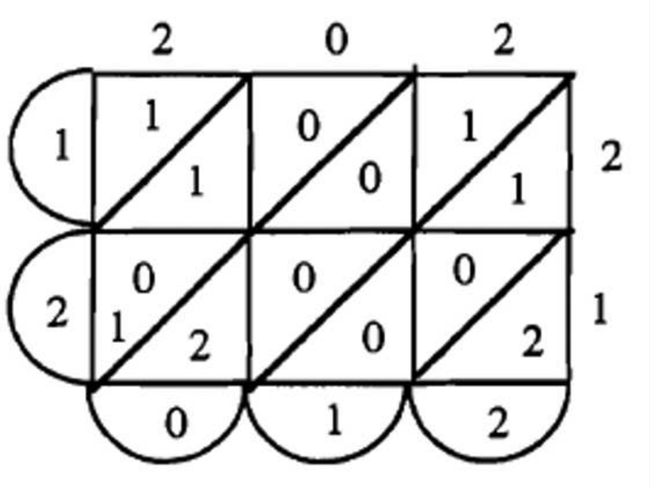
$54_{\text{six}} \times 23_{\text{six}} = 2210_{\text{six}}$



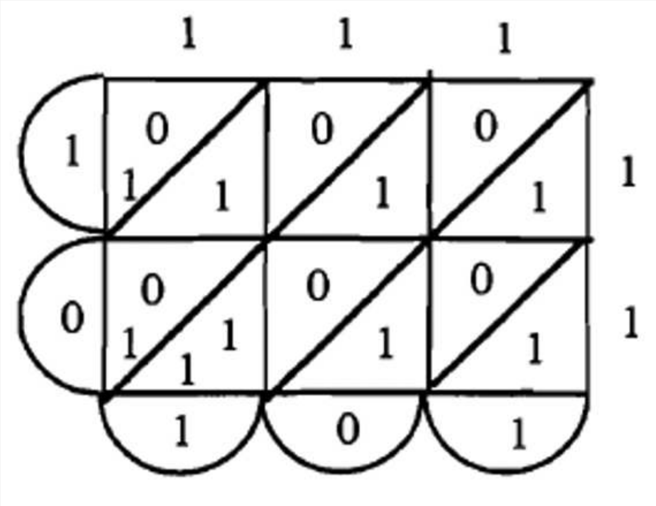
$42_{\text{twelve}} \times 58_{\text{twelve}} = 1E74_{\text{twelve}}$



$202_{\text{three}} \times 21_{\text{three}} = 12012_{\text{three}}$

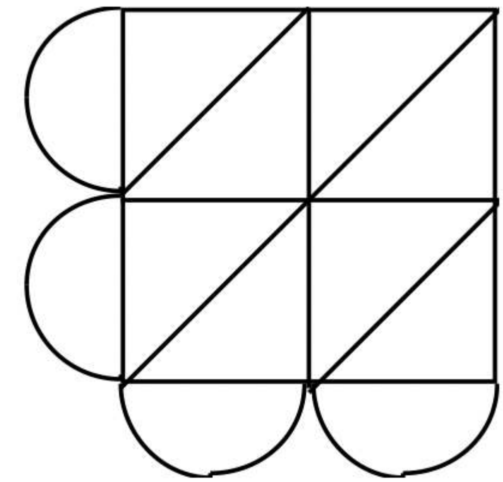

2
0
2
1
1
1
0
0
1
1
2
2
0
1
2
0
0
0
2
1
0
1
2

$111_{\text{two}} \times 11_{\text{two}} = 10101_{\text{two}}$

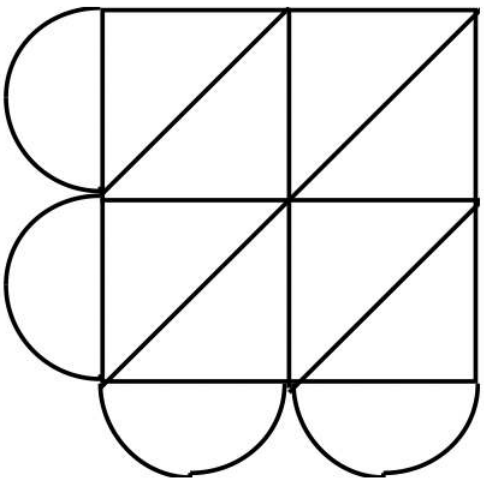

1
1
1
1
0
1
1
0
1
0
1
1
0
0
1
1
1
0
1
0
1
1
1
1
0
1

? Exercise 27

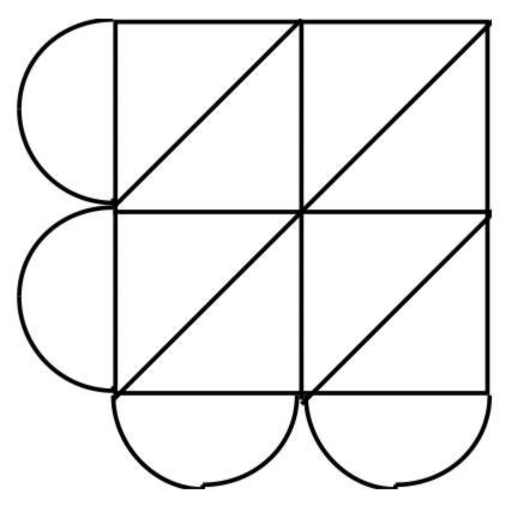
Use a lattice to multiply each of the numbers in the base given.



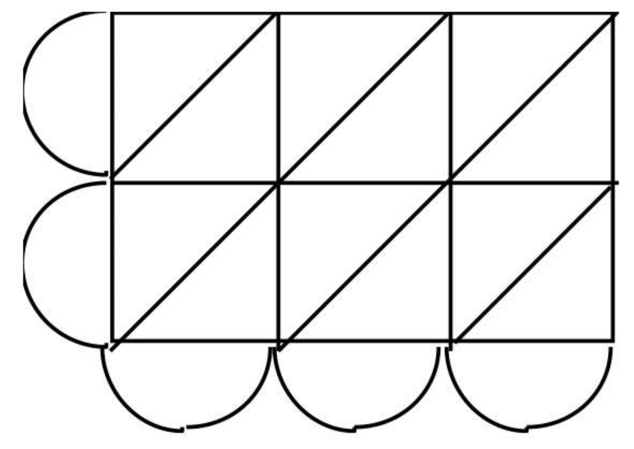
a. $32_{\text{five}} \times 14_{\text{five}} =$



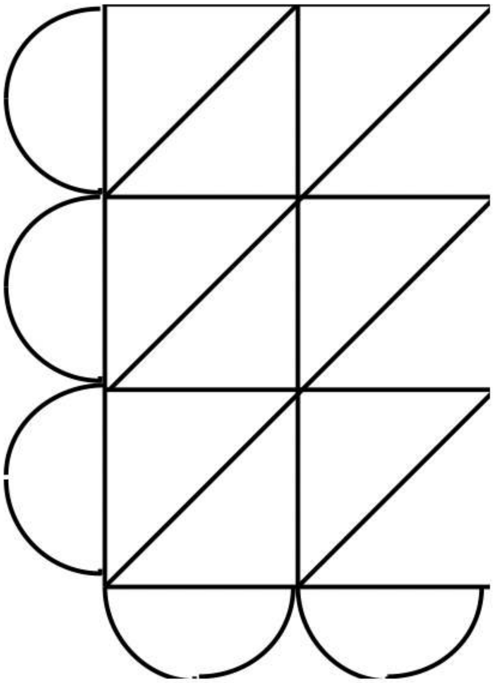
b. $93_{\text{thirteen}} \times 27_{\text{thirteen}} =$



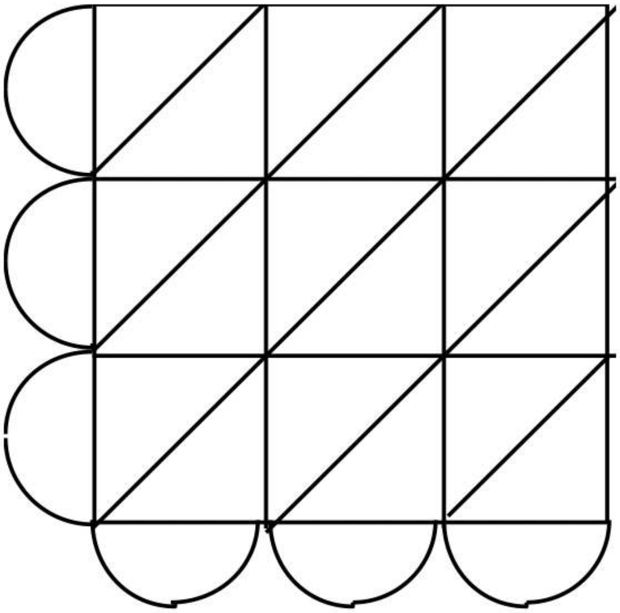
c. $12_{\text{three}} \times 21_{\text{three}} =$



d. $524_{\text{seven}} \times 23_{\text{seven}} =$



e. $23_{\text{seven}} \times 524_{\text{seven}} =$



f. $212_{\text{four}} \times 132_{\text{four}} =$

Finally, you could use the standard multiplication algorithm with carrying. This is the shortest algorithm in terms of space on the paper, but you have to be fairly adept at mental calculations, which doesn't come naturally to most of us when working in other bases.

Before going on to the standard multiplication algorithm, we need to notice a few more things. First of all, remember when we were working with the blocks to figure out a $L \times L$, etc.? When something was multiplied by a long, everything moved up one place value. In base ten, when you multiply a whole number by 10, the effect is that the answer is the original number with a zero tacked on to the end. That is, every digit moved up a place value. This is true in all bases. For example, $524_{\text{seven}} \times 10_{\text{seven}} = 5240_{\text{seven}}$ and $1T6_{\text{twelve}} \times 10_{\text{twelve}} = 1T60_{\text{twelve}}$. If you are multiplying by 100 in any base, two zeroes are tacked on to the end, and so on. This principle can be used to multiply 2000×400 by taking advantage of the commutative and associative properties of multiplication as shown below:

$2000 \times 400 = 2 \times 1000 \times 4 \times 100 = 2 \times 4 \times 1000 \times 100 = 8 \times 100000 = 800000.$

Of course, there is no need to write all the steps out. Simply multiply 2 times 4 and tack on five zeroes. Here's one in base eight: $3000_{\text{eight}} \times 500_{\text{eight}} = 1700000_{\text{eight}}$.

The **Partial Products Algorithm** uses the above fact. Doing the partial products algorithm is similar to how we used the distributive property to multiply with the blocks (back in exercise 22). For instance, to multiply, rewrite the problem like this:

$$\begin{aligned} &346 \times 72 = (300+40+6) \times (70+2) \\ &= (300+40+6) \times 70 + (300+40+6) \times 2 \\ &= 300 \times 70 + 40 \times 70 + 6 \times 70 + 300 \times 2 + 40 \times 2 + 6 \times 2 \\ &= 21000 + 2800 + 420 + 600 + 80 + 12 \\ &= 24912 \end{aligned}$$

The partial products are shown in the third line in the example above. They are:

$300 \times 70, 40 \times 70, 6 \times 70, 300 \times 2, 40 \times 2$ and 6×2

It's easier to write this in a vertical format. I'll show this problem two ways. It doesn't matter which partial products you multiply first. The second way it is shown is more similar to our standard algorithm, where we start with the unit's value of the bottom number and eventually work up to the larger place values.

$\begin{array}{rr} 346 & \\ \times \underline{72} & \\ 21000 & (300 \times 70) \\ 2800 & (40 \times 70) \\ 420 & (6 \times 70) \\ 600 & (300 \times 2) \\ 80 & (40 \times 2) \\ \underline{12} & (6 \times 2) \\ 24912 & \end{array}$	$\begin{array}{rr} 346 & \\ \times \underline{72} & \\ 12 & (2 \times 6) \\ 80 & (2 \times 40) \\ 600 & (2 \times 300) \\ 420 & (70 \times 6) \\ 2800 & (70 \times 40) \\ \underline{21000} & (70 \times 300) \\ 24912 & \end{array}$

On the left the first three partial products are 346×70 and the second three partial products are 346×2. On the right, the first three partial products are 2×346 and the second three partial products are 70×346.

Our standard algorithm is simply a shortening up of the partial products algorithm. We don't write all the zeroes and we doing the carrying involved with adding more than one partial product at a time in our head. Sure, it's shorter, but it is easier to make mistakes.

$$\begin{array}{rr} 346 & \\ \times \underline{72} & \\ 692 & (2 \times 346) \\ \underline{24220} & (70 \times 346) \\ 24912 & \end{array}$$

Usually we just move over one place to the left to account for the zero in the one's place and don't write the zero in 24220. We "think" 7 times 346.

? Exercise 28

For each problem, write out the partial products in a vertical format to multiply the numbers together. Show your work, identifying the multiplication for each partial product as shown in the example on the bottom of the previous page.

a. $\begin{array}{r} 538 \\ \times \underline{34} \end{array}$	b. $\begin{array}{r} 257 \\ \times \underline{941} \end{array}$

Finally, I'll provide some examples of what the multiplication looks like when the standard algorithm is used to multiply numbers in different bases. You have to remember to write down the number and all carries in the base you are working in. We tend to always put the numeral with the most digits on the top, but this is not necessary. Here are three ways to multiply $563_{\text{eight}} \times 24_{\text{eight}}$.

IMPORTANT NOTE:

Each numeral below is in base eight, even though the "eight" isn't written in each intermediate step. E.g., 4×3 in the first multiplication below, is done in base 8; therefore $4 \times 3 = 14$.

$\begin{array}{r} 563_{\text{eight}} \\ \times\underline{24}_{\text{eight}} \\ 2714 \\ \underline{13460} \\ 16374_{\text{eight}} \end{array}$	$\begin{array}{r} 24_{\text{eight}} \\ \times\underline{563}_{\text{eight}} \\ 74 \\ 1700 \\ \underline{14400} \\ 16374_{\text{eight}} \end{array}$	$\begin{array}{rr} 563_{\text{eight}} & \\ \times\underline{24}_{\text{eight}} & \\ 14 & (4\times 2) \\ 300 & (4\times 60) \\ 2400 & (4\times 500) \\ 60 & (20\times 3) \\ 1400 & (20\times 60) \\ \underline{12000} & (20\times 500) \\ 16374_{\text{eight}} & \end{array}$

Multiply the following numbers together using **two different algorithms** (or switching the order of the numbers around as shown above). Show your work. Do not do it by converting each number to base ten, multiplying them together, and then convert back to the given base. Use each algorithm at least one time; lattice, partial products, standard algorithm, using the blocks with charts, using the distributive property with or without the blocks. You have lots of choices, but always keep in mind what base you are working in. You need to know all the algorithms for the exam so be sure to try them all. Okay, have fun!

? Exercise 29

$23_{\text{five}} \times 43_{\text{five}}$

First way:	Second way:

? Exercise 30

$612_{\text{seven}} \times 35_{\text{seven}}$

First way:	Second way:

? Exercise 31

$1011_{\text{two}} \times 1101_{\text{two}}$

First way:	Second way:

? Exercise 32

$2121_{\text{three}} \times 202_{\text{three}}$

First way:	Second way:

? Exercise 33

$4T1_{\text{tweleve}} \times 27_{\text{tweleve}}$

First way:	Second way:

? Exercise 34

$423_{textsix} \times 145_{\text{six}}$

First way:	Second way:

Exercise 35

$2212_{\text{three}} \times 210_{\text{three}}$

First way:	Second way:

Exercise 36

$5421_{\text{six}} \times 44_{\text{six}}$

First way:	Second way:

Exercise 37

$10110_{\text{two}} \times 1111_{\text{two}}$

First way:	Second way:

Exercise 38

$E4T_{\text{twelve}} \times E2_{\text{twelve}}$

First way:	Second way:

Exercise 39

$6661_{\text{eight}} \times 72_{\text{eight}}$

First way:	Second way:

Exercise 40

$1212_{\text{four}} \times 302_{\text{four}}$

First way:	Second way:

Exercise 41

$4234_{\text{five}} \times 244_{\text{five}}$

First way:	Second way:

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4.3: Homework

- Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)
- Start a new module on the front side of a new page and write the module number on the top center of the page.
- Answers without supporting work will receive no credit.
- Some solutions are given in the solutions manual.
- You may work with classmates but do your own work.

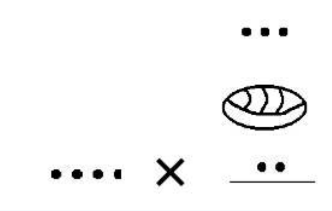
? HW #1

Use the repeated addition definition of multiplication to compute the following. First, write out the meaning of the multiplication, and then compute the answer.

a. 8×4	b. 4×11

? HW #2

Use the repeated addition definition of multiplication to compute the following. Make sure you write out the meaning of the multiplication **in the system given** showing all of the work and exchanges. Do not do the problem in base ten.



? HW #3

Use the definition of multiplication for trains to compute the following. Then translate to make an equation using numbers.

a. $P \times W =$ _____ translates to _______________
b. $K \times W =$ _____ translates to _______________

? HW #4

Write the Cartesian product

a. {3, x} × {0, 1, 6}	b. {a, b, 0} × {1, 2}

? HW #5

Use the set theory definition of multiplication to verify $3 \times 2 = 6$

? HW #6

Complete the following using your base blocks. Show all work.

a. $F \times L$	b. $B \times L$	c. $F \times F$	d. $F \times B$	e. $B \times F$

? HW #7

Write the base four multiplication table

? HW #8

Compute the following $2506_{\text{seven}} \times 451_{\text{seven}}$ using the lattice method.

a. 2506×451	b. $2506_{\text{seven}} \times 451_{\text{seven}}$

? HW #9

Compute 19×24 and 24×19 using the Duplation method. Show all steps.

? HW #10

For each set of three numbers given, illustrate an example of the associative property of multiplication, and then illustrate an example of the distributive property of multiplication over addition. Follow procedures outlined in this module

a. $2_{\text{five}}, 3_{\text{five}}, 4_{\text{five}}$	b. $3_{\text{six}}, 4_{\text{six}}, 5_{\text{six}}$

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CHAPTER OVERVIEW

5: Binary Operations

The teacher said to the student,

"Multiply 3×7*."*

The teacher announced to the students that were going to be learning about operations.

The students didn't understand. This was a mathematics class. They didn't want to do operations. That was for doctors to learn.

The teacher said, "You do operations in this class all the time."

"WE DO?" answered the puzzled students.

Now they were a little skeptical.

But they were a little curious, too.

When they went home that night, they proudly told their parents that they performed operations in school.

Most parents looked at each other and smiled weakly.

5.1: Operations and Properties
5.2: Homework

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5.1: Operations and Properties

This exercise set is designed to give you an understanding of what "binary operations" are, and to give you a deeper understanding for the commutative, associative and distributive properties. To do this, we're going to define and work with some nonsense operations. First, we need to get a good grasp on what an operation is. The binary operations you are familiar with are addition, subtraction, multiplication and division. This means that you are performing a rule using two numbers. For instance, we know what to do when we see the plus sign (+), the subtraction sign (–), the multiplication sign (× or •) or the division sign (÷) between two numbers. There is a specific "rule" that we apply.

Suppose you were asked to compute 5 ♓ 3. You wouldn't know what to do unless someone told you what ♓ meant. It's like asking someone who has never heard of addition, or seen an addition sign (+) to compute 5 + 3. In order to do a computation, the operation used must be defined. The operations you already know about are addition, multiplication, subtraction and division. You also know how to compute with exponents, and how to compare numbers (<, = or >).

Let's start off by defining what ♓ means. This is just a made-up operation that we're defining to be used in this workbook, so that you'll have a deeper understanding of binary operations. There really isn't such an operation as i in the real world.

Define ♓ like this: M ♓ N = 3M + 2N + 8.

The variables I use to define this binary operation are arbitrary. I could use any letters or symbols I want. Or I could explain how to perform the binary operation ♓ , which is much more cumbersome.

This is how I could explain how to compute: To compute with ♓ , multiply the number before the ♓ by 3 and add this to twice the number after the ♓ , and then add 8. As you can see, it is easier to "explain" by using variables. Here are three more ways I could have defined ♓ , without changing the meaning of ♓.

a ♓ b = 3a + 2b + 8	b ♓ a = 3b + 2a + 8	x ♓ y = 3x + 2y + 8

Notice that in each case, one multiplies the first symbol by 3, adds it to twice the second symbol, and then adds 8! It's the same rule, but I chose different variables to "explain" the rule. You have to pay careful attention to what the ***meaning*** of the operation is, and follow the rule exactly. It's like working with functions in algebra.

Let's compute and simplify a few problems with ♓ :

5 ♓ 3 = 3(5) + 2(3) + 8 = 15 + 6 + 8 = 29
3 ♓ 5 = 3(3) + 2(5) + 8 = 9 + 10 + 8 = 27
r ♓ s = 3r + 2s + 8

The rule for the operation doesn't necessarily depend on both of the variables used, and in fact, may not depend on either of them. On the next page, several new operations are defined; some computations use only one of the variables, some use neither. Suppose there are eight new binary operations that are defined as follows:

♓	a ♓ b = 3a + 2b + 8
*	a * b = a + 2b
,	a, b = $a^2 + b^2$
!	a ! b = 2 (Notice the answer doesn't depend on a or b)
$\oplus$	$a \oplus b$ = 3ab
#	a # b = the smaller value of a or b
@	a @ b = 2b (Notice the answer doesn't depend on a)
⊙	a ⊙ b = 2a + 2b
$\boxtimes$	a $\boxtimes$ $b = a^2 + b$

Remember that a and b are just "dummy" variables. Any variables could have been used to define the above functions. The first operation, *, could have been defined like this: m * n = m + 2n. The meaning of the definition is exactly the same. To apply the definition of * to get the answer, it says to take the first number and add it to twice the second number.

Here are a several examples for you to study before going on to the next page

6 ♓ 7 = 3(6)+2(7)+8 = 18+14+8 = 40	2 ⊕ 5 = 3(2)(5) = 30
7 ♓ 6 = 3(7)+2(6)+8 = 21+12+8= 41	r ⊕ s = 3rs
4 ♓ 4 = 3(4)+2(4)+8 = 12+8+8 = 28	4 # 7 = 4
v ♓ z = 3v + 2z + 8	5 # 2 = 2
5 * 3 = 5 + 2(3) = 5 + 6 = 11	7 # 4 = 4
4 * 7 = 4 + 2(7) = 4 + 14 = 18	c # d = the smaller value of c or d
3 * 5 = 3 + 2(5) = 3 + 10 = 13	5 @ 6 = 2(6) = 12
v * z = z + 2z	7 @ 3 = 2(3) = 6
5 , 3 = $5^2 + 3^2 = 25 + 9 = 34$	6 @ 5 = 2(5) = 10
4 , 3 = $4^2 + 3^2$ = 16 + 9 = 25	p @ q = 2q
3 , 5 = $3^2 + 5^2$ = 9 + 25 = 34	6 ⊙ 3 = 2(6) + 2(3) = 12 + 6 = 18
v , z = $v^2 + z^2$	5 ⊙ 8 = 2(5) + 2(8) = 10 + 16 = 26
5 ! 3 = 2	3 ⊙ 6 = 2(3) + 2(6) = 6 + 12 = 18
8 ! 7 = 2	h ⊙ k = 2h + 2k
(junk) ! (stuff) = 2	4 $\boxtimes$ $7 = 4^2$ + 7 = 16 + 7 = 23
w ! q = 2	6 $\boxtimes$ $9 = 6^2$ + 9 = 36 + 9 = 45
$5 \oplus 2 = 3(5)(2) = 30$	7 $\boxtimes$ $4 = 7^2$ + 4 = 49 + 4 = 53
$4 \oplus 7 = 3(4)(7) = 84$	z $\boxtimes$ $n = z^2 + n$

Exercise 1

Compute the following. Show all of the steps. If you need help, look at the examples on the previous page

7 ♓ 6 = ____________________	4 ⊕ 7 = ____________________
4 ♓ 4 = ____________________	2 ⊕ 5 = ____________________
v ♓ z = ____________________	r ⊕ s = ____________________
5 * 3 = ____________________	4 # 7 = ____________________
4 * 7 = ____________________	5 # 2 = ____________________
3 * 5 = ____________________	7 # 4 = ____________________
v * z = ____________________	c # d = ____________________
5 , 3 = ____________________	5 @ 6 = ____________________
4 , 3 = ____________________	7 @ 3 = ____________________
3 , 5 = ____________________	6 @ 5 = ____________________
v , z = ____________________	p @ q = ____________________

5 ! 3 = ____________________	6 $\odot$ 3 = ____________________
8 ! 7 = ____________________	5 $\odot$ 8 = ____________________
(junk) ! (stuff) = ____________	3 $\odot$ 6 = ____________________
w ! q = ____________________	h $\odot$ k = ____________________
5 $\oplus$ 2 = ____________________	4 $\boxtimes$ 7 = ____________________
	6 $\boxtimes$ 9 = ____________________
	7 $\boxtimes$ 4 = ____________________
	z $\boxtimes$ n = ____________________

The above problems are the same examples that were done on the previous page. If you need help, look back at the examples again. Don't go on until you can get them all right.

? Exercise 2

Compute and simplify the following. Show all of the steps.

a. 8 * 4
b. 4 , 7
c. 79 ! 88
d. 7 $\oplus$ 2
e. 6 # 4
f. 4 @ 9
g. 5 $\odot$ 2
h. 6 $\boxtimes$ 5
i. 2 ♓ 3

An operation, ♦, is **commutative** if for any two values, X and Y, X ♦ Y = Y ♦ X.

Again, ♦ is just a "dummy" operation and "X" and "Y" are dummy variables. For a particular operation to be commutative, the equation must always be true no matter what values are used for X and Y.

For instance, the operation * is commutative only if m * n = n * m is always true no matter what values are put in for m or n.

To show that an operation ***is not*** *commutative*, all you need to do is provide a counterexample (with particular values) that shows the equation is not true for at least those particular values. *To prove an operation* ***is*** *commutative* is more involved because you must prove it is always true no matter what values you use. You would have to switch the order of the original values (a and b, or X and Y, etc.), and show algebraically that both expressions simplify to the same thing.

From my examples after defining the operations and the problems you worked in exercise 2, it should be clear which of the eight operations are not commutative.

Let @ be defined as follows: m @ n = 2n. Is @ commutative?

Solution: If @ is commutative, then m @ n = n @ m for all values m and n.

But, 5 @ 6 = 12 and 6 @ 5 = 10.

Therefore, @ is not commutative since 5 @ 6 $\neq$ 6 @ 5.

Let & be defined as follows: m & n = 2mn. Is & commutative?

Solution: If & is commutative, then m & n = n & m for all values m and n. First, I'd try some numbers in for a and b to see if I might come up with a counterexample. For instance, 5 & 6 = 2(5)(6) = 60 and 6 & 5 = 2(6)(5) = 60. No counterexample here. So, I use algebra to prove that m & n = n & m. Since m & n = 2mn, and n & m = 2nm, the question is: Does 2mn = 2nm? Yes, it does! Since m & n = n & m for all m and n, then & is commutative.

Exercise 3

For each operation listed, determine whether it is commutative or not. *If it* ***is not commutative****, give a* ***counterexample****,* like I did for @. *If it* ***is commutative, prove it is*** *commutative,* like I did for & above. Begin each problem by stating what equation must be true if the operation listed is commutative

a. ! is defined like this: m ! n = 2. Determine if ! is commutative.

Write the general equation that is true if ! is commutative: ______________

Is ! commutative? ________. If you answered yes, prove ! is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

b. ⊕ is defined like this: m ⊕ n = 3mn. Determine if ⊕ is commutative.

Write the general equation that is true if ⊕ is commutative: _______________

Is ⊕ commutative? ________. If you answered yes, prove ⊕ is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

c. # is defined: m # n = the smaller value of m or n. Determine if # is commutative.

Write the general equation that is true if # is commutative: _______________

Is # commutative? ________. If you answered yes, provide an example. If you answered no, provide a counterexample to illustrate it is not commutative

d. ⊙ is defined like this: m ⊙ n = 2m + 2n. Determine if ⊙ is commutative.

Write the general equation that is true if ⊙ is commutative:

Is ⊙ commutative? ________. If you answered yes, prove ⊙ is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

e. ⊠ is defined like this: m ⊠ $n = m^2 + n$. Determine if ⊠ is commutative.

Write the general equation that is true if ⊠ is commutative:

Is ⊠ commutative? ________. If you answered yes, prove ⊠ is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

f. $m, n = m^2 + n^2$. Determine if o is commutative.

Write the general equation that is true if , is commutative:

Is , commutative? ________. If you answered yes, prove , is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

g. m * n = m + 2n. Determine if * is commutative.

Write the equation that must be true if * is commutative:

Is * commutative? ________. If you answered yes, prove * is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

h. m ♓ n = 3m + 2n + 8 . Determine if ♓ is commutative.

Write the general equation that is true if ♓ is commutative:

Is ♓ commutative? ________. If you answered yes, prove ♓ is commutative. If you answered no, provide a counterexample to illustrate it is not commutative.

Before going on to determining whether an operation is associative or distributive, we should compute a few more problems, which are a bit more involved. Make sure you follow the order of operations as you work through these next few problems. Look at the examples first

Example 1: $(2 \oplus 3) \oplus 4$ (first do $2 \oplus 3)$ $18 \oplus 4$ 216	Example 2: 4, (3, 2) (first do 3 , 2) 4, 13 185

Exercise 4

Simplify each of the following. Do the order of operations ***(do what is in parentheses first)*** and show each step.

a. (3 * 5) * 2	b. (3 @ 5) @ 2	c. (3 ! 5) ! 7
d. (3 $\oplus$ 4) $\oplus$ 2	e. (3 $\odot$ 5) $\odot$ 2	f. (3 $\boxtimes$ 2) $\boxtimes$ 4

Exercise 5

Compute the following, using the definitions for the operations as shown above. Note that more than one operation is in some of the problems. When simplifying, use the order of operations ***(do what is in parentheses first)*** and show each step.

a. 3 $\oplus$ (5 $\oplus$ 2)	b. 3 $\odot$ (5 $\odot$ 2)	c. 3 $\boxtimes$ (4 $\boxtimes$ 2)
d. 8 # (9 # 6)	e. (8 # 9) # 6	f. (4 , 3) , 2
g. 2 @ (3 # 4)	h. (2 @ 3) # 4	i. (1 $\odot$ 5) # 40

An operation, ♦, is associative if (X ♦ Y) ♦ Z = X ♦ (Y ♦ Z) for values of X, Y and Z.

Again, ♦ is just a "dummy" operation and "X" and "Y" and "Z" are dummy variables. For a particular operation to be associative, the equation must always be true no matter what values are used for X, Y and Z.

For instance, the operation * is associative only if (v * w) * x = v * (w * x) is always true no matter what values are put in for v, w or x.

To show that an operation ***is not*** *associative*, all you need to do is provide a counterexample (using actual numbers) that shows the equation is not true for at least those particular numbers. *To prove an operation* ***is*** *associative* is more involved because you must prove it is always true no matter what values you use. You would have to switch the parentheses, and *show algebraically* that both expressions always simplify to the same thing.

Example

Let @ be defined as follows: m @ n = 2n. Is @ associative?

If @ is associative, then (a @ b) @ c = a @ (b @ c) for all values a,b and c. First, I'd try some numbers in for a, b and c to see if I might come up with a counterexample: (2 @ 3) @ 4 = 6 @ 4 = 8, and 2 @ (3 @ 4) = 2 @ 8 = 16. This shows @ is not associative and provides us with a counterexample: Since $(2@3)@4 \neq 2@(3@4)$, @ is not associative.

Example

Let & be defined as follows: m & n = 2mn. Is & associative?

If & is associative, then (a & b) & c = a & (b & c) for all values a,b and c. First, I'd try some numbers in for a, b and c to see if I might come up with a counterexample: (2 & 3) & 4 = 12 & 4 = 96, and 2 & (3 & 4) = 2 & 24 = 96. No counterexample here. I might want to try another example with numbers, or I can go directly to using algebra to see if I can prove that it is always true that (a & b) & c = a & (b & c). First, we need to simplify the left side: (a & b) & c = 2ab & c = 4abc. Now, we have to simplify the right side: a &(b & c)=a &2bc = 4abc. Since (a &b)& c = a & (b & c), then & is associative.

Exercise 6

For each operation listed, determine whether it is associative or not. If it **is not associative**, give a **counterexample**, like I did for @ and ♓. If it **is** associative, **prove it is associative**, like I did for & above. Begin each problem by stating the general equation that is true if the operation listed is associative.

a. ! is defined like this: m ! n = 2. Determine if q is associative.

Write the general equation that is true if ! is associative:

Is ! associative? ________. If you answered yes, prove ! is associative. If you answered no, provide a counterexample to illustrate it is not associative

b. ⊕ is defined like this: m ⊕ n = 3mn. Determine if ⊕ is associative.

Write the general equation that is true if ⊕ is associative:

Is ⊕ associative? ________. If you answered yes, prove ⊕ is associative. If you answered no, provide a counterexample to illustrate it is not associative.

c. m # n = the smaller value of m or n. Determine if # is associative.

Write the general equation that is true if # is associative:

Is # associative? ________. If you answered yes, provide an example. If you answered no, provide a counterexample to illustrate it is not associative.

d. ⊙ is defined like this: m ⊙ n = 2m + 2n. Determine if ⊙ is associative.

Write the general equation that is true if ⊙ is associative:

Is ⊙ associative? ________. If you answered yes, prove ⊙ is associative. If you answered no, provide a counterexample to illustrate it is not associative.

e. $\boxtimes$ is defined like this: m $\boxtimes$ $n = m^2 + n$. Determine if $\boxtimes$ is associative.

Write the general equation that is true if $\boxtimes$ is associative:

Is $\boxtimes$ associative? ________. If you answered yes, prove $\boxtimes$ is associative. If you answered no, provide a counterexample to illustrate it is not associative.

f. m, $n = m^2 + n^2$. Determine if , is associative.

Write the general equation that is true if , is associative:

Is , associative? ________. If you answered yes, prove , is associative. If you answered no, provide a counterexample to illustrate it is not associative.

g. a * b = a + 2b. Determine if * is associative.

Write the general equation that is true if * is associative:

Is * associative? ________. If you answered yes, prove * is associative. If you answered no, provide a counterexample to illustrate it is not associative.

An operation, ♦, distributes over another operation, ϕ if for any values of X, Y and Z:

X ♦ (Y ϕ Z) = (X ♦ Y) ϕ (X ♦ Z). This is a **Left-Hand** Distributive Property, because the symbol on the LEFT (in this case an X) is being distributed across the parentheses to the right. The Right-Hand Distributive Property states: An operation, ♦, distributes over

another operation, ϕ if for any values of X, Y and Z: (Y ϕ Z) ♦ X = (Y ♦ X) ϕ (Z ♦ X).

Unless otherwise stated, assume that the distributive property refers to the Left-Hand Distributive Property.

Remember that ♦ and ϕ are just "dummy" operations and "X" and "Y" and "Z" are dummy variables. For a particular operation to distributive over another operation, the equation

must always be true no matter what values or variables are used for X, Y and Z.

For instance, the operation * distributes over + only if v * (w + x) = (v * w) + (v * x) is always true no matter what value you put in for v, w and x.

*To show that an operation **does not*** distribute over another operation, you only need to provide a counterexample (using actual numbers) that shows the equation is not true for at least those particular numbers. *To prove an operation **does*** distribute over another operation is more involved because you must prove it is always true no matter what values you use. You would first have to work the left side of the equation (by using order of operations — simplifying in parentheses first), and then work the right side of the equation (by using order of operations by simplifying in parentheses first), and finally you would need to *show algebraically* that both expressions always simplify to the same thing.

? Exercise 7

Let @ be defined as follows: m @ n = 2n. We're going to determine if @ distributes over addition. Write the equation that would be true if @ distributed over addition:

I'll help you with the rest of the solution. If @ distributes over addition, then a @ (b + c) = (a @ b) + (a @ c) for all values a, b and c. First, I'd try some numbers in for a, b and c to see if I might come up with a counterexample: 5 @ (3 + 4) = 5 @ 7 = 14 and (5 @ 3) + (5 @ 4) = 6 + 8 = 14. No counterexample here. I can try another example with numbers or try proving it algebraically. First, simplify the left side using the definition of @: a @ (b + c) = 2(b + c) = 2b + 2c. Now to simplify the right side: (a @ b) + (a @ c) = 2b + 2c. Since both expressions equal the same thing (2a + 2b), a @ (b + c) = (a @ b) + (a @ c), and therefore, we say that YES, @ distributes over addition.

? Exercise 8

Let @ be defined as follows: m @ n = 2n. We are going to determine if addition distributes over @. Write the equation that is true if addition distributes over @:

I'll help you with the rest of the solution. If addition distributes over @, then a + (b @ c) = (a + b) @ (a + c) for all values a,b and c. First, I'd try some numbers in for a, b and c to see if I might come up with a counterexample: 5 + (3 @ 4) = 5 + 8 = 13 and (5 + 3) @ (5 + 4) = 8 @ 9 = 18. This shows that addition does not distribute over @ and provides us with a counterexample.
Since $5+(3@4) \neq (5+3)@(5+4)$, addition does not distribute over @.

? Exercise 9

Let & be defined by: m & n = 2mn and let $ be defined by: m $ $n=m^2$. We are going to determine if & distributes over $ or if $ distributes over &. First, let's determine if & distributes over $.

a. Write the equation that would be true if & distributed over $:

I'll help you with the rest of the solution. If & distributes over $, then this equation is true: a & (b $ c) = (a & b) $ (a & c). Let's compute each side of the equation by putting in some values for a, b and c to see if we find a counterexample. We'll see if 2 & (3 $ 4) and (2 & 3) $ (2 & 4) are equal. Since 2 & (3 $ 4) = 2 & 9 = 36, and
(2 & 3) $ (2 & 4) = 12 $ 16 = 144, the equation isn't true and we have a counterexample. Therefore, & does not distribute over $.

Next, we'll determine if $ distributes over &.

b. Write the equation that is true if $ distributes over &:

I'll help you with the rest of the solution. If $ distributes over &, then this equation is true for all values of a, b and c: a $ (b & c) = (a $ b) & (a $ c). Let's compute each side of the equation by putting in some values for a, b and c to see if we find a counterexample. Let's see if 2 $ (3 & 4) and (2 $ 3) & (2 $ 4) are equal. Since 2 $ (3 & 4) = 2 $ 24 = 4, and (2 $ 3) & (2 $ 4) = 4 & 4 = 32, the equation isn't true (since $4 \neq 32$) and we have a counterexample. Therefore, $ does not distribute over &.

Exercise 10

Let ! and ⊕ be defined as follows: a ! b = 2 and $a \oplus b$ = 3ab.

a. Write the general equation that is true if ! distributes over ⊕.

b. Does ! distribute over ⊕? __________

c. If ! distributes over ⊕, prove it. Otherwise, provide a counterexample to illustrate that !does not distribute over ⊕.

Continuation of exercise 10 where a ! b = 2 and a ⊕ b = 3ab.

d. Write the general equation that is true if ⊕ distributes over !.

e. Does ⊕ distribute over ! ? ___________

f. If ⊕ distributes over ! , prove it. Otherwise, provide a counterexample to illustrate that ⊕ does not distribute over !.

Exercise 11

a. Write the general equation that is true if addition distributes over multiplication.
b. Does addition distribute over multiplication? __________
c. If addition distributes over multiplication, prove it. Otherwise, provide a counterexample to illustrate that addition does not distribute over multiplication.

Exercise 12

a. Write the general equation that is true if multiplication distributes over subtraction.
b. Does multiplication distribute over subtraction? __________
c. If multiplication distributes over subtraction, prove it. Otherwise, provide a counterexample to illustrate that multiplication does not distribute over subtraction.

Exercise 13

Let , and @ be defined as follows: a , $b = a^2 + b^2$ and a @ b = 2b

a. Write the general equation that is true if , distributes over @.
b. Does , distribute over @? ___________
c. If , distributes over @, prove it. Otherwise, provide a counterexample to illustrate that , does not distribute over @.
d. Write the general equation that is true if @ distributes over ,.
e. Does @ distribute over ,? ___________
f. If @ distributes over ,, prove it. Otherwise, provide a counterexample to illustrate that @ does not distribute over ,.

Exercise 14

a. Make up and define two new operations.
b. Write a general equation that is true if one operation distributes over the other one.
c. Determine if the distributive property holds for your operations by proving it or providing a counterexample illustrating the equation in part b is not true.

Exercise 15

Define $\oint$ and $\boxed{\wedge}$ as follows: m $\oint$ n = 2m + 3n and m $\boxed{\wedge}$ n = mn + 2

a. State the equation that is true if $\oint$ is commutative:

Is $\oint$ commutative?

Prove it is commutative or provide a counterexample if it is not commutative.

b. State the equation that is true if $\boxed{\wedge}$ is commutative:

Is $\boxed{\wedge}$ commutative?

Prove it is commutative or provide a counterexample if it is not commutative.

c. State the equation that is true if $\oint$ is associative:

Is $\oint$ associative?

Prove it is commutative or provide a counterexample if it is not commutative.

d. State the equation that is true if $\boxed{\wedge}$ is associative:

Is $\boxed{\wedge}$ associative?

Prove it is commutative or provide a counterexample if it is not commutative.

e. State the equation that is true if $\oint$ distributes over addition:

Does $\oint$ distribute over addition?

Prove it is commutative or provide a counterexample if it is not commutative.

f. State the equation that is true if $\boxed{\wedge}$ distributes over addition:

Does $\boxed{\wedge}$ distribute over addition?

Prove it is commutative or provide a counterexample if it is not commutative.

g. State the equation that is true if $\oint$ distributes over $\boxed{\wedge}$:

Does $\oint$ distribute over $\boxed{\wedge}$?

Prove it is commutative or provide a counterexample if it is not commutative.

h. State the equation that is true if $\boxed{\wedge}$ distributes over $\oint$:

Does $\boxed{\wedge}$ distribute over $\oint$?

Prove it is commutative or provide a counterexample if it is not commutative.

For these last few exercises, you'll be working with the Right-Hand Distributive Property. For clarification, " # right-hand distributes over @ "means the same thing as "# distributes over @, using the Right-Hand Distributive Property." Again, here is the definition of the Right-Hand Distributive Property: An operation, $\blacklozenge$, distributes over another operation, ϕ if for any values of X, Y and Z: (Y ϕ Z) $\blacklozenge$ X = (Y $\blacklozenge$ X) ϕ (Z $\blacklozenge$ X).

? Exercise 16

State the equation that is true if multiplication right-hand distributes over addition:

Does multiplication right-hand distribute over addition?

If multiplication right-hand distributes over addition, provide an example. Otherwise, provide a counterexample if multiplication does not right-hand distribute over addition

? Exercise 17

State the equation that is true if addition right-hand distributes over multiplication:

Does addition right-hand distribute over multiplication?

Prove addition right-hand distributes over multiplication or provide a counterexample if addition does not right-hand distribute over multiplication:

Exercise 18

State the equation that is true if division right-hand distributes over addition:

Does division right-hand distribute over addition?

If division right-hand distributes over addition, provide an example. Otherwise, provide a counterexample if division does not right-hand distribute over addition:

Exercise 19

State the equation that is true if division left-hand distributes over addition:

Does division left-hand distribute over addition?

Prove division left-hand distributes over addition or provide a counterexample if division does not left-hand distribute over addition:

Exercise 20

Define $\oint$ and $\boxed{\wedge}$ as follows: $\mathbf{m \oint n = 2m + 3n}$ and $\mathbf{m \boxed{\wedge} n = mn + 2}$

a. State the equation that is true if $\oint$ right-hand distributes over addition:

Does $\oint$ right-hand distribute over addition?

Prove $\oint$ right-hand distributes over addition or provide a counterexample if $\oint$ does not right-hand distribute over addition.

b. State the equation that is true if $\boxed{\wedge}$ right-hand distributes over addition:

Does $\boxed{\wedge}$ right-hand distribute over addition?

Prove $\boxed{\wedge}$ right-hand distributes over addition or provide a counterexample if $\boxed{\wedge}$ does not right-hand distribute over addition.

c. State the equation that is true if $\oint$ right-hand distributes over $\boxed{\wedge}$:

Does $\oint$ right-hand distribute over $\boxed{\wedge}$?

Prove $\oint$ right-hand distributes over $\boxed{\wedge}$ or provide a counterexample if $\oint$ does not right-hand distribute over $\boxed{\wedge}$.

d. State the equation that is true if $\boxed{\wedge}$ right-hand distributes over $\oint$:

Does $\boxed{\wedge}$ right-hand distribute over $\oint$?

Prove $\boxed{\wedge}$ right-hand distributes over $\oint$ or provide a counterexample if $\boxed{\wedge}$ does not right-hand distribute over $\oint$.

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5.2: Homework

- **Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)**
- **Start a new module on the front side of a new page and write the module number on the top center of the page.**
- **Answers without supporting work will receive no credit.**
- **Some solutions are given in the solutions manual.**
- **You may work with classmates but do your own work**

? Exercise 1

Compute and simplify the following. Show all of the steps. Use the binary operations defined at the beginning of this module.

a. 5 * 9	b. 4 * 8	c. b * a	d. 1 , 5
e. 7 , 4	f. b , a	g. 88 ! 79	h. m ! n
i. b ! a	j. $6 \oplus 5$	k. $2 \oplus 7$	l. $b \oplus a$
m. 8 # 1	n. 4 # 6	o. b # a	p. 8 @ 5
q. 9 @ 4	r. b @ a	s. 6 ⊙ 7	t. 2 ⊙ 5
u. b ⊙ a	v. 3 ⊠ 8	w. 5 ⊠ 6	x. b ⊠ a
y. 7 ♓ 8	z. 3 ♓ 2	aa. b ♓ a	

? Exercise 2

Simplify each of the following. Show each step.

a. 3 * (4 * 2)	b. 3 @ (5 @ 2)	c. 3 ! (5 ! 2)

? Exercise 3

Compute the following, using the definitions for the operations defined at the beginning of this module. Show each step.

a. (1 ⊠ 2) ⊙ 7	b. (3 * 1) , c	c. (8 ! 5) ⊕ 6
d. 7 ⊠ (6 ! 13)	e. 5 * (2 ⊙ 2)	f. 8 ! (5 ⊕ 6)

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CHAPTER OVERVIEW

6: Integers

The parent said, "What? You're studying what?"

"Integers," said the child.

"What's an integer?" asked the parent.

The child replied, "Well, like 2 or -2 or 11 or -11."

"Oh," said the parent."Why didn't you just say you were studying about numbers?"

The child just stared back.

- 6.1: Addition and Subtraction
- 6.2: Multiplication
- 6.3: Homework

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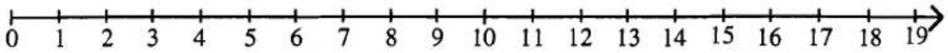
6.1: Addition and Subtraction

You will need*: Positive and Negative Counters (Material Cards 18A and 18B)*

Remember that whole numbers are made up of zero and the counting numbers: 0, 1, 2, 3, 4, 5, . . . When school children first see a number line, they usually focus on the whole numbers and the number line looks something like this:

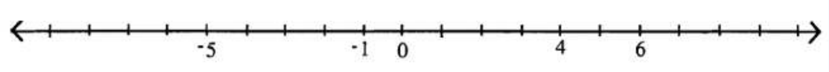


In this model, the number line starts at zero, and the arrow to the right indicates that it goes on indefinitely to the right. Actually, there are infinitely many numbers (or points) between any two numbers on the number line. For example, a few of the infinite number of points between 1 and 2 are 3/2, 4/3, 5/4, 6/5, 7/5, 8/5, etc. On the number line, there are infinitely many irrational numbers as well. Irrational numbers cannot be written as the ratio of two whole numbers, like the way we represent reduced fractions. Examples of some irrational numbers between 3 and 4 are $\sqrt{11}$, $\sqrt{15}$, 3.10110111..., and π. Contrary to popular belief, π *does not* equal 3.14 and π. *does not* equal 22/7. Both 3.14 and 22/7 are rational numbers (unlike π which is an irrational number, NOT a rational number) and 3.14 and 22/7 are only ***approximations*** commonly used when calculating with π. Irrational numbers will be explored and discussed in more detail in a different module.

Okay, let's get back to our discussion of the number line. The number line shown at the beginning of this exercise set is really only a half-line, so to call it a number *line* is really a misnomer. A horizontal line goes on indefinitely in *both* the left and right direction, not just to the right. Numbers to the right of zero are called positive numbers. Numbers to the left of zero are called *negative numbers*. Zero is the only number that has no sign – it is neither positive, nor negative. For each positive number, there is a corresponding negative number, which on a given number line, is the same distance from zero as the positive number, but it is to the left side of zero. Numbers the same distance from but on opposite sides of zero are called "opposites." Negative numbers are represented like this: -5 is read "negative five", -20 is read "negative twenty", etc. So, 5 and -5 are opposites, 20 and -20 are opposites, and zero is the opposite of itself.

? Exercise 1

For each marking on the number line, fill in the missing numbers



Definitions: The set of **positive integers** (which are also commonly called the **natural numbers** or **counting numbers**) is written as: {1, 2, 3, 4, ... }. Positive numbers may also be written with a positive sign before it. For instance, 4 may be written a +4.

The set of **whole numbers** contains both zero and the natural numbers and is written as: {0, 1, 2, 3, ... }.

Definition: The set of negative integers contains exactly all the opposites of all the positive integers.

1. Write the set of **negative integers**:

Definition: The set of **integers** is made up of zero, the positive integers and the negative integers.

2. Write the set of **integers**:

Definition: The **absolute value** of a number on the number line is defined to be the distance that point is from zero. Since absolute value is defined in terms of distance, it can never be negative. To evaluate the absolute value of a number, state the distance it is from zero.

✓ Examples

The absolute value of 5 is 5, since the distance from 5 to 0 is 5 units. The absolute value of -5 is also 5, since the distance from -5 to 0 is also 5 units.

Exercise 3

Evaluate the absolute value of each of the following numbers:

a. 4: ____	b. -8: ____	c. 2: ____	d. -1: ____	e . 0: ____

To indicate the absolute value of a number, we enclose it between two vertical lines. "The absolute value of -15 is 15" is written like this: | -15 | = 15. In order to simplify a problem with an absolute value sign, the part inside the absolute value sign must be simplified first. So, if the part inside the absolute value sign is not simply a numeral, think of the part inside as being in parentheses, and simplify it first. Then, take the absolute value of the numeral. You'll have to do this for part c and f of exercise 4.

Exercise 4

Simplify each of the following:

a. $\lvert 7 \rvert =$ ______		c. $\lvert 7 - 3 \rvert =$ _______
d. $\lvert \frac{3}{7} \rvert =$ ______	e. $\lvert - \frac{3}{7} \rvert =$ ______	f. $\lvert 2{\cdot}5 - 4 \rvert =$ _______

Exercise 5

Write two numerals that have an absolute value of 6 ______ and _______

Exercise 6

Write two numerals that have an absolute value of 19 ______ and ______

Exercise 7

Write all numerals that have an absolute value of zero _______

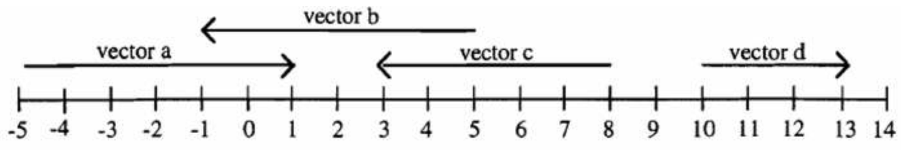
Exercise 8

Write all numerals that have an absolute value of -10 (negative 10) _______

Each positive and negative number can be represented with a **directed line segment** called a **vector**.

Basically, a vector looks like an arrow. It has two properties: it has a certain length (called the "magnitude") and it points in a certain direction. Since we'll be using horizontal number lines, we'll be drawing horizontal vectors. An arrow pointing to the right will denote a positive number and an arrow pointing to the left will denote a negative number.

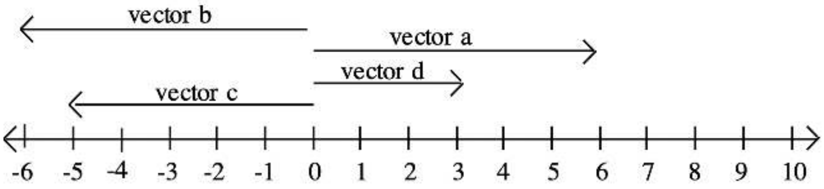
Below is a number line, with some vectors shown above. Vector a is 6 units long and the arrow points to the right. Therefore, it represents the number +6. Vector b is also 6 units long, but it points to the left. Therefore, it represents the number -6. Both vectors have a magnitude of 6.



Exercise 9

a. What number does vector c represent? __________
b. What number does vector d represent? __________

Each vector has an initial point (the starting point of the vector) and a terminal point (the ending point, where the arrow is shown). If the initial point of the vector is placed at zero, the number line can be used as a convenient marker – the terminal point of the vector is simply placed on the number being represented. But it *isn't necessary to* represent it that way, as you could see from all the vectors shown in the above examples. Vectors a, b, c and d could have been drawn with the initial point at zero as shown below. If you draw a vector with its initial point at zero, the number that vector represents is simply the same number on the number line where its terminal point lands.



Another way of thinking about numbers is to think of them in terms of "**actions**". You could think of the number "5" as the action of moving 5 spaces to the right. When you draw the vector representing 5, that is a way of showing that particular action. You could think of the number "-3" as the action of moving 3 spaces to the left. When you draw the vector representing -3, that is a way of showing that particular action.

The number line, along with this idea of thinking about numbers in terms of actions, can be used to add numbers together.

Definition: **To add two numbers, m and n, you DO the first action, m. The addition sign means NOW DO the second action, n. The answer to the problem (which is the sum: m + n) is the single action that could have been done to achieve the same result as doing the separate actions in succession.**

Consider this practical way of thinking about it. Think of walking east as representing a positive number and walking west as representing a negative number. If you walked 5 blocks east, then 7 blocks west, then 3 blocks east, you would be one block east of your original starting point. This can be a way to think about the problem: 5 + (-7) + 3 = 1.

Procedure for using vectors as actions to add two numbers, m and n

1. Pick a starting point and draw the vector m (which is a way of representing the action of m) with its initial point at the designated starting point.
2. The addition sign means starting at the terminal point of the previous action DO the next action. So, draw the vector n, placing the initial point of n at the terminal point of vector m.
3. The sum, m + n, is the same number that would be represented by a vector with its initial point placed on the original starting point and its terminal point on the final terminal point after the last action took place.

For CONVENIENCE only, it is easiest to always use a number line and pick zero as the initial starting point. By doing that, the answer to the problem will be wherever the terminal point of the last action **landed** on the number line.

Before going on, it is important to note that it is not necessary to mark every point on a number line. In fact, this is impossible since there are infinitely many points on a number line. However, it is usually a good idea to mark zero and at least one point on either side of zero. Mark the number line in a way that makes it convenient to do the problem.

We'll use the above procedure to add the following numbers. For these examples and for the exercises, start at zero for convenience. Draw the first vector. Then move up a little to draw the second vector so it is obvious which vector was drawn first and which one was drawn last. There should be one vector shown for each number in the sum. Since we are placing the initial point of the first vector on zero, we can state the answer by noting where the terminal point of the last vector landed on the number line.

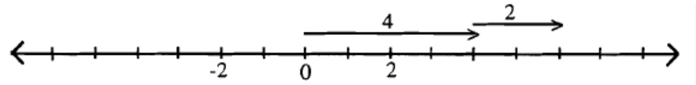
Example: $4+2$

4 + 2.

Solution

Starting at zero, DO the action 4 (which is shown by drawing the vector representing 4), then starting at the point where the first action ended, DO the action 2 (shown by drawing the vector representing 2).

The answer is **6** since the terminal point of the last vector landed on 6.



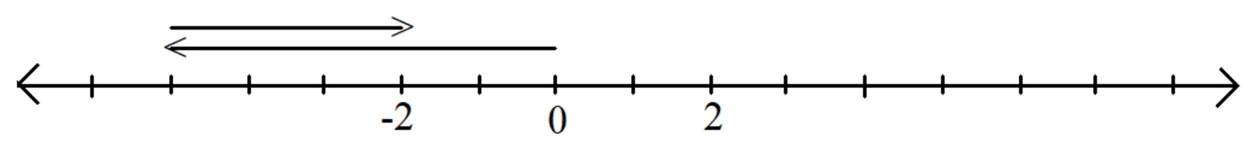
✓ Example: $-5+3$

-5 + 3.

Solution

Starting at zero, DO the action -5 (which is shown by drawing the vector representing -5), then DO the action 3 (shown by drawing the vector representing 3).

The answer is **-2** since the terminal point of the last vector landed on -2.



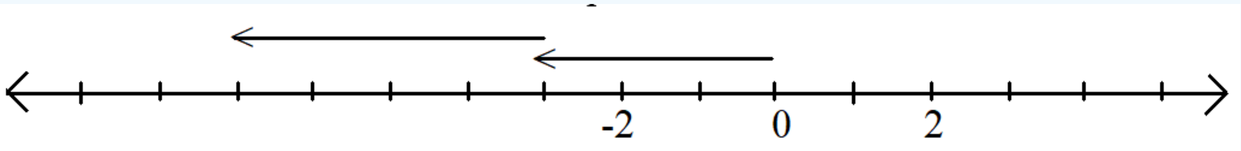
✓ Example: $-3+-4$

-3 + -4.

Solution

Starting at zero, DO the action -3 (shown by drawing the vector representing -3), then DO the action -4 (shown by drawing the vector representing -4).

The answer is **-7** since the terminal point of the last vector landed on -7.



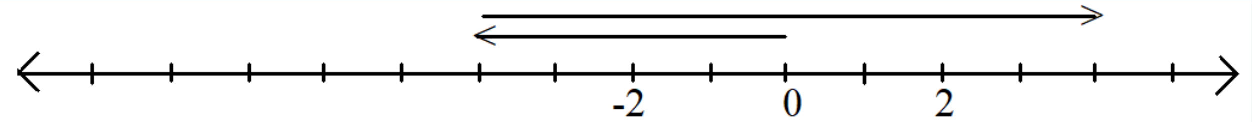
✓ Example: $-4+8$

-4 + 8.

Solution

Starting at zero, DO the action -4 (which is shown by drawing the vector representing -4), then DO the action 8 (shown by drawing the vector representing 8).

The answer is **4** since the terminal point of the last vector landed on 4.

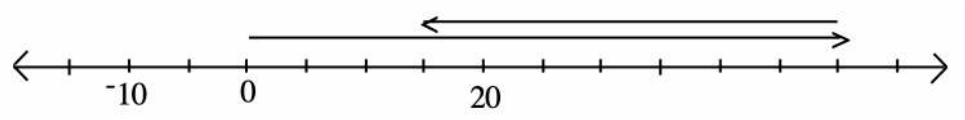


✓ Example: $50+-35$

50 + -35.

Starting at zero, DO the action 50 (shown by drawing the vector representing 50), then DO the action -35 (shown by drawing the vector representing -35).

The answer is **15** since the terminal point of the last vector landed on 15.



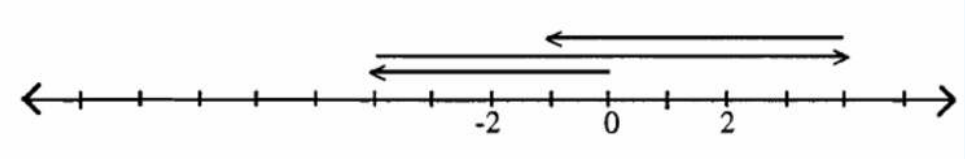
It is IMPERATIVE that you LABEL your number line. Otherwise, it is not clear what each vector represents. Do not assume that each mark stands for 1 unit. Without the label of at least one number on each side of zero on the last example, one might assume the last example shown represented 10 + -7 and that the answer was 3.

? Exercise 10

Use vectors on the number line to add the following numbers together. Mark and LABEL your number line with at least zero and a point on either side.

Here is an example of how to do the following problems:

-4 + 8 + (-5) = -1 since *the terminal point of the last vector landed on -1.*



a. 5 + (-8) = _____ since ____

b. -7 + (-2) = _____ since ____

c. -5 + 9 = _____ since ____

d. 7 + (-6) = _____ since ____

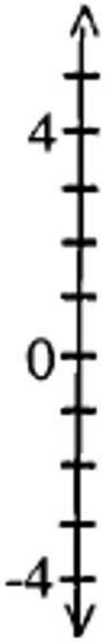
e. -7 + 5 = _____ since ____

f. -100 + 40 = _____ since ____

g. -3 + (-2) + 4 = _____ since ____

h. 4 + (-12) + 5 = _____ since ____

Sometimes, you'll see vertical number lines instead of horizontal number lines. In this case, the positive numbers go up, and are represented by vertical vectors with the arrow pointing up. Likewise, negative numbers go down and are represented by vertical vectors pointing down. An example of a vertical number line is shown on the right.



Exercise 11

There is a common use of using a vertical number line. Can you think of one? See if you can think of any and write any down here.

If you need a hint for Exercise 11, think about the weather.

We'll continue to think about actions as we use vectors to do subtraction problems.

Definition: **To subtract a number, m, you UNDO the action, m which means you DO the OPPOSITE of the action m.**

Whether or not you are doing a subtraction problem, an addition problem, or both in the same problem, the answer to the problem is the single action that could have been done to achieve the same result as doing the separate actions in succession.

You can think of subtraction as undoing some mistake.

Consider a variation on the original example given for addition. To refresh your memory, here was that problem: Think of walking east as representing a positive number and walking west as representing a negative number. If you walked 5 blocks east, then 7 blocks west, then 3 blocks east, you would be one block east of your original starting point. This can be a way to think about the problem: 5 + (-7) + 3 = 1.

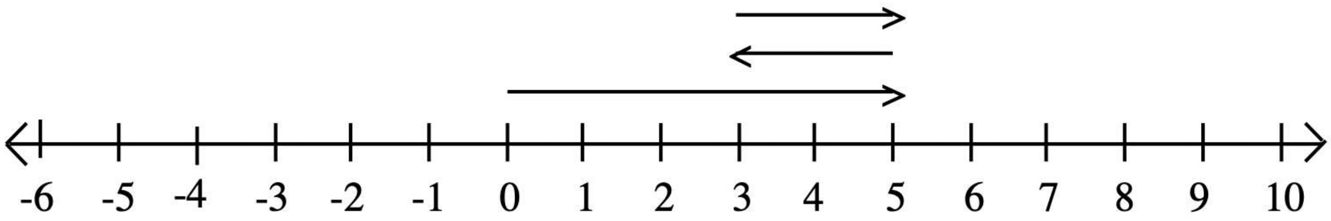
Exercise 12

Imagine you walked 5 blocks east, then 2 blocks west, and then realized you had made a mistake. You weren't supposed to walk 2 blocks west. You would need to undo that mistake. What would you do to undo that mistake and get back to where you were before you walked 2 blocks west on accident?

In exercise 12, I hope you realized you needed to walk 2 blocks east to undo your mistake. To translate the steps taken in #12, we could write 5 + (-2) – (-2) = 5. Here is how we would explain and show the actions using vectors on the number line.

5 + (-2) – (-2): Starting at zero, DO the action 5 (shown by drawing the vector representing 5), then DO the action -2 (shown by drawing the vector representing -2), then UNDO the action of -2 (which means to DO the OPPOSITE of the action -2, which is to DO the action +2, shown by drawing the vector representing +2).

The answer is **5** since the terminal point of the last vector landed on 5.



Procedure for using vectors to add or subtract two numbers:

1. If the first number is m, pick a starting point and draw vector m (which is a way of representing the action m) with its initial point at the designated starting point.
2. If the next sign is addition, DO the next action by drawing the vector representing the next number, placing the initial point of this vector at the terminal point of the previous vector. If, on the other hand, the next sign is subtraction, UNDO the next action. So, draw the vector representing the opposite of the next number, placing the initial point of this vector at the terminal point of the previous vector.
3. If there is another addition or subtraction sign, go back to step 2. Otherwise, at this point, we can find the answer. The answer is the same number that would be represented by a vector with its initial point placed on the original starting point and its terminal point on the final terminal point after the last action took place.

Again, using zero as the starting point is CONVENIENT because the answer to the problem will be wherever the terminal point of the last action **landed** on the number line.

We'll use the above procedure to compute the following problems. For these examples and for the exercises, start at zero for convenience. Draw the first vector. Then move up a little to draw the second vector so it is obvious which vector was drawn first and which one was drawn last. There should be one vector shown for each number in the problem. Since we are placing the initial point of the first vector on zero, we can state the answer by noting where the terminal point of the last vector landed on the number line.

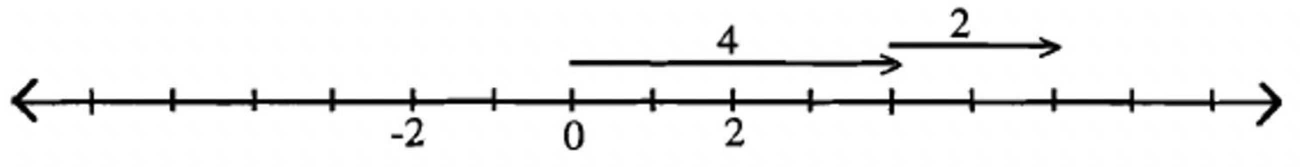
✓ Example

4 – (-2).

Solution

Starting at zero, DO the action 4 (which is shown by drawing the vector representing 4), then UNDO the action -2 (which means to DO the OPPOSITE of the action -2, which is to DO the action +2, shown by drawing the vector representing +2).

The answer is **6** since the terminal point of the last vector landed on 6.



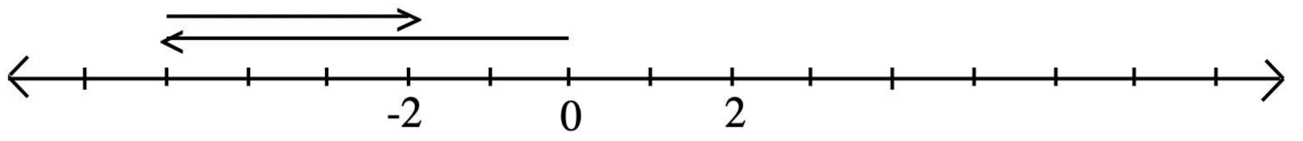
? Example

-5 – (-3).

Solution

Starting at zero, DO the action -5 (shown by drawing the vector representing -5), then UNDO the action -3 (which means to DO the OPPOSITE of the action -3, which is to DO the action +3, shown by drawing the vector representing +3).

The answer is **-2** since the terminal point of the last vector landed on -2.



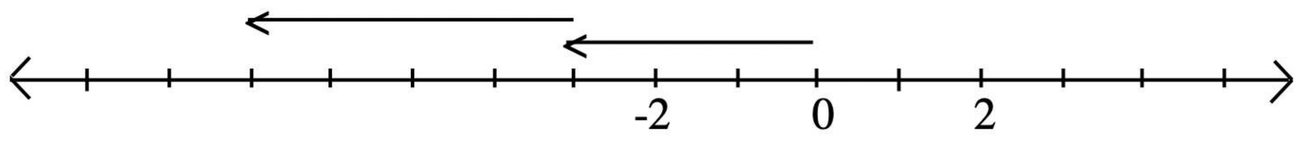
Example

-3 – (+4).

Solution

Starting at zero, DO the action -3 (shown by drawing the vector representing -3), then UNDO the action +4 (which means to DO the OPPOSITE of the action +4, which is to DO the action -4, shown by drawing the vector representing -4).

The answer is **-7** since the terminal point of the last vector landed on -7.



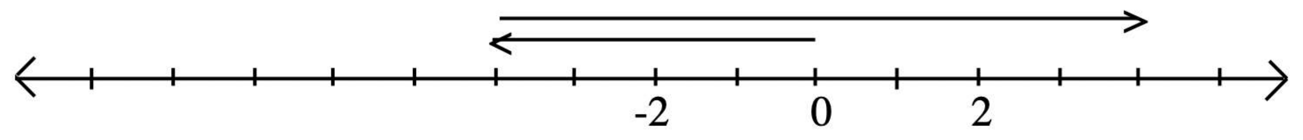
Example

-4 – (-8).

Solution

Starting at zero, DO the action -4 (shown by drawing the vector representing -4), then UNDO the action -8 (which means to DO the OPPOSITE of the action -8, which is to DO the action +8, shown by drawing the vector representing +8).

The answer is **4** since the terminal point of the last vector landed on 4.



Example

50 + (-35).

Solution

Starting at zero, DO the action 50 (shown by drawing the vector representing 50), then DO the action -35 (shown by drawing the vector representing -35).

The answer is **15** since the terminal point of the last vector landed on 15.

It is IMPERATIVE that you LABEL your number line. Otherwise, it is not clear what each vector represents. Do not assume that each mark stands for 1 unit. Without the label of at least one number on each side of zero on the last example, one might assume the last example shown represented 10 + (-7) and that the answer was 3.

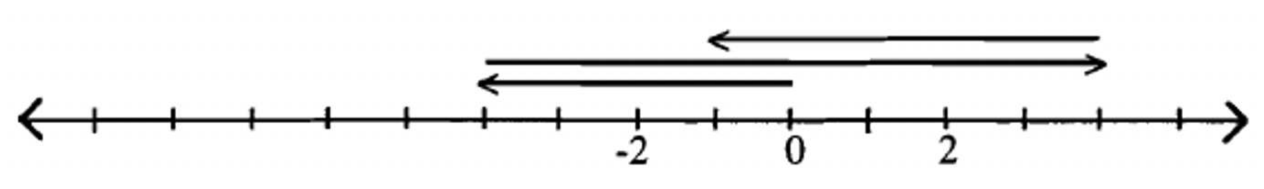
Exercise 13

Use vectors on the number line to add the following numbers together. Mark and LABEL your number line with at least zero and a point on either side.

Here is an example of how to do the following problems:

Example

-4 + 8 + (-5) = -1 since *the terminal point of the last vector landed on -1.*



a. 5 – (+8) = _____ since ____ .

b. -7 – (+2) = _____ since ____ .

c. -5 – (-9) = _____ since ____ .

d. 7 – (+6) = _____ since ____ .

e. -7 – (-5) = _____ since ____ .

f. -100 – (-40) = _____ since ____ .

g. -3 – (+2) – (-4) = _____ since ____ .

h. 4 – (+12) + 5 = _____ since ____ .

At this point, it would be beneficial if you looked back at the examples on page 4 and 5 and compared them with the recent examples shown on page 8 and 9. The examples on page 8 and 9 include subtraction, whereas the examples on page 4 and 5 are addition problems. Using the idea of adding and subtracting using vectors and the action approach (either DO or UNDO an action), the problems when done with vectors on the number line look exactly the same. The same is true if you look at the problems in

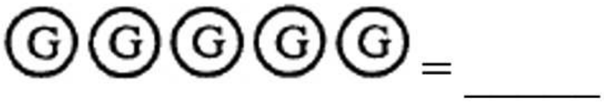
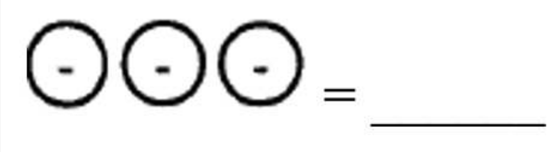
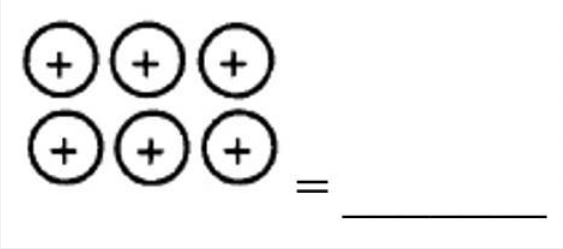
Exercise 10 and 13. Somehow, then, there seems to a connection between subtraction and addition. We'll put it all together after we explore a very different way of adding and subtracting, this time using manipulatives.

Another way to represent positive and negative numbers is by using a collection of positive and negative counters. You need to use counters of one color to represent positive numbers (each counter = +1) and counters of another color to represent negative numbers. If you use the material cards, each green square will represent +1 and each red square will represent -1. We'll discover that there are many ways to represent the same number using these counters. Get out your counters to do the following exercises.

The simplest way to represent a positive number, x, is to have a collection of x green counters. The simplest way to represent a negative number, -y, is to have a collection of y red counters. For example, 4 red counters will represent -4 and 3 green counters will represent +3. In these exercises, I'll sometimes represent a positive counter by either a square or circle with a positive sign (+) or a "G" inside, or by just a "G". Similarly, a negative counter will be represented by either a square or circle with a negative sign (-) or an "R" inside, or by just an "R".

Exercise 14

What number is represented by each of the following collections?

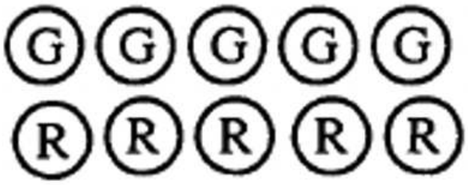
a. 	b. RRRRRRRRR = ______
c. 	d. 

Exercise 15

Under each number, represent the number by showing a collection of counters.

	b. 2	

What number do you think the collection below represents? ______ Explain why.



In exercise 14, there were 5 positives and 5 negatives. Did you come up with an answer of zero? If so, you're on the right track. Basically, when a positive counter is paired with a negative counter, they nullify each other. Any time there are the same number of positive counters as negative counters, you have a representation of the number, zero.

? Exercise 17

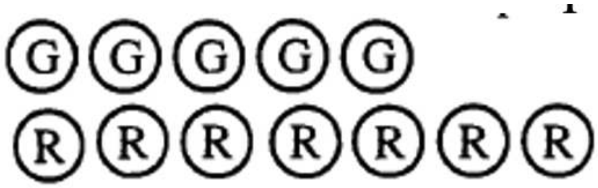
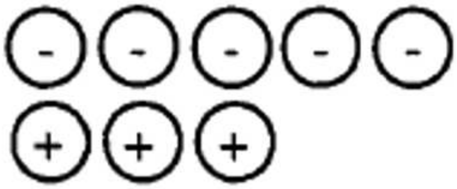
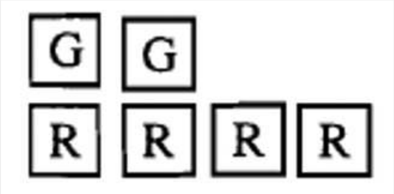
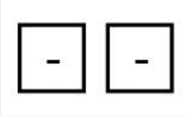
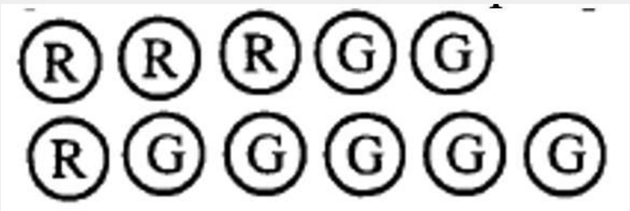
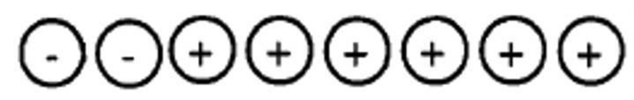
Using positive and negative counters, show three different representations of zero.

a.	b.	c.

Let's see if you can figure out what number each collection represents.

? Exercise 18

State the number represented by each collection

a. The collection below represents _____ 	b. The collection below represents _____ 
c. The collection below represents _____ 	d. The collection below represents _____ 
e. The collection below represents _____ RRRRRRGGG	f. The collection below represents _____ GRRGRGRRGGGGG
g. The collection below represents _____ 	h. The collection below represents _____ 

How do you think you did? Hopefully, you came up with an answer of -2 for a-d. If so, you should be convinced that different collections can represent the same number. Each pairing of a negative with a positive basically "cancels each other out". Therefore, to figure out what number a particular collection represents, simply pair as many positives with negatives as possible (red-green pairs), and remove or ignore those counters. Then, count the counters of the one "color" or "sign" that remains. You may have noticed it was easier to compute a-d because the pairings were more easily seen.

? Exercise 19

Using positive and negative counters, find 3 different representations of 4. Do these with your actual positive and negative manipulatives first. Then, show it on paper.

a.	b.	c.

Exercise 20

Using positive (green) and negative (red) counters, show 3 different representations of -4. Then, show your three different representations below.

a.	b.	c.

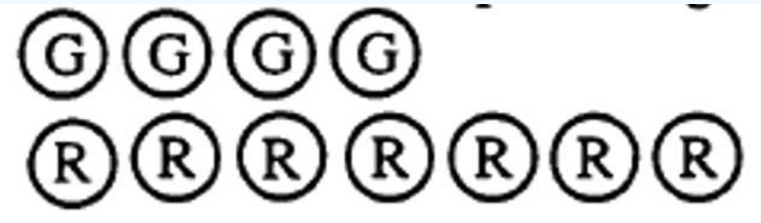
Let's move on to how we can use the counters to add integers. To add integers, simply make a collection of counters representing each number to be added and combine the counters. Then, find the number that the resulting collection represents. Do this by first pairing off any positives and negatives and removing them from the pile.

Example

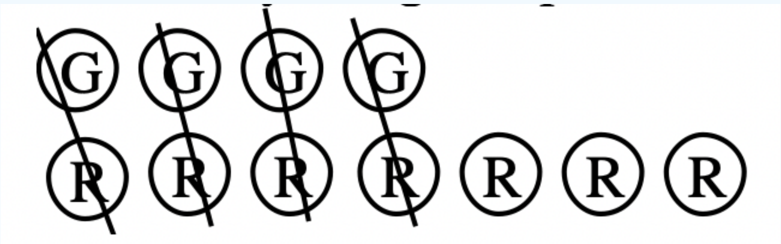
Show two different ways to use the counters to add the 4 + (-7).

First Way:

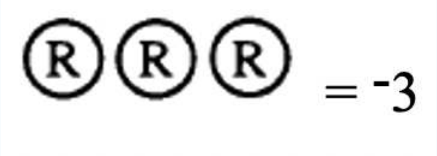
First, combine a collection representing 4 (4G) and a collection representing -7 (7R):



Second, remove any red-green pairs:



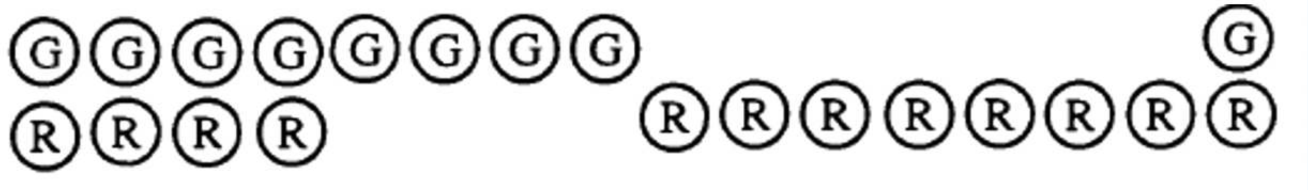
Third, count the counters remaining:



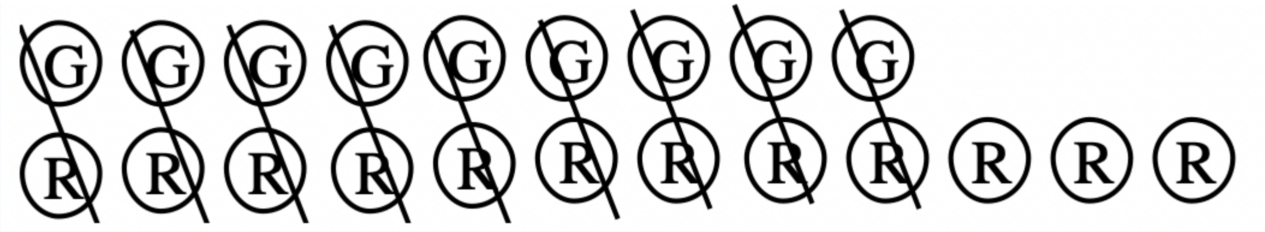
Therefore, 4 + (-7) = -3.

Second Way:

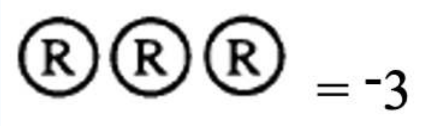
First, combine a collection representing 4 (8G and 4R) and a collection representing 7 (8R and 1G):



Second, remove any red-green pairs:



Third, count what counters are remaining:



Therefore, 4 + (-7) = -3.

Of course, the simpler and more direct method is the first way shown, but the result is the same even when complicated representations of the numbers are used in the first step.

Exercise 21

Use red and green counters to add the following integers. **Do these problems using the actual manipulatives**. Then, ***explain*** and ***show*** the steps below

Although subtraction is a little trickier using counters, they provide a really interesting way of showing how to subtract negative numbers.

Using the counters, m – n means remove a collection of counters representing n from a collection of counters representing m. The number that the resulting collection represents is the difference (answer).

This means we have to start with a collection of counters representing m. In order to remove a collection representing n, it must be possible to remove a collection representing n from the collection used for m. Sometimes, the original collection chosen for m will not suffice, and the collection must be altered by adding some red-green pairs to the collection representing m. Usually, we start by putting in a simple representation of m counters and see if it is possible to remove or taking away a representation of n counters. If so, we do than and find the number that the resulting collection represents. Let's try this on some examples using the physical models (manipulatives) and then practice showing it on paper.

Example: $4 - 2$

4 – 2

This means to remove a collection representing 2 from a collection representing 4.

Step 1: Make a collection representing 4 (4 green counters): $\not GG$GG

Step 2: Take out a collection representating 2 (2 green counters):

Step 3: The collection remaining (GG) represents 2. So, 4 – 2 = 2.

Example: $-5 - (-3)$

-5 – (-3)

This means to remove a collection representing -3 from a collection representing -5.

Step 1: Make a collection representing -5 (5 red counters): RRRRR

Step 2: Take out a collection representing -3 (3 red counters):

Step 3: The collection remaining (RR) represents -2. So, -5 – (-3) = -2.

✓ Example: 3 – 5

3 – 5

This means to remove a collection representing 5 from a collection representing 3.

Step 1: Make a collection representing 3 (3 green counters): GGG

OH NO! *The next step would be to take out a representation of 5 (5 green counters), but notice this is impossible because there are only 3 green counters to begin with.* ***An extra step must be inserted here.*** *We need to alter the representation of 3 so that 5 greens can be taken out. It should make sense that we can always insert some green-red pairs to the representation of 3 (each green-red pair is the effect of adding zero) without changing the representation of 3. In this example, it makes sense to add 2 more red-green pairs, so that there will be 5 greens and 2 reds in the collection. It will still be a collection representing 3. This extra step is often necessary when doing subtraction problems with integers.*

Step 2: Add 2 red-green pairs to the collection representing 3: GGGRGRG

Step 3: Take out a collection representing 5 (5 green counters): $\begin{array}{l}\boxed{\text{GGGGG}}\\ \text{RR}\end{array}$

Step 4: The collection remaining (RR) represents -2. So, 3 – 5 = -2

? Exercise 22

For each problem, state what the problem means (For instance, 4 – (- 9) means remove a collection of counters representing -9 from a collection of counters representing 4), and then explain and show each step you need to take to find the answer. Each problem requires 3 or 4 steps. You should do these problems using the manipulatives, and as you do it, explain and show the steps on paper.

a. 6 – 3
This means
Show all of the steps below.

b. 4 – 6
This means
Show all of the steps below.

c. -7 – (- 6)
This means
Show all of the steps below.

d. -3 – (- 7)
This means
Show all of the steps below.

e. 4 – (- 3)
This means
Show all of the steps below.

f. -2 – 5
This means
Show all of the steps below.

g. 5 – 5
This means
Show all of the steps below.

h. 5 – (- 5)
This means
Show all of the steps below.

i. -4 – (- 4)
This means
Show all of the steps below.

j. -6 – 6
This means
Show all of the steps below.

? Exercise 23

State what the problem means in terms of using positive and negative counters and show all steps to (a) subtract 5 - 8, and then (b) add 5 + (-8).

a. 5 – 8

c. What did you notice about your answers to a and b?

Hopefully, you noticed that removing a collection representing 8 from a collection representing 5 produced the same result as combining a collection representing 5 and -8.

In other words, subtracting 8 produced the same result as adding -8.

? Exercise 24

a. Use the actual manipulatives. Start with a collection of at least 4 red and some green counters and write what you have below:
b. Remove 4 red counters from your collection. Show this step by crossing out or circling 4 red counters in your collection above. After removing those red counters, what number does the collection in part a now represent? ______
c. Start over. Begin by combining the original collection you had in part a with 4 green counters (add 4 green counters to your original collection). Below, show what the collection now looks like.
d. What number does your final collection represent? ______

Exercises 21 and 22 illustrate a very important fact about subtraction. Subtracting a number gives the same result as adding the opposite number. Exercises 10 and 13, and the examples preceding each of those exercises also illustrated this fact.

Note that red and green counters, or positive and negative counters are opposites.

✓ Example

Explain what 7 – 4 means, and then state another way to get the same result.

Solution

7 – 4 means remove a collection representing 4 from a collection representing 7. This gives the same result as combining a collection representing 7 with a collection representing the opposite of 4, which is to combine a collection representing 7 with a collection representing -4 (which means combine 7G with 4R.) So, 7 – 4 = 7 + (- 4).

✓ Example

Explain what 2 – (- 6) means, and then state another way to get the same result.

Solution

2 – (- 6) means remove a collection representing -6 from a collection

representing 2. This gives the same result as combining a collection representing 2 with a collection representing the opposite of -6, which is to combine a collection representing 2 with a collection representing 6 (which means combine 2G with 6G.) So, 2 – (- 6) = 2 + 6.

? Exercise 25

Explain what each problem means and then state another way to get the same result

	b. 4 – 8		

? Exercise 26

Rewrite each subtraction as an equivalent addition problem.

a. 4 – 5 = _____________		

? Exercise 27

Rewrite each problem so it is only in terms of addition. In other words, only change each subtraction part to an equivalent addition part. Do not compute the answer.

? Exercise 28

Fill in the blanks:

a. Subtracting 6 is the same as adding _____ .
b. Subtracting -8 is the same as adding _____ .
c. Subtracting a positive number is the same as adding a _______________ number, where that ______________ number is the opposite of the original positive number.
d. Subtracting a negative number is the same as adding a _______________ number, where that ______________ number is the opposite of the original negative number.
e. Subtracting a number is the same as adding the ________________ of that number.

The one thing you must pay careful attention to is the difference between the subtraction sign and the number being subtracted. When you rewrite a subtraction problem as an equivalent addition problem, you have to remember to change the number that was being

subtracted to its opposite. It is certainly not true that 6 – 3 is the same as 6 + 3. But it is true that 6 minus (positive 3) is equal to 6 plus (negative 3); or 6 – 3 = 6 + (- 3).

It isn't necessary to change subtraction problems to equivalent addition problems. It is, however, sometimes very useful because the addition problem may be easier to work with. And if all subtraction can be written as addition, then one only needs to learn rules for addition and apply them to all addition or subtraction problems. Let me say that although it's a good idea to know the rules for addition and subtraction, it's even more important to be able to go back to the basic concepts and be able to use manipulatives to understand what addition and subtraction really mean.

If you filled in the correct answer to 28e, then you've got it. The relationship between addition and subtraction lies in this fact.

Every subtraction problem can be rewritten as an equivalent addition problem. Instead of subtracting a number, you can add the opposite of that number.

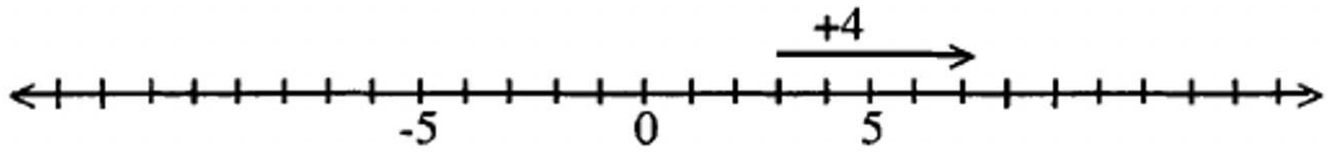
Definition: **Let m and n be any integer. Then, m – n = m + (- n). This is called Adding the Opposite.** Remember: -n is not necessarily negative! If n is positive, -n is negative but if n happens to be negative (n is just a variable), then -n would be positive.

Let's return to the number line and using vectors in conjunction with the *missing addend approach* to compute a subtraction problem. Note that this is an entirely different approach from the way we did it at the beginning of the exercise set, when we did it with doing and undoing actions. First, let's review the definition of subtraction using the missing addend approach.

Definition: Subtraction *(Missing Addend Approach)* The difference of two numbers, m and n, written **m – n** is the number **c** such that **n + c = m**. In other words, if **c** is added to the subtrahend, **n**, then the sum is the minuend, **m**. The answer, **c,** is called the *missing addend.*

Consider the problem 7 – 3. The missing addend approach states the answer to this subtraction problem will be the number that when added to the number shown after the subtraction sign equals the number that precedes the subtraction sign. Since 3 is the number after the subtraction sign, we have to find a number to add to 3 that will give an answer of 7. Using a number line, this means if you place the initial point of a vector at 3, what vector must be drawn (the missing addend can still be thought of as which action must take place) so that the terminal point will land on 7? Draw a vector with its initial point at 3 and its terminal point at 7. Make sure the arrow on the end of the vector *points* to 7.

The number that the vector represents is the answer to the problem. We are trying to find that missing addend, which is the vector we draw. The illustration for 7 – 3 is shown below. The vector has length 4 and points to the right, so the answer is 4, which I've written above the vector (labeled and marked the missing addend) to indicate the answer to the problem.



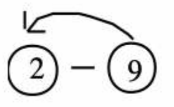
Again, note that this is a **completely different approach to subtraction**. It is not at all like the way we did it at the beginning of the exercise set, when we did it by doing one action, and then undoing another and then seeing what one action could have achieved the same result.

Using the missing addend approach, only one vector is shown, and *the number that vector represents is the answer* to the problem. In other words, the vector *represents* the missing addend you are looking for, which is the difference of the two numbers. You can find the answer to the subtraction problem by drawing a vector, placing initial point of the vector on the subtrahend (the number shown right after the subtraction sign), and placing the terminal point on the minuend (the number preceding the subtraction sign). Note that this definition only make sense when subtracting two numbers, as opposed to adding and/or subtracting several numbers.

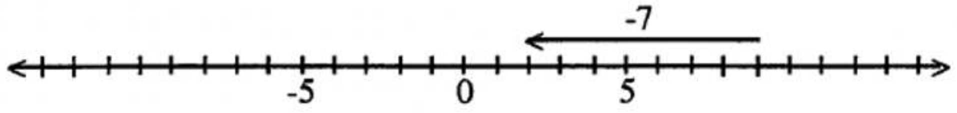
Procedure for subtracting two integers using vectors and the missing addend approach: To find the difference of m and n, draw a vector with its initial point on n and its terminal point on m. The vector drawn is the missing addend, and is the answer to the problem: m – n.

It's important to distinguish between the subtraction sign and the subtrahend (the number shown after the subtraction sign). Also it's important to remember to put the initial point of the vector that will represent the missing addend on the subtrahend (the number AFTER the subtraction sign, and the terminal point on the minuend (the number shown PRECEDING the subtraction sign). One method I sometimes use is to circle the minuend and subtrahend and then draw an arrow from the subtrahend to the minuend so I remember to draw the vector with its initial point on the subtrahend and its terminal point on the minuend.

Let's try the problem 2 – 9. Then, 9 + ____ = 2. Circle the minuend and subtrahend and draw an arrow from the subtrahend to the minuend before you draw the vector.



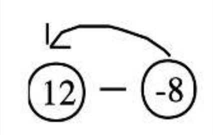
On the number line, draw a vector with its initial point at 9 and its terminal point at 2. The directed vector drawn represents the answer: -7.



Example

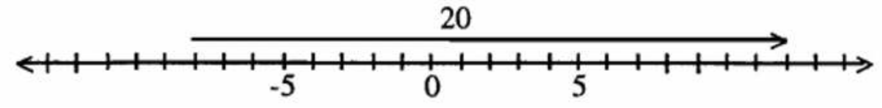
12 – (- 8). Then, -8 + ____ = 12.

Circle the minuend and subtrahend and draw an arrow from the subtrahend to the minuend before you draw the vector.



Note when you draw an arrow from the subtrahend to the minuend, the arrow points the the left in the actual subtraction problem. But that has NOTHING to do with whether or not the arrow on the vector you draw on the number line will point left or right. It points to the minuend. For this example, note that on the missing addend vector drawn below, it points to the right, and that the answer to this problem is a positive number.

Place the initial point of the vector on -8 and the terminal point on 12. The vector drawn below shows the answer: 20.



Exercise 29

Before going on, think about the possibility of using the missing addend approach and not even getting a vector.

a. In other words, what if the initial and terminal point of the vector was on the same number? It would look like a point, not a vector. When would this happen?
b. What would be the answer to a problem where this happened?

Exercise 30

Find the answer to the following subtraction problems by *using vectors on the number line in conjunction with the missing addend approach.* Draw an arrow from the subtrahend to the minuend as shown in the last two examples. Before drawing the missing addend representing the answer to the problem, label and mark your number line with at least 0 and one point on each side of zero. Then, draw the vector that represents the answer to each problem and **label** the vector with that number.

a. 10 – 4 = ______

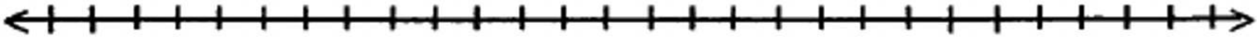
b. -7 – 5 = ______

c. 8 – (- 3) = ______

d. -9 – (- 1) = ______

e. 3 – 10 = ______

f. -5 – (- 8) = ______



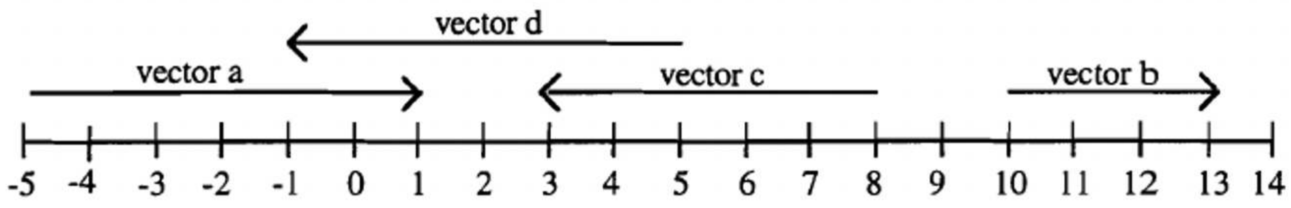
g. 6 – (- 6) = ______

h. 9 – 9 = ______

Exercise 31

By looking at the vectors a - c below, figure out what subtraction problem was being performed, and then state the answer. For instance, vector d came from doing the problem -1 – 5 since the initial point of that vector is on 5 and the terminal point is on -1. Since vector d is 6 units long and goes left, it represents the number -6. Therefore, vector d would be: -1 – 5 = -6.

a. Vector a:	b. Vector b:	c. Vector c:



One thing I love about using the missing addend model to show subtraction is that it works equally well when subtracting with larger integers. Just approximate where the integers are on the number line in relation to themselves and zero. The answer makes sense using the model.

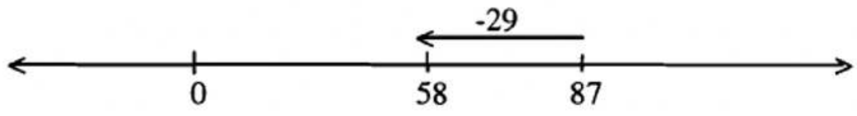
The following examples show how to use vectors on the number line in conjunction with the missing addend approach to find these differences: 58 – 87, -130 – 55 and -130 – -55.

Example 1: 58 – 87

58 – 87.

Solution

Draw a vector from 87 to 58. Since the vector is pointing to the left, the answer is negative and the length of the vector is 29 units long, 87 – 58. So, 58 – 87 = -29.

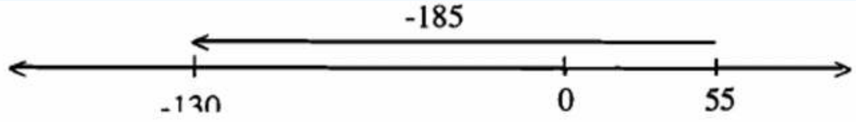


Example 2: $-130 - 55$

-130 – 55.

Solution

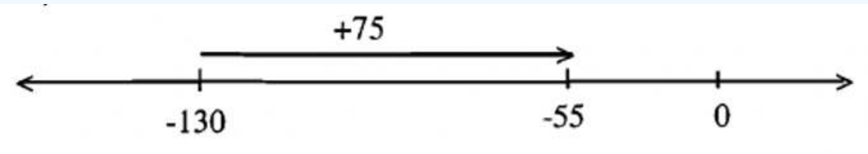
Draw a vector from 55 to -130. Since the vector is pointing to the left, the answer is negative and the length of the vector is 185 units long, since it is 130 units from -130 to 0 plus another 55 units from 0 to 55. So, -130 – 55 = -185.



Example 3: $-55-(-130)$

-55 – (-130).

Draw a vector from -130 to -55. Since the vector is pointing to the right, the answer is positive and the length of the vector is 130 – 55 , or 75 units long. So, -55 – (-130) = 75.



Exercise 32

Mark and label a number line with at least zero, the minuend (the number in front of the subtraction sign) and the subtrahend (the number being subtracted). Then, use a vector to find the answer to the problem. Label the vector that is the answer

a. -92 – (-71) = ________

b. 92 – (- 81) = ________

c. -110 – (- 200) = ________

d. -122 – 76 = ________

e. 208 – 389 = ________

Finally, here are a set of rules that can always be used to add and subtract integers without the use of manipulatives. While these rules will always work, it is better to first understand where these rules came from and why they work. From doing the exercises in this exercise set, you should now have a better understanding of what it means to add and subtract with positive and negative numbers.

- **Any subtraction problem can be rewritten as an equivalent addition problem using the definition of adding the opposite. Once all subtraction problems are rewritten as equivalent addition problems, there is only addition to be done, and so therefore, we only need rules for addition, which follow.**
- **To add numbers that have like signs (either all positive or all negative), add the absolute values of the numbers. The sign of the answer will be the sign of the original numbers.**
- **To add two numbers with unlike signs (one positive and one negative), take the difference of the absolute values of the two numbers. The sign of the answer will be the same as the sign of the number with the larger absolute value. If the absolute values of the numbers are the same, the answer is zero.**

I never really liked the way these definitions were worded, but this is the mathematical way of expressing exactly how to get the answer. This is how **I** think about the rules.

First, I do usually rewrite each subtraction as an equivalent addition problem, unless it is two whole numbers being subtracted. In that case, if the first whole number is larger than the second whole number, I do regular whole number subtraction. If the larger whole number is being subtracted from the smaller whole number, I know the answer is negative and simply put a negative number in front of the answer.

*This is how **I** think about adding numbers that have like signs:*

- Let's say you were adding 8 + 9. These are both positive. So you just add the two whole numbers together as usual, and the answer is still positive. It's like having 8 dollars in the bank, and depositing 9 more dollars. You now have 17 dollars in the bank. So, I hope it's no surprise that 8 + 9 = 17.
- Let's say you were adding -8 + (- 9). These are both negative. This is like being in debt, and going deeper into dept. So you just add the two whole numbers together as usual, but the answer is negative. It's like owing someone 8 dollars, and then borrowing 9 more dollars. You now owe that person 17 dollars. So, -8 + (- 9) = - 17.

*This is how **I** think about adding numbers that have unlike signs.*

- Let's say you were adding 24 + (-10). Now this is an example of adding two numbers with unlike signs. Since the signs are DIFFERENT, I ignore their signs and take their DIFFERENCE – *that's the same thing as just subtracting their absolute values*. When I ignore their signs, I subtract the smaller number from the larger number. Then, to determine if the sign of the answer is positive or negative, I see which one outweighs the other (are there more negatives or positives?) – *that's the same thing as taking the sign of the one with the absolute value*. 24 + (-10) could be thought of as having 24 dollars in the bank and writing a check for 10 dollars. You still have $14 left in the bank. So, 24 + (-10) = 14.
- If you were adding -24 + 10, the problem would be done the same as the previous one except that in this case, the negatives outweigh the positives, so the final answer is negative. It's like owing someone 24 dollars and paying them 10 dollars. Now, you only owe them 14 dollars. So, -24 + 10 = -14.
- Let's say you were adding 14 + -14 or adding -83 + 83. In both of these examples, I'm adding opposites. In a way, one undoes the other. So both equal zero. This also could be done by thinking about which sign outweighs the other. But neither outweighs the other, and that is another reason the answer is neither negative nor positive, but is zero.

Using the rules at the beginning of the previous page, or the way I think about it, or maybe some other way you think about it that you are sure works and makes sense, you should be able to add and subtract integers with confidence. You should be able to explain the meaning of the operations, and be able to use manipulatives and number lines to help others learn how to add and subtract with integers and understand

? Exercise 33

Perform the following arithmetic problems by first changing subtraction to adding the opposite. Then add two numbers at a time, working from left to right. Show your steps.

? Exercise 34

Perform the following arithmetic problems by first changing subtraction to adding the opposite. Then add numbers with like signs together so that you have the sum of a positive and a negative number. Then add those two numbers together. Show your steps.

Remember that for a set to be closed under addition, the sum of any two elements in the set will always be in the set. For a set to be closed under subtraction, the difference of any two elements in the set will always be in the set. To prove a set is not closed under addition or subtraction, you need to provide a counterexample.

? Exercise 35

For each of the following sets, determine if the set is closed under the operation given. Provide a counterexample if it is not closed

a. Integers under addition?	b. Integers under subtraction?
c. Positive integers under addition?	d. Positive integers under subtraction?
e. Negative integers under addition?	f. Negative integers under subtraction?
g. {-1, 0, 1} under addition?	h. {-1, 0, 1} under subtraction?

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6.2: Multiplication

***You will need**: Positive and Negative Counters (Material Cards 18A and 18B)*

Now, we'll explore how to multiply positive and negative numbers using the counters. Let's look again at a definition of multiplication for whole numbers.

Definition: Multiplication of Whole Numbers using the Repeated-Addition Approach

If m and n are whole numbers, then

$$m \times n = n + n + n + n + \ldots + n,$$

where there are m add ends of n in this sum.

This definition makes sense as long as m and n are positive numbers. In fact, we can even make sense of and modify this definition using counters as long as m is positive by rewording the definition slightly, as stated below:

Definition: Multiplication of a Whole Number times an Integer using the Repeated-Addition Approach, using positive and negative counters

If m is a whole number and n is any integer, $m \times n$ is obtained by combining m subsets of a collection of counters representing n. The number that the resulting collection represents is the answer to the problem $m \times n$.

Exercise 1

This exercise shows you one way to use the above definition to multiply 3×-4. In this problem, the definition can be used because 3 is a whole number and -4 is an integer. We need to combine 3 subsets of a collection of counters that represent -4.

a. Use your positive and negative counters to represent a collection of counters that represent -4. For this exercise, choose a collection of 5 negatives and 1 positive. Show what your collection looks like below:
b. Now, form two more collections (for a total of 3 subsets) of counters that you had for part a. Combine the counters together and show what the large collection looks like below:
c. After removing any red-green pairs (zero) from your collection in part b, show the collection that remains below. What number does this represent? _____

Exercise 2

Do 3×-4 again, using a different collection of counters to represent -4.

a. Use your positive and negative counters to represent a collection of counters that represent -4. This time, choose a collection of 6 negatives and 2 positives. Show what your collection looks like below:
b. Now, form two more collections (for a total of 3 subsets) of counters that you had for part a. Combine the counters together and show what the large collection looks like below:
c. After removing any red-green pairs (zero) from your collection in part b, show the collection that remains below. What number does this represent? _____

Exercise 3

Okay, let's do 3×-4 one more time, choosing the easiest way to represent -4.

a. Form a collection of counters to represent -4. Do it the easy, natural way, using the least number of counters possible. Show what your collection looks like below:
b. Now, form two more collections (for a total of 3 subsets) of counters that you had for part a. Combine the counters together and show what the large collection looks like below:
c. What number does the collection in part b represent? _____

Well, I hope you got the answer of -12 for exercises 1, 2 and 3, since $3 \times -4 = -12$! This illustrates that it doesn't matter exactly which collection of counters you use to represent -4, as long as the collection really is -4. To compute 3×-4, you could combine 3 subsets of a collection of 8 reds and 4 greens, or you could combine 3 subsets of a collection of 7 reds and 3 greens, etc. You'll always end up with a collection that represents -12. For exercise 3, did you choose 4 negatives as your representation? If so, did you notice you didn't have to remove any red-green pairs to answer part c? From now on, let's do it the easy way, using the simplest collection possible.

? Exercise 4

Use your counters to do each of the following multiplication problems using the definition of multiplying a whole number by an integer. Then, explain what the multiplication problem given *means* in terms of the counters, and explain and show the individual steps. Use the following example as a model.

? Example

Multiply $\mathbf{2 \times -6}$.

Solution

Multiplying 2×-6 means to combine 2 subsets of 6 red counters. The number that the resulting collection represents is the product (answer). 2×-6 = RRRRRR + RRRRRR = RRRRRRRRRRRR = -12

a. Multiply 4×-2
This means to

b. Multiply 3×5
This means to

c. Multiply 5×-3
This means to

d. Multiply 7×2
This means to

e. Multiply 0×-3
This means to

Okay, now that you've mastered how to multiply a whole number by an integer, let's work on how we can use the counters to multiply a negative integer by an integer. Let's look once more at the definition for multiplying $m \times n$, when m is a whole number.

Definition: Multiplication of a Whole Number times an Integer using the Repeated-Addition Approach, using positive and negative counters

If m is a whole number and n is any integer, $m \times n$ is obtained by combining m subsets of a collection of counters representing n. The number that the resulting collection represents is the answer to the problem $m \times n$.

If m is negative, this definition doesn't make sense since you certainly can't combine a negative number of subsets! The way we'll revise this definition to include the possibility that m may be negative is to agree that if m is negative, we REMOVE m subsets of a collection of counters representing n. The trick to doing this is to remove the subsets from a collection of counters representing zero. So, here is the comprehensive definition for multiplying any two *integers*, using positive and negative counters.

Multiplication of two Integers, using positive and negative counters

- ***Case 1:*** If m is a whole number and n is any integer, $m \times n$ is obtained by combining m subsets of a collection of counters representing n. The product of m and n, $m \times n$, is the number that the resulting collection represents.

- ***Case 2:*** If m is negative and n is any integer, $m \times n$ is obtained by removing |m| subsets of a collection of counters representing n from a collection of counters representing zero. The product of m and n, $m \times n$, is the number that the resulting collection represents.

? Exercise 5

This exercise shows you how to use the above definition to multiply -4×3. For this problem, we need to remove 4 subsets of a collection of counters that represents 3 from a collection of counters representing zero. The simplest collection to represent 3 is 3 positives, or 3 green counters. (We *could* use a more complicated collection and still arrive at the same answer. We did this kind of an exercise in exercises 1 - 3. Convince yourself it wouldn't matter here, either.)

a. We first need to form a collection of counters that represents zero so that it will be possible to remove 4 subsets of 3 green counters. For this example, make a collection of 14 red and 14 green counters. Write down what your collection looks like here:
b. From your collection, remove a subset of 3 green counters. Then, remove 3 more subsets of 3 green counters. You have just removed 4 subsets of 3 green counters from zero. To show it on paper, circle a subset of 3 green counters in the collection above in part a. Then, circle 3 more subsets of 3 green counters so that four separate subsets are circled. After you remove the counters (which are shown by what you circled in part a), show what is left in your collection below.
c. Remove any red-green pairs (zero) from your remaining collection. Show this on paper by crossing off or circling any red-green pairs (zero) from your collection shown in part b. Show the collection that remains below. What number does this collection of counters represent?

? Exercise 6

Use the definition again to multiply -4×3. Remember, you need to remove 4 subsets of a collection of counters that represents 3.

a. We first need to form a collection of counters that represents zero so that it will be possible to remove 4 subsets of 3 green counters. This time, put in the smallest collection of counters possible so that you'll be able to remove 4 subsets of 3 green counters. Write down what your collection looks like here:
b. From your collection, remove four subsets of 3 green counters (take out one subset of 3 green counters at a time). You have just removed 4 subsets of 3 green counters from zero. To show it on paper, circle 4 different subsets of 3 green counters in the collection shown in part a. After you remove the counters (which are shown by what you circled in part a), show what is left in your collection below.
c. What number does the above collection represent?

Did you notice that if you start out with a minimal collection to represent zero, you don't have to remove any red-green pairs when you get to part c? That is the easiest way to do it because you know how to calculate what to take out before you start the problem. However, if you were teaching this to someone who couldn't figure that out ahead of time, you might always start out with zero being 20 (or some other agreed upon number) of each counter.

On the next page, the steps are shown for the example below– how to use your counters to do a multiplication problem when the first number is a negative integer. It's necessary to explain what the multiplication problem given means in terms of the counters, and then explain and show the individual steps. Use the example as a model for the exercises that follow.

✓ Example

Multiply -4×-3 by using the definition of multiplying integers.

Solution

Multiplying -4×-3 means to remove 4 subsets of 3 red counters from a representation of zero. The number that the resulting collection represents is the answer.

Since there are 12 greens remaining, the answer is +12. Didn't you always wonder why a negative number times a negative number equaled a positive number? Using the definition of multiplying integers with counters, you can really see why it's true. After you do a few more problems, feel free to go out and show all of your friends who still don't understand it or just memorized the rules because someone told them that's the way it is and to just accept it. Okay, enough of that unbridled enthusiasm from me for now.

It's time for you to work a few problems. Some have negative numbers before the multiplication sign (start with a representation of zero and remove subsets as in the last example) and some have whole numbers before the multiplication sign (so just combine subsets together like you did in the earlier exercises of this exercise set). It's the number *before* the multiplication sign that will indicate which case of the definition you will use.

Exercise 7

Use your counters to do each of the following multiplication problems using the definition of multiplying two integers with positive and negative counters. Then, *explain what the multiplication problem given* ***means*** in terms of the counters, and explain and show each of the individual steps. Use the example above as a model when the first number is negative.

a. -5×3 ____ This means to ____

Show work and all steps below. Then, state the answer to the problem.

b. -3×2 ____ This means to ____

Show work and all steps below. Then, state the answer to the problem.

c. 2×-3 ____ This means to ____

Show work and all steps below. Then, state the answer to the problem.

NOTE: Although the answer to part b is the same as part c due to the commutative property of multiplication, the problems **mean** different things, the steps are not alike and the problems are done differently.

d. -2×3 ____ This means to ____

Show work and all steps below. Then, state the answer to the problem.

e. 3×2 ____ This means to ____

Show work and all steps below. Then, state the answer to the problem.

f. 0×-4 ____ This means to ____

Show work and all steps below. Then, state the answer to the problem.

g. -4×0 (this means something different than 0×-4) ____ This means to ______________________________

Show work and all steps below. Then, state the answer to the problem.

Let's do some more problems when the first integer is negative, using a chart to keep track of the counters. The first column will be the multiplication problem, the second will show what representation of zero is being used, the third will explain the meaning of the multiplication problem (what has to be done in all of these cases, subsets are being removed from the representation of zero), the fourth will show how many of each of the counters are left after the removal of the subsets, and the last column will be the answer, obtained from the representation shown in the fourth column.

Exercise 8

Fill in all of the blanks. FILL IN THE ENTIRE TABLE ALL THE WAY TO THE BOTTOM!! You should be using your real manipulatives (red and green counters) as you do (most of) these problems. I've done the first one for you to use as a model.

problem	Counters for Zero	**The meaning of the problem**	counters remaining	answer
-3×6	20G 20R	remove 3 sets of 6 G from zero	2G 20R	-18
-3×6	23G 23R			
-3×6	18G 18R			

problem	Counters for Zero	**The meaning of the problem**	counters remaining	answer
-4×4	18G 18R			
-4×-4	18G 18R			
-5×2	14G 14R			
-2×5	13G 13R			
-5×-2	11G 11R			
-2×-5				
-3×3	10G 10R			
-3×-3	12G 12R			
-7×2				
-2×8	18G 18R			
-2×-8	18G 18R			

After doing all of the exercises in this Exercise Set, the rule for multiplication of integers should make sense.

Rule for Multiplying Two Integers Together:

- **To multiply two integers, first multiply the absolute values of the integers. To determine the sign of the product: it is positive if both integers have the same sign (both positive or both negative); otherwise it is negative (if one of the integers is positive and the other integer is negative). If one of the integers is zero, the answer is zero.**

If you are multiplying more than two integers together, the sign of the product can be determinined by how many negative numbers are being multiplied. For every two negative numbers being multiplied together, the answer is positive. Therefore, if there are an even number of negative numbers being multiplied, the sign of the product is positive. If there are an odd number of negative numbers being multiplied, the sign of the product is negative. If one of the integers is zero, the answer is zero.

Although we haven't covered division in this exercise set, the rule for determining the sign of a quotient is the same as the rule for determining the sign of a product.

? Exercise 9

Determine if each product is negative (–), positive (+) or zero (0). Do not compute

Remember that for a set to be closed under multiplication, the product of any two elements in the set must be in the set. To prove a set is not closed under multiplication, you need to provide a counterexample.

? Exercise 10

For each of the following sets, determine if the set is closed under multiplication. Provide a counterexample if it is not closed.

a. Integers
b. Positive Integers
c. Negative Integers
d. {-1, 0}
e. {1, -1}
f. {-1, 0, 1}

Less than (<) and Greater than (>) signs are used to order numbers. $a < b$ if a is to the left of b on the number line. $a < b$ (read "a is less than b") can also be written as $b > a$ (read "b is greater than a.)

? Exercise 11

Decide which of the following are true if a, b and c are any integers, p is a positive integer and n is a negative integer. Provide a counterexample if it is false.

a. If $a < b$ and $b < c$, then $a < c$
b. If $a < b$, then $a + c < b + c$
c. If $a < b$, then $ap < bp$
d. If $a < b$, then $ap > bp$
e. If $a < b$, then $an > bn$
f. If $a < b$, then $an < bn$

Exercise 11a, 11b, 11c, and 11e are the four properties of Ordering for Integers. 11a is called the Transitive Property for Less Than. 11b is called the Property of Less Than and Addition. 11c is the Property of Less Than and Multiplication by a Positive 11e. is the Property of Less Than and Multiplication by a Negative. If the less than symbols in 11a, 11b and 11c were replaced with greater than symbols, you would have the corresponding properties of greater than. For part 11e, if both signs were switched, you would have the Property of Greater Than and Multiplication by a Negative.

? Exercise 12

Fill in the following properties, if a, b and c are any integers, p is a positive integer and n is a negative integer.

a. Transitive Property for Greater Than:
b. Property of Greater Than and Addition:
c. Property of Greater Than and Multiplication by a Positive:
d. Property of Greater Than and Multiplication by a Negative:

11.a,b and c are similar to properties of equality. The difference between equalities and inequalities (whether there is a less than or greater than symbol) if that when both sides of an inequality are multiplied by a negative number, the inequality sign changes directions.

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6.3: Homework

- **Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)**
- **Start a new module on the front side of a new page and write the module number on the top center of the page.**
- **Answers without supporting work will receive no credit.**
- **Some solutions are given in the solutions manual.**
- **You may work with classmates but do your own work.**

? HW #1

How would you explain to someone what the "absolute value of a number" means? Explain without using examples.

? HW #2

Simplify each of the following:

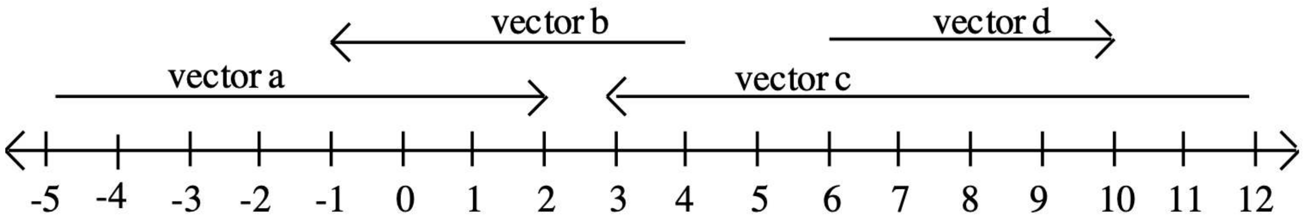
	b. | 8 | – | 10 |	c. | 8 – 10 |
d. | 35 |	e. | 10 – 18 |	f. | 10 | – | 18 |

? HW #3

Write any and all numerals that have the given absolute value:

a. 4	b. 0	

For each vector drawn, write the number that the vector represents:



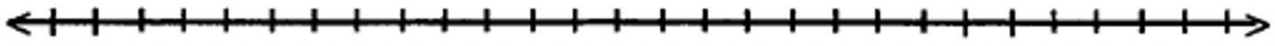
a. Vector a	b. Vector b	c. Vector c	d. Vector d

? HW #5

Look again at the four vectors shown in exercise 4. If someone had drawn those vectors to find the difference of two numbers using the missing addend approach, write the four subtraction problems which generated those vectors.

? HW #6

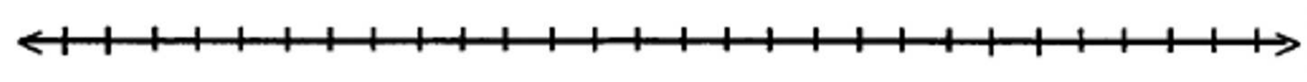
Use vectors on the number line to add -8 + 5 + (-3) using vectors as actions. Mark and LABEL your number line with at least zero and a point on either side. Explain how to read the answer.



-8 + 5 + -3 = _____ since ____

HW #7

Use vectors on the number line to compute the 8 – 5 + -7 – -4 using vectors to DO and UNDO actions. Mark and LABEL your number line with at least zero and a point on either side. Explain how to read the answer.



____8 – 5 + -7 – -4 = _____ since ____.

HW #8

Use the missing addend approach to perform the following subtractions. Label the vector on the number line.

a. 5 – 11	
c. 5 – (-11)	

Use red and green counters to add the following integers. For each problem, explain and show each of the steps involved.

HW #10

For each problem, use positive and negative counters. First, state what the problem means, and then explain and show each step you need to take to find the answer. Each problem requires 3 or 4 steps.

a. 6 – 8	

For each of the following sets, determine if the set is closed under the operation given. Provide a counterexample if it is not closed.

a. Negative Integers under addition
b. Positive Integers under subtraction
c. {..., -10, -5, 0, 5, 10, ...} under addition
d. {0, 3, 6, 9, ...} under subtraction
e. Positive Integers under multiplication
f. Negative Integers under multiplication

HW #12

Use your counters to do each of the following multiplication problems using the definition of multiplying two integers with positive and negative counters. Then, explain what the multiplication problem given means in terms of the counters, and explain and show each of the individual steps.

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CHAPTER OVERVIEW

7: Division

The teacher said to the student, "Do the following division: $12 \div 3$*."*

The student replied, "But I don't know how to divide."

Then the teacher asked, "Do you know how to subtract?"

The student replied, "Sure, that's easy."

Then the teacher said, "Well, then, you can divide. Just count how many threes you can subtract from twelve!"

The student marveled at this notion. The student simply noticed 3 could be subtracted four times from 12, and answered, "Is $12 \div 3$ *equal to 4?"*

The teacher said , "Yes! I knew you could divide. I'm very proud of you."

The student was proud too...and very happy.

- 7.1: The Meaning of Division
- 7.2: Division Algorithms
- 7.3: Division in other Bases
- 7.4: Homework

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7.1: The Meaning of Division

You will need: ***Centimeter Strips*** *(Material Cards 16A-16L) and* ***Base Blocks*** *(Material Cards 3 - 15)*

In these first few exercises, you'll be using c-strips to explore division.

? Exercise 1

In this first exercise, we will divide $H \div Y$. H, the part before the division sign, is called the **DIVIDEND**, and Y, the part after the division sign, is called the **DIVISOR**. One way to do this division is to find the maximum number of times the divisor (Y) can be subtracted from the dividend (H). Take out a hot pink c-strip (H). To figure out how many yellow c-strips could be subtracted, make a train consisting of as many yellow c-strips (Y) as possible such that the train you make is shorter or equal to the length of H, *and* if you were to add one more yellow c-strip, the train would be longer than the hot pink c-strip.

a. How many yellow c-strips did you use? ______

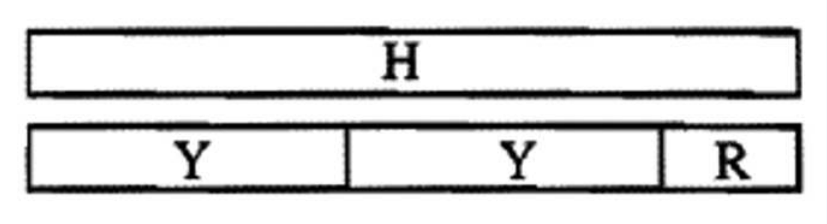
This number is called the **QUOTIENT**.

The **quotient** is the maximum number of times the **divisor** can be subtracted from the **dividend**.

b. If the train you made is shorter than H, what color c-strip could
you add to that train to make it equal in length to the hot pink c-strip? ______

This piece, which must be shorter than the divisor, is called the **REMAINDER**.

Put the hot pink strip next to the train you formed. It should look like this:



To compute $H \div Y$, we need to figure out what goes in the blanks to make the equation in part c, below, true. The first blank must contain the largest c-strip possible that will make the equation true, *and* the second blank must be shorter than the yellow c-strip. We made a train of yellows, with at most one extra piece to form the length of H. Notice this is a train of yellows (which can be formed into a rectangle) plus an extra piece. Form the two yellow strips into a rectangle and find the c-strip that goes across the width. That c-strip goes in the first blank, and the extra piece (R) goes in the second blank. Fill in the blanks of the equation below. This equation should make sense by looking at the picture above showing H and the train are the same length.

c. H = ____ $\cdots Y$ + ____

The **QUOTIENT** is in bold, and the **REMAINDER** is underlined.

We can now conclude that: $H \div Y = \mathbf{R}$ remainder $\underline{R}$, since $H = \mathbf{R} \cdots Y + \underline{R}$

This can be written as a statement about numbers by translating each c-strip to a number.

Using numbers in the above equation, $12 \div 5 = \mathbf{2}$ remainder $\underline{2}$, since $12 = \mathbf{2} \cdots 5 + \underline{2}$

? Exercise 2

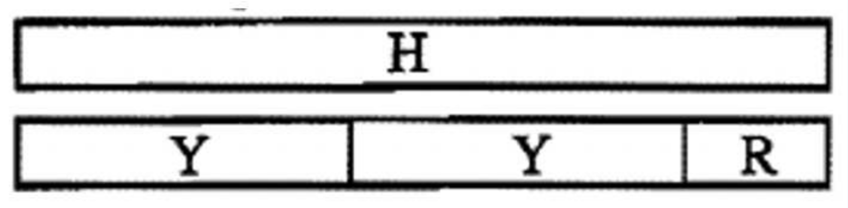
For a - e, do the division using the c-strips to make a train of as many of the divisor as possible such that the length of the train you make is less than or equal to the length of the dividend, *and* if you were to add one more of the divisor to the train, it would be longer than the dividend. If the train ends up being the same length as the dividend, there will be no remainder and the second blank will be empty. Draw a diagram showing the c-strip that is the dividend and under it, draw the train you formed. Do this the same way we did it for exercise 1. Follow the example below.

Example: $H \div Y$

$H \div Y = \mathbf{R}$ remainder $\underline{R}$, since H = $\mathbf{R} \cdots$ Y + $\underline{R}$

$12 \div 5 = \mathbf{2}$ remainder $\underline{2}$, since $12 = \mathbf{2} \cdots 5 + \underline{2}$

Show diagram below:



a. $B \div R$:
$B \div R$ = __ remainder __ since B = __ ⋯ R + __
__ ÷ __ = __ remainder __ since __ = __ ⋯ __ + __
Show diagram below:

b. S ÷ L:
S ÷ L = __ remainder __ since S = __ ⋯ L + __
__ ÷ __ = __ remainder __ since __ = __ ⋯ __ + __
Show diagram below:

c. $N \div P$:
$N \div P$ = __ remainder __ since N = __ ⋯ P + __
__ ÷ __ = __ remainder __ since __ = __ ⋯ __ + __
Show diagram below:

d. $K \div P$:
$K \div P$ = __ remainder __ since K = __ ⋯ P + __
__ ÷ __ = __ remainder __ since __ = __ ⋯ __ + __
Show diagram below:

e. $D \div L$:
$D \div L$ = __ remainder __ since D = __ ⋯ L + __
__ ÷ __ = __ remainder __ since __ = __ ⋯ __ + __
Show diagram below:

Exercise 3

Count out 36 unit blocks. How would you divide the 36 blocks by 2? Explain how you would do it, and show a picture of what you did. What is the answer?

Exercise 4

Get out your base four blocks now, and trade in 36 units for a number in base four.

a. Write 36 as a base four numeral: ______________

Decide how to divide this by 2 using base four blocks.

b. Explain how you would do it, and show a picture of what you did.

c. What is the answer in base four? __________

d. Write the division problem and the quotient (answer) in base four: _____________$\div 2_{\text{four}}$ = _____________

? Exercise 5

Get out your base three blocks now, and trade in 36 units for a number in base three.

a. Write 36 as a base three numeral: _____________

Decide how to divide this by 2 using base three blocks.

b. Explain how you would do it, and show a picture of what you did.

c. What is the answer in base three? _________

d. Write the division problem and the quotient (answer) in base three: ____________$\div 2_{\text{three}}$ = ____________

? Exercise 6

Count out 36 unit blocks. How would you divide the 36 blocks by 3? Explain how you would do it, and show a picture of what you did. What is the answer?

? Exercise 7

Get out your base four blocks now, and trade in 36 units for a number in base four.

a. Write 36 as a base four numeral: _____________

Decide how to divide this by 3 using base four blocks.

b. Explain how you would do it, and show a picture of what you did.

c. What is the answer in base four? _________

d. Write the division problem and the quotient (answer) in base four: ____________$\div 3_{\text{four}}$ = ____________

? Exercise 8

Count out 48 unit blocks. How would you divide the 48 blocks by 16? Explain how you would do it, and show a picture of what you did. What is the answer?

There are two basic ways of thinking about division. The two word problems below both require the division $18 \div 3$, but the problems are really quite different.

✓ Word Problem A:

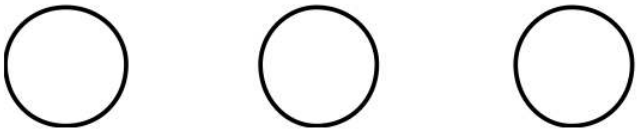
If you have 18 cookies and want to divide the cookies up equally amongst your three children, how many cookies should you give to each child?

✓ Word Problem B:

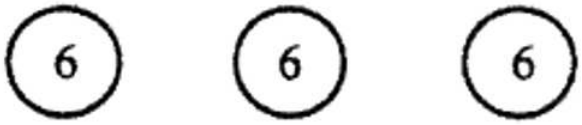
If you have 18 cookies and want to make baggies having three cookies in each to put in lunch boxes, how many baggies will you be able to make?

Both require the division problem $18 \div 3$, and the answer to each problem is 6. But the picture used to solve each is very different.

In Word Problem A, *you are told how many kids there are, and you are asked to figure out how many cookies each child should be given.* Imagine 3 children in front of you. You need to disburse the cookies amongst the 3 children so that each receives the same number of cookies. To begin, you might draw a model to represent three kids. It might look something like this:



You know there are 3 kids and each will be given the same number of cookies so that the total number of cookies they all have is 18. After disbursing, you might show the picture like this:



This is a way to illustrate that each of the 3 children has 6 cookies. If you prefer, you can actually draw 6 dots or 6 cookies in each circle. It's up to you.

Note that if you look at the model drawn, it is simply the multiplication problem $3 \cdots 6$, since there are 3 sets of 6 added together. The *interpretation* of division Word Problem A represents is called:

Partitioning into (Equal) Subsets

In this interpretation, *the divisor is the number of subsets in which to disburse something*. The dividend, 18, must be divided amongst 3 equal subsets (3 is the divisor). The quotient (answer to the division problem) will be the number of elements (6) placed in each subset (the kids, in this case.) The task is to **partition 18 cookies into 3 equal subsets**. In this interpretation, you are given how many subsets there are (from reading the problem in this case, each kid represents one subset), but you do not how many are in each subset (that is the question being asked in this case, how many cookies each child should receive). You do the division to figure out how many elements will end up in each subset. After doing the division, **the number of elements *IN* each subset is the answer to the problem.**

One way to distinguish that a problem is suggesting this interpretation is to consider whether the information in the problem provides you with the *number of subsets*. If it does, as in Word Problem A, then it is the *partitioning into equal subsets* interpretation.

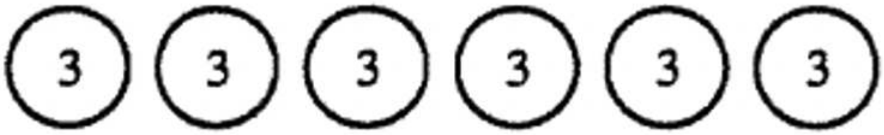
Okay, let's look at Word Problem B now.

✓ **Word Problem B:**

If you have 18 cookies and want to make baggies having three cookies in each to later put in lunch boxes, how many baggies will you be able to make?

In Word Problem B, *you don't know how many subsets there will be*. In fact, you are being ***asked*** how many subsets you will be able to make, so the partitioning into subsets interpretation wouldn't apply.

What you **do** know is how many of something will be in each subset. You would place three cookies in a baggie, then 3 more in another baggie, and so on. You won't know how many subsets you will have until you finish disbursing the cookies. You would take three cookies away at a time and put them in a baggie, until you ran out of cookies. After doing this, a picture you might draw to illustrate this procedure might look like this:



Again, if you prefer to show three dots in each circle, as opposed to just writing the numeral 3, in each circle, that is perfectly fine. Either way, it should be clear from the drawing what procedure was used.

After disbursing all the cookies, you would count up how many baggies you were able to make, and that would give you the answer. Note the answer is 6, in this case. The model shown represents the multiplication $6 \cdots 3$, since there are 6 baggies of 3 cookies each.

The *interpretation* of division Word Problem B represents is called:

Repeated Subtraction

In this interpretation, *the divisor is the number of elements that will go into each subset*. Each time the divisor (3) is subtracted from the dividend (18), a subset is formed. After all elements have been disbursed, the quotient (answer) is found by counting how many subsets were formed. This interpretation is used when you are given how many elements are in each subset, and you are trying to find out how many subsets can be made (that is the question being asked). You do the division (by using repeated subtraction) to figure out how many subsets can be made. After doing the division, **the number of subsets you made is the answer to the problem.**

One way to distinguish that a problem is suggesting this interpretation is to consider whether the information in the problem provides you with *how many elements* go into a subset. If so, as in Word Problem B, then it is the *repeated subtraction* interpretation.

Word Problems requiring division generally fall into the category of Repeated Subtraction or Partitioning into Subsets. After reading a word problem, you should be able to analyze and interpret it as a problem that requires **Partitioning into Subsets** or **Repeated Subtraction** to solve the problem, and then be able to draw a picture using that interpretation. Also, for any division problem given, you should be able to write two distinct word problems, one for each interpretation, and draw a picture for each interpretation, and be able to explain and show all the steps necessary to solve the problem.

? Exercise 9

Terry wanted to divide 100 marbles amongst his three best friends and himself. How many marbles would each of them get?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? ______

? Exercise 10

Jordan made 80 ounces of jello, and wanted to put them in 8 ounce containers. How many containers can she fill?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? ______

? Exercise 11

Angie had 144 pages of blank paper. She was making copies of a 16 page short story she wrote to send to magazines. How many complete copies could she make?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem? ______

? Exercise 12

A group of four friends pooled their money together for some lottery tickets and won \$500. How much does each person receive from the winnings?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.

? Exercise 13

Sheila wanted to make as many cakes from a recipe requiring $1\frac{1}{3}$ cups of flour for each cake. She had 8 cups of flour. How many cakes could she make?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? ______

? Exercise 14

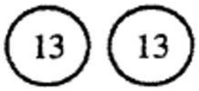
Tom won 65 baseballs and decided to divide them up evenly amongst the 13 boys on his baseball team. How many baseballs will each kid get?

a. What division problem do you need to do to find the answer?
b. What interpretation of division does this word problem represent?
c. Explain how to use the correct interpretation to draw a picture to find the answer.
d. Draw a picture using the correct interpretation to help find the answer.
e. Write the multiplication problem shown by the picture:
f. Explain how you use the picture to find the answer to the problem. Be specific.
g. What is the answer to this problem? ______

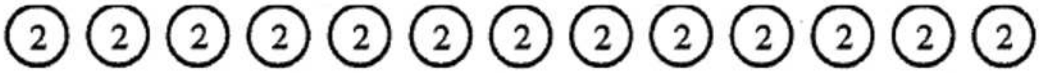
If you are *not* presented with a word problem, but are simply given a division problem, you can use either interpretation to think about and solve the problem. Many times, it is easier to use one interpretation over another, especially if you need to draw a picture to show how you solved the problem.

Consider the following division problem: $26 \div 2$. Let’s say the 26 represented 2 pennies. Would it be easier to partition 26 pennies into 2 piles and count how many are in each pile; or would it be easier to make several equal subsets containing 2 pennies each from the 26 pennies, and then to count how many subsets there are? The picture you draw to represent each interpretation is quite different. Look at the difference:

A picture using partitioning into equal subsets interpretation is shown below:



A picture using the Repeated Subtraction interpretation is shown below:



Exercise 15

For parts a - h of this exercise, consider the division problem: $48 \div 12$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to use.
c. Make up a word problem requiring the division $48 \div 12$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. Draw a picture to solve your problem using the Partitioning into Subsets interpretation:
e. The multiplication for the picture shown in part d is: ____________________
f. Make up a word problem requiring the division $48 \div 12$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture shown in part g is: ____________________

Exercise 16

For parts a - h of this exercise, consider the division problem: $200 \div 4$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $200 \div 4$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. Draw a picture to solve your problem using the Partitioning into Subsets interpretation:
e. The multiplication for the picture using Partitioning into Subsets is: ________
f. Make up a word problem requiring the division $200 \div 4$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Repeated Subtraction interpretation. ***Do not actually draw the picture — just explain.***
h. The multiplication for a picture using Repeated Subtraction would be:

Exercise 17

For parts a - h of this exercise, consider the division problem: $150 \div 50$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $150 \div 50$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Partitioning into Subsets interpretation. ***Do not actually draw the picture — just explain!***
e. The multiplication for a picture using Partitioning into Subsets would be: ____
f. Make up a word problem requiring the division $150 \div 50$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture using Repeated Subtraction is: ___________

Exercise 18

For parts a - h of this exercise, consider the division problem: $\mathbf{140 \div 35}$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $140 \div 35$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Partitioning into Subsets interpretation. ***Do not actually draw the picture* — just explain!**
e. The multiplication for a picture using Partitioning into Subsets would be: ____
f. Make up a word problem requiring the division $140 \div 35$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture using Repeated Subtraction is:

Exercise 19

For parts a - h of this exercise, consider the division problem: $\mathbf{95 \div 5}$.

a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $95 \div 5$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. Draw a picture to solve your problem using the Partitioning into Subsets interpretation:
e. The multiplication for the picture using Partitioning into Subsets is: ________
f. Make up a word problem requiring the division $95 \div 5$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Repeated Subtraction interpretation. ***Do not actually draw the picture* — just explain!**
h. The multiplication for a picture using Repeated Subtraction would be: ______

Exercise 20

For parts a - h of this exercise, consider the division problem: $\mathbf{800 \div 160}$.

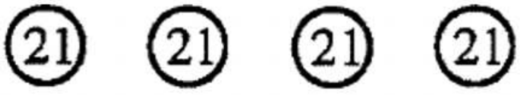
a. Which interpretation do you think would be easier to use if you needed to draw a picture?
b. **Explain why** the interpretation you chose in part a would be easier to solve this division problem. Contrast it with the other interpretation.
c. Make up a word problem requiring the division $800 \div 160$, such that the partitioning into subsets interpretation is used. Make sure your problem asks a question!
d. **Explain** what you would have to do if you had to draw a picture to solve your problem using the Partitioning into Subsets interpretation. ***Do not actually draw the picture* — just explain!**
e. The multiplication for a picture using Partitioning into Subsets would be: ____
f. Make up a word problem requiring the division $800 \div 160$, such that the repeated subtraction interpretation is used. Make sure your problem asks a question!
g. Draw a picture to solve your problem using the Repeated Subtraction interpretation:
h. The multiplication for the picture using Repeated Subtraction is:

Exercise 21

For each division problem, a picture is shown. Based on the picture, state which interpretation of division was used.

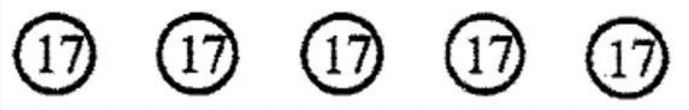
a. $84 \div 21$

Interpretation: _______



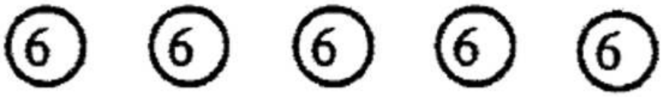
b. $85 \div 5$

Interpretation: _______



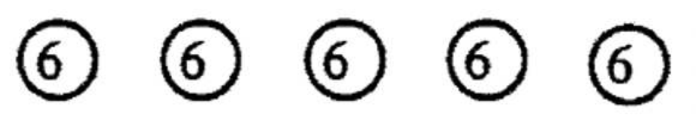
c. $30 \div 5$

Interpretation: _______



d. $30 \div 6$

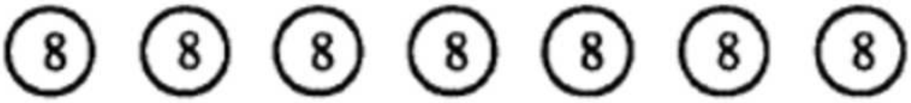
Interpretation: _______



Exercise 22

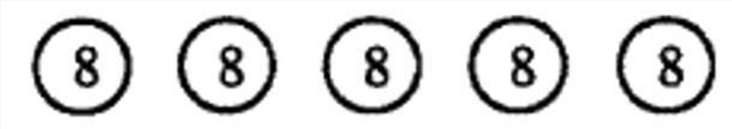
For each picture, a particular interpretation of division was used to solve a division problem. Based on the interpretation and picture, decide what the original division problem was, and circle the correct one of the two choices given.

a. Repeated Subtraction



Circle the correct division problem: $56 \div 8$ or $56 \div 7$

a. Repeated Subtraction

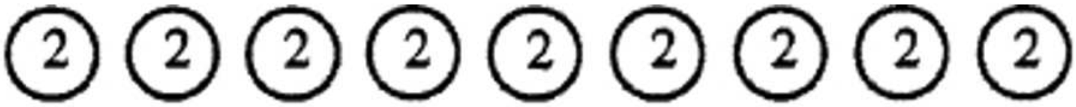


Circle the correct division problem: $56 \div 8$ or $56 \div 7$

? Exercise 23

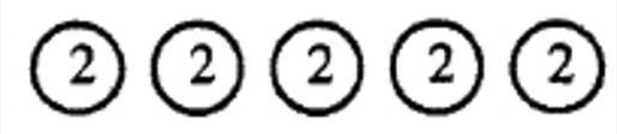
For each picture, a particular interpretation of division was used to solve a division problem. Based on that interpretation and picture, state the original division problem.

a. Partitioning into Subsets



If partitioning into subsets was used, state the division problem: ____________

b. Repeated Subtraction



If repeated subtraction was used, state the division problem: ____________

? Exercise 24

If you drew a picture for the multiplication problem, a $\cdot$ b, which letter represents how subsets are made, and which letter represents how many are in each subset?

Division is the operation that undoes mutiplication. Students can rely on this fact to learn basic division facts. For instance, since 6 · 7 = 42, then $42 \div 6 = 7$ and $42 \div 7 = 6$. ***Below is a formal definition of division***

Definition: a $\div$ b is defined to be the unique number, c, under the following conditions: $a \div b = c$ **if and only if a = b $\cdot$ c.** If there is no number, c, that exists, or if there is more than one unique number that can be put in for c, then $a \div b$ is said to be ***undefined***.

Instead of writing c, it is sometimes easier to write a blank or box in the equations to see if a unique solution exists. If you can put a unique solution in the blank to the multiplication problem, then you can go back and fill in the solution to the division problem.

✓ Example

Use the definition of division to find the answer to $21 \div 3$.

First, note that $21 \div 3$ = ____ means 21 = 3 $\cdot$ _____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 21 = 3 $\cdot$ _____, true: 21 = 3 $\cdot \underline{7}$. If the number filled in is the only possible solution (unique), then that is the answer to the division problem. Therefore, $21 \div 3$ = $\underline{7}$. *You do not need to write this paragraph for your solution. Below is how to write the solution.*

Solution

Since $21 = 3 \cdot \underline{7}$, then $21 \div 3 = \underline{7}$.

? Exercise 25

Use the definition of division to find the following quotients:

a. $32 \div 8$
b. $56 \div 8$
c. $32 \div 2$
d. $0 \div 13$

e. $12 \div 1$
f. $X \div 1$ (Assume X is any number.)
g. $0 \div Y$ (Assume Y is any nonzero number.

Exercise 25f illustrates that any number divided by 1 is itself.

In other words, **$a \div 1 = a$ for all values of a**.

Exercise 25g illustrates that zero divided by any *nonzero* whole number is 0.

In other words, **$0 \div a = 0$, for all nonzero values of a**.

In the above problem, all of the answers were whole numbers. If you were asked to use the definition to find only whole number solutions, then some divisions would be undefined under whole numbers. There may, however, be rational or irrational solutions depending on the problem.

✓ Example

Use the definition of division to find if $21 \div 5$ is defined under whole numbers. Show the solution if it is defined, or explain why not if it is not defined.

First, note that $21 \div 5$ = ____ means 21 = 5 · _____ Second, try to fill in the blank in the multiplication problem with a unique whole number to make the equation, 21 = 5 · _____, true. There is no whole number that can be put in the blank to make that equation true. Therefore, $21 \div 5$ is not defined under whole numbers. Here is how you would show the answer.

Solution

Since there is no whole number solution to make the equation, 21 = 5 · ____ true, $21 \div 5$ is not defined under whole numbers.

? Exercise 26

Use the definition of division to find which of the following quotients are defined under whole numbers. Show the solution if it is defined, or explain why not if it is not defined. Part a and b are done for you. Use these as examples.

a. $45 \div 5$ Solution: Since $45 = 5 \cdot \underline{9}$, then $45 \div 5 = \underline{9}$.
b. $42 \div 9$ Solution: Since there is no whole number solution to make the equation, 42 = 9 · ___ true, $42 \div 9$ is not defined under whole numbers. *(Yes, you should write all of this out!)*
c. $48 \div 6$ Solution:
d. $35 \div 4$ Solution:
e. $48 \div 1$ Solution:
f. $55 \div 7$ Solution:
g. $0 \div 8$ Solution:

Let's explore what happens when we divided by zero.

✓ Example

Use the definition of division to find the quotient: $8 \div 0$.

First, note that $8 \div 0 =$ ____ means 8 = 0 · _____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 8 = 0 · _____, true. There is no number that can be put in the blank to make that equation true since any number times 0 is 0, and will never equal 8. This is how you could write the solution:

Solution

There is no number that will make the equation, 8 = 0 · ____ true, since any number put in the blank will make the right hand side of the equation zero, which can never equal the left-hand side of the equation, which is 8. Therefore, $8 \div 0$ is not defined.

? Exercise 27

Use the definition of division to show that the following quotients are undefined:

a. $6 \div 0$

b. $18 \div 0$

c. $M \div 0$ (Assume $M \neq 0$)

In exercise 27, you showed that dividing a nonzero number by 0 was undefined. Now, let's explore what happens if we divide zero by zero.

✓ Example

Use the definition of division to find the quotient: $0 \div 0$.

First, note that $0 \div 0 =$ ____ means 0 = 0 · _____ Second, try to fill in the blank in the multiplication problem with a unique number to make the equation, 0 = 0 · _____, true. There is no unique number that can be put in the blank to make that equation true since any number put in the blank will make the equation true, and therefore there is no unique number that will work. This is how you could write the solution:

Solution

There is no unique number that can be put in the blank to make the equation 0 = 0 · ____ true, since any number will make it true. Therefore, $0 \div 0$ is called indeterminate.

In exercise 27, you showed that a nonzero number divided by zero is undefined. The next example showed that zero divided by zero is indeterminate.

To prove that division by zero is undefined, two cases must be shown. Below is a complete proof to show that division by zero is undefined.

Problem: Prove that division by zero is undefined

Solution: The goal is to show that $a \div 0$ is undefined for all values of a. There are two cases to consider Case 1: $a \neq 0$; Case 2: a = 0.

Case 1: Let $a \neq 0$. There is no unique number that could be put in the blank to make the equation, a = 0 · ____, true because the left-hand side of the equation is not zero (since $a \neq 0$), and any number put in the blank will make the right hand side of the equation zero. Therefore, when $a \neq 0$, $a \div 0$ is not defined.

Case 2: Let a = 0. There is no unique number that can be put in the blank to make the equation 0 = 0 · ____ true, since any number will make it true. Therefore, $0 \div 0$ is indeterminate.

It takes practice writing this proof. You'll need to practice writing it several times to master the problem. One option is to memorize the proof shown above by writing it out a few times until you really get the idea. This would make a good exam question.

? Exercise 28

Prove that division by zero is undefined.

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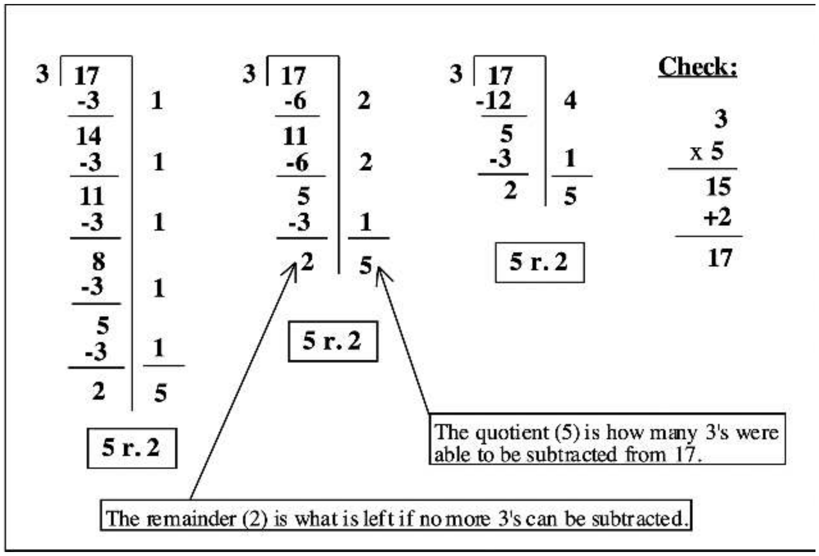
7.2: Division Algorithms

As we've seen, division can be viewed as the opposite of multiplication.

Another way to think about it is as **REPEATED SUBTRACTION**.

Let's do DIVISION the Easy Way!

That's right. Division is really just "repeated subtraction." If you are asked to divide 17 by 3, which is written $17 \div 3$, you have to figure out how many 3's can be subtracted from 17. You probably already know the answer is 5 r. 2, but let's see how this works by counting how many times 3 can be subtracted from 17. Look at the work below. Three different ways are shown. You make a scaffold (looks like a hangman) and keep track of how many 3's you subtract at each step to the right of the scaffold. In the first one, 3 is subtracted one at a time, and a total of five 3's were subtracted. In the second one, first two 3's are subtracted, then 1 more, then 2 more, for a total of five 3's being subtracted. In the third one, four 3's are subtracted, and then 1 more, again for a total of five 3's being subtracted. Each time you subtract, look to see what number is left. If it is 3 or more, you can subtract some more 3's. But when you get a number smaller than 3, you know 3 can't be subtracted anymore, so that is your remainder. For this problem, 2 was the remainder. You need to count how many 3's you subtracted. To the right of the scaffold, you write *how many* 3's you are subtracting as you go along. Simply add those numbers to find out how many 3's you subtracted. The answer to the problem is how many 3's you subtracted, and then you need to also write the remainder. In this case, $17 \div 3 = 5$ r. 2. Always check your answer my multiplying the divisor (3) times the quotient (5) and adding the remainder (2): $3 \times 5 + 2 = 15 + 2 = 17$. It checks!



3
17
-3
1
14
-3
1
11
-3
1
8
-3
1
5
-3
1
2
5
5 r. 2
3
17
-6
2
11
-6
2
5
-3
1
2
5
5 r. 2
3
17
-12
4
5
-3
1
2
5
5 r. 2
Check:
3
x 5
15
+2
17
The quotient (5) is how many 3's were able to be subtracted from 17.
The remainder (2) is what is left if no more 3's can be subtracted.

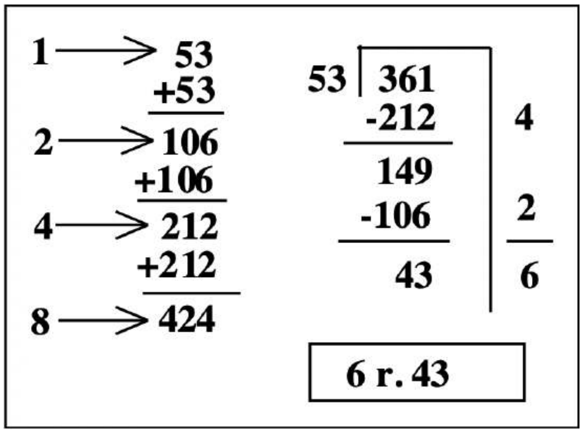
Doing long division with large numbers is often difficult because you have to multiply in your head, and it's easy to make mistakes. But, by using this new method of repeated subtraction, along with a partial multiplication you make up ahead of time using repeated addition, you never have to multiply in your head (or multiply at all), and you never have to guess how many times one number "goes into" another number. This method takes a little more paper, but it makes the problem easier and it's practically foolproof. Lots of people don't like to do long division, because it's hard to always guess the exact right number and sometimes they get a headache from thinking too hard. We're going to eliminate that problem. The important thing is that you know what division means (repeated subtraction), you know how to get the right answer using a method that makes sense to you, and you know how to check your answer to make sure it is right.

Okay, so let's say you had to do this division problem: $361 \div 53$.

YUK! This doesn't look like any fun. Actually, all this is asking is how many 53's can be subtracted from 361. We can subtract 53 one at a time, but it might take a long time. We might decide to subtract more than one 53 at a time.

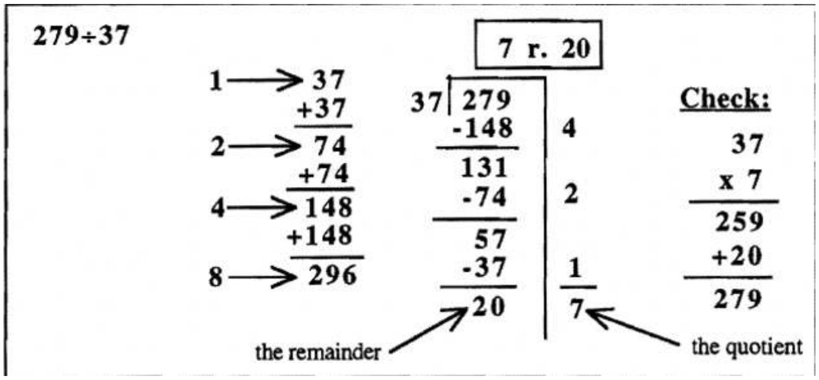
Before we doing the division, let's make a quick partial multiplication table up for 53, by using REPEATED ADDITION! This table is your prep work for division. A simple way to make up a partial multiplication table for 53 without ever multiplying entails doubling numbers. We know that $1 \times 53 = 53$. To figure out 2×53, just add two 53's together. Two 53's added to two more 53's would equal four 53's, or 4×53. Four 53's added to four more 53's would equal eight 53's, or 8×53. Okay, so how do you show this? Start with one 53 and double it to find two 53's. Double that number to find four 53's. Double that number to find eight 53's.

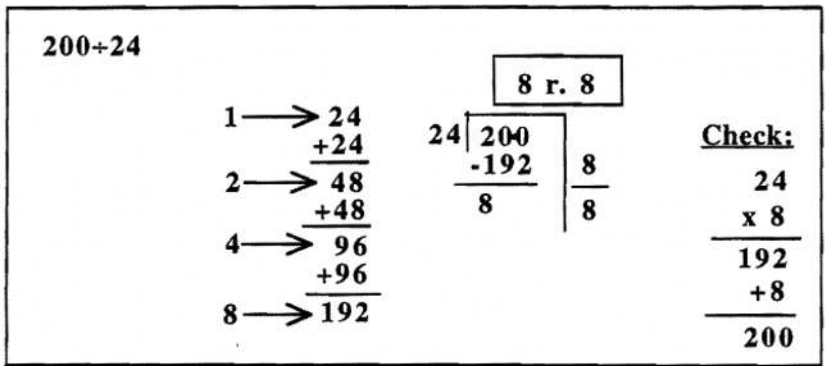
Look in the left part of the box shown below. See how there is a 1 pointing to 53 that means 1×53. If you double the 53, (53 + 53), then you now know $2 \times 53 = 106$, so a 2 points to 106. If you double that (106 + 106), you now know $4 \times 53 = 212$, so a 4 points to 212. And if you double this number (212 + 212), then you now know $8 \times 53 = 424$, so an 8 points to 424. Isn't that an easy way to find 8×53? But the cool thing is that you also know $1 \times 53, 2 \times 53, 4 \times 53,$ AND 8×53! This is useful to know to do the long division. Instead of guessing how many times 53 "goes into" 361, we note how many 53's can be subtracted from 361 (underestimating is okay). No multiplication needs to be done in your head, either! The box to the right shows how to use repeated subtraction to do the division problem: $361 \div 53$. First, make up a partial multiplication table. Then, write the division problem using a scaffold. Look at the multiplication table and pick the biggest number you can subtract from 361, which is 212. Subtract 212 from 361 and write 4 on the right because you just subtracted 4 53's. The biggest number you can subtract from 149 is 106. If you do that, you've subtracted 2 more 53's, so write a 2 on the right. After subtracting, 43 is left and no more 53's can be subtracted. Therefore, 43 is the remainder. Add the numbers on the right to see how many 53's you subtracted. The answer is 6 r. 43. To check, multiply 53×6 and add 43. The answer is 361.

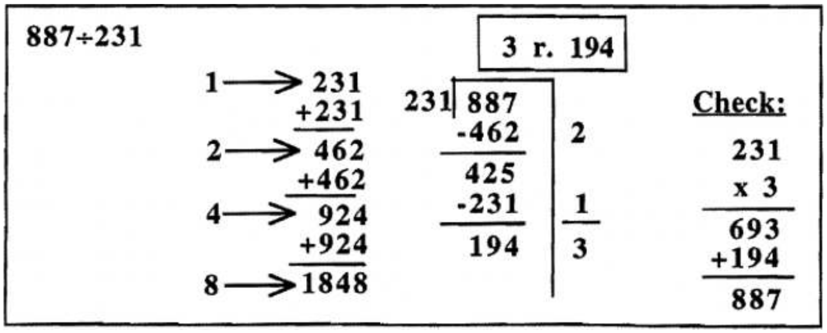


P.S. You can multiply 53 times 6 by adding 106 + 106 + 106, since that is two 53's added together 3 times, which is six 53's!!!

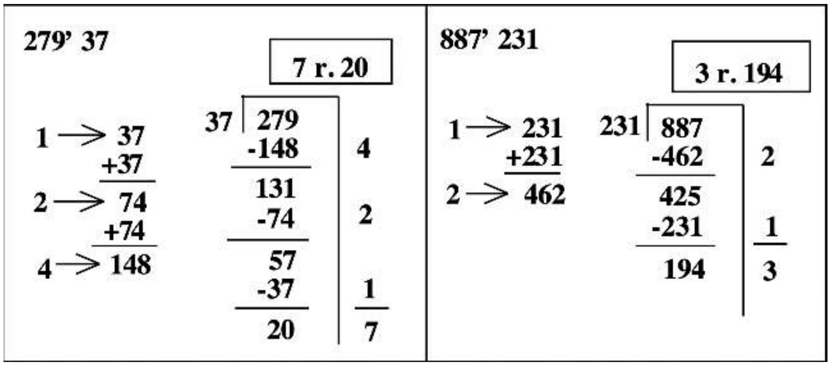
Here are some more examples for you to study. A partial multiplication table is made up first. This prep work which involves a few additions makes the division much easier. Then, the problem is done using repeated subtraction (with help from the partial multiplication table) using a scaffold. After adding up how many times the divisor was subtracted from the dividend (the numbers to the right of the scaffold), the answer (quotient and remainder) is written in a box up above the problem. Finally, the check is shown. Also, check to make sure the remainder is less than the divisor.




200÷24
8 r. 8
1 → 24
+24
2 → 48
+48
4 → 96
+96
8 → 192
24 | 200
-192
8
8
8
Check:
24
x 8
192
+8
200


887÷231
3 r. 194
1 → 231
+231
2 → 462
+462
4 → 924
+924
8 → 1848
231 | 887
-462
425
-231
194
2
1
3
Check:
231
x 3
693
+194
887

In the examples so far, all of the quotients were less than 10. It's possible to continue doubling the divisor in the partial multiplication table, so that you can find the divisor times any power of 2 — 1, 2, 4, 8, 16, 32, 64, etc. You would stop doubling if you notice that doubling once more gives a number bigger than the quotient. In fact, in the first example on the previous page, $279 \div 37$, I really didn't have to double 148 to figure out 8×37, since doubling 148 gives me a number bigger than 279. In the third example, $887 \div 231$, I didn't have to double 462 to figure out 4×231, since doubling 462 gives a number bigger than 887. The problems could have been shortened slightly had I noticed that ahead of time. They would look like this (the check is omitted).


279’ 37
7 r. 20
1 → 37
+37
2 → 74
+74
4 → 148
37 | 279
-148
131
-74
57
-37
20
4
2
1
7
887’ 231
3 r. 194
1 → 231
+231
2 → 462
231 | 887
-462
425
-231
194
2
1
3

Okay, now that you've studied some examples, it's time for you to try using this method. The first four problems are the same exact ones done in the previous examples. If you get stuck, look back at them and copy how I did them on another piece of paper. Then, try again.

? Exercise 1

For each division problem, do your prep work by using the doubling method (double as far as necessary) to make a partial multiplication table first. Then, draw a scaffold, and use repeated subtraction to do each division. When you are done, make sure the remainder is less than the divisor. Then, check the answer. All work should be shown in the space provided. After checking, write the answer on the space provided at the beginning of the problem.

a. $361 \div 53 =$ ________________

Partial multiplication table:	Scaffold:	Check:

b. $279 \div 37 =$ ________________

Partial multiplication table:	Scaffold:	Check:

c. $200 \div 24 =$ ____________________

Partial multiplication table:	Scaffold:	Check:

d. $887 \div 231 =$ _______________

Partial multiplication table:	Scaffold:	Check:

e. $415 \div 72 =$ ________________

Partial multiplication table:	Scaffold:	Check:

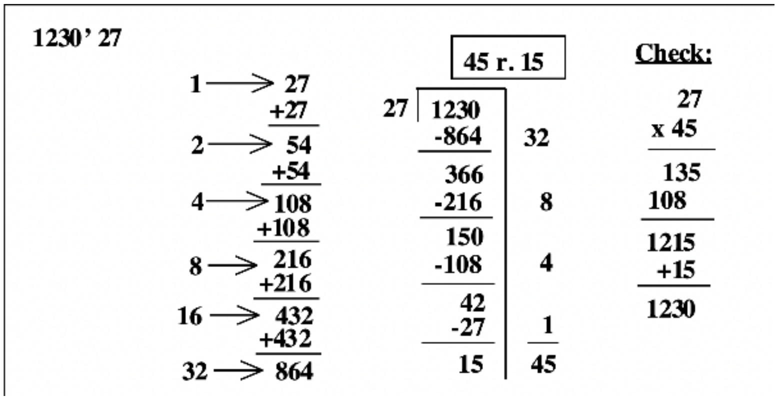
f. $1235 \div 214 =$ ______________

Partial multiplication table:	Scaffold:	Check:

g. $3128 \div 321 =$ ______________

Partial multiplication table:	Scaffold:	Check:

The following is an example to show that the doubling can continue in this way. You stop doubling if one more double gives a number bigger than the quotient. Below is $1230 \div 27$.



I thank and credit my son, Jakob, with this method of doubling to figure out more than eight times the divisor. I had always used a method that employed the use of place values and multiples of 10, which we'll get to after doing a few this way. Jakob, nine years old and in fourth grade, hated long division because it didn't make sense to him, and had trouble with the "guessing" involved. Before I was able to show him my method where you use the doubling method to make a partial multiplication table only up to 8 times the divisor, he did all of the problems like the way shown above. It finally made sense to him, and he did them all correctly!!! It opened up a whole new way of thinking for me. Kids and students can be great innovators! It's wonderful when you get a chance to explore and discover on your own, and find your own solution that makes sense. We have to remember that there is nothing sacred about any one algorithm for division, or any other operation.

Okay, it's time for you to try some more problems, using the doubling method, where the quotient is bigger than 10. The first problem is the same exact one done above. If you get stuck, look back at it and copy how I did it on another piece of paper. Then, try again.

Exercise 2

For each division problem (starting on the next page), use the doubling method (double as far as necessary) to make a partial multiplication table. Then, draw a scaffold, and use repeated subtraction to do each division. Then, check the answer, and make sure the remainder is less than the dividend. All work should be shown. After checking, write the answer on the space provided at the beginning of the problem.

$$\text{Check: } \begin{array}{r} 27\\ \underline{\times 45}\\ 1215\\ \underline{+15}\\ 1230 \end{array}$$

A calculator may be used to check the answer. For instance, to check that 45 r. 15 is the correct answer to $1230 \div 27$, you need to multiply 27 by 45 and add 15. The answer should be 1230. You can use the calculator to do the multiplication and addition, but write down the steps.. It would look the way it is shown to the right.

a. $1230 \div 27 =$ _______________

Partial multiplication table:	Scaffold:	Check:

b. $603 \div 48 =$ _______________

Partial multiplication table:	Scaffold:	Check:

c. $1346 \div 41 =$ _______________

Partial multiplication table:	Scaffold:	Check:

d. $3512 \div 43 =$ _______________

Partial multiplication table:	Scaffold:	Check:

You may have noticed the partial multiplication table could get long if you have to keep doubling for quite awhile. Also, it's not as easy to add the numbers up to the right of the scaffold. Another approach is to take advantage of the fact that multiplying a whole number by 10 or 100, etc., simply adds zeroes to the end of the number. For instance, if you know 53 times 2 is 106, then 53 times 20 is 1060, and 53 times 200 is 10600. We'll use this fact to divide using repeated subtraction when the quotient is more than 10.

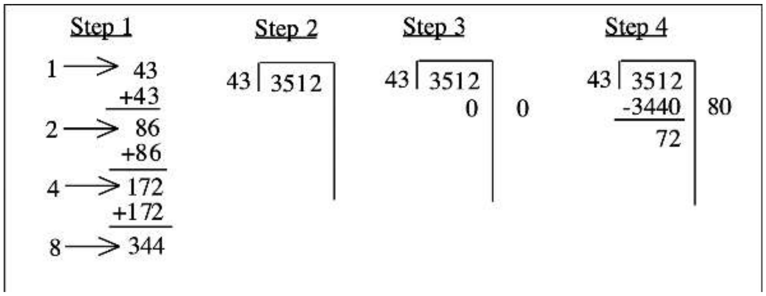
Look back at the last problem you did in Exercise 2: $3512 \div 43$. You had to double until you figured out what 64 times 43 was. In the box at the bottom of this page is a different way of doing the same problem. The explanation and steps are on this and the next page.

Step 1: Use doubling to make up a partial multiplication table up to 8 times 43.

Step 2: Draw the scaffold.

Step 3: Scan 3512 starting with the left-most digit until you get a number greater or equal to 43. Try 3, then 35, then 351. Note how many more digits are after 351. In this case there is only one (the 2). This means you can take a multiple of ten times 43 away at a time. (In the old standard algorithm, when you guess and put the number above the 1 in 3512, you are putting it in the tens place value.) So we put a zero under the extra numbers in the dividend (the 2) and the same number of zeroes to the right of the scaffold. Leave a space for a digit before the 0 on the right of the scaffold.

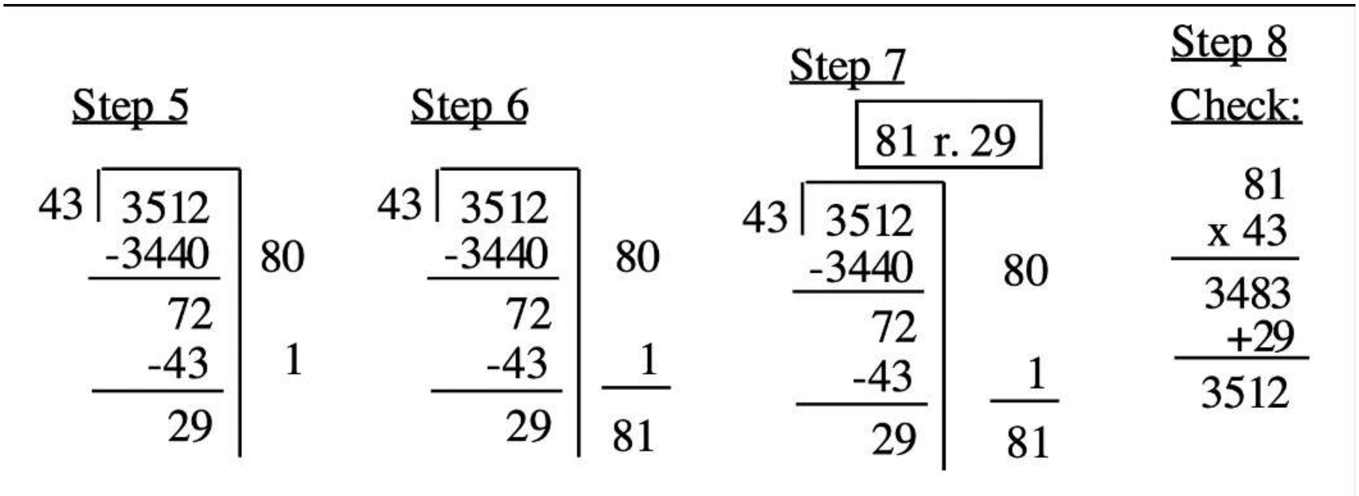
Step 4: Now, look at the partial multiplication table to determine the largest number you can subtract from 351. It is 344, which is 8 times 43. Note that if 8 times 43 is 344, then 80 times 43 is 3440. You've already put the extra zero under the 2, so just put the 344 in front of the 0 that you already put down in step 3. Note that 3440 is 80 times 43, and you need to keep track of that to the right of the scaffold. You've already written the zero down in step 3, so simply put an 8 in front of that 0. You've just subtracted eighty 43's from the dividend, since 80 times 43 is 3440. Subtract 3440 from 3512 to get 72.


Step 1
1 → 43
+43
2 → 86
+86
4 → 172
+172
8 → 344
Step 2
43 | 3512
Step 3
43 | 3512
0
0
Step 4
43 | 3512
-3440
72
80

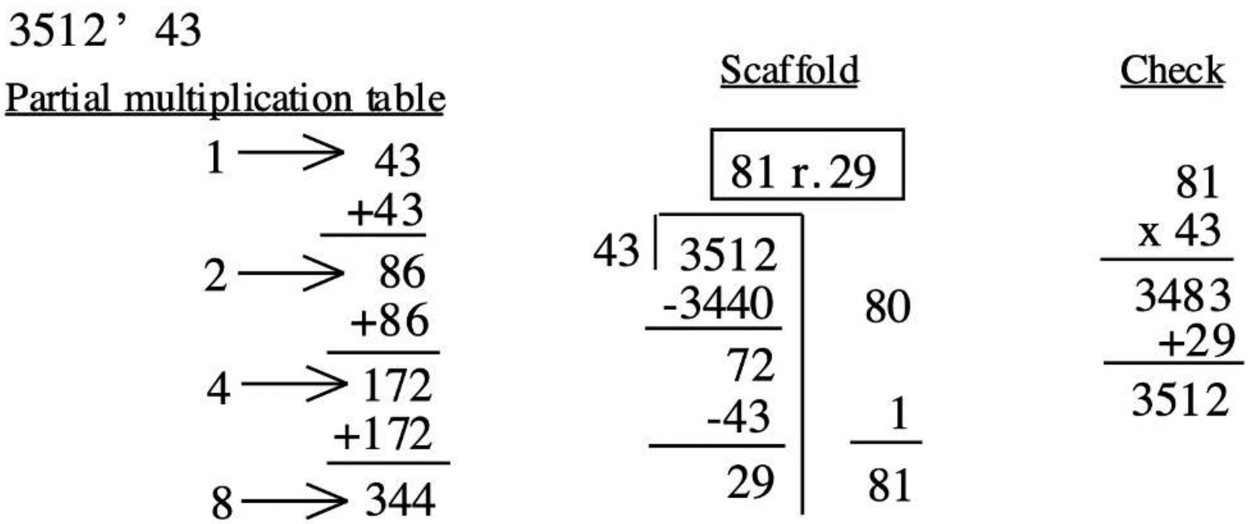
Step 5: Now we have to see how many more 43's can be subtracted from 72. There are no extra zeroes to tack on this time, as in step 4. We can subtract 1 more 43 from 72, so write a 1 to the right of the scaffold, and subtract 43 from 72 to get 29.

Steps 6, 7 and 8: Since 29 is less than 43, that is the remainder, so add up the numbers on the right of the scaffold to get the quotient.Write the answer as a quotient and remainder in a box up at the top of the problem. Check the answer. You can use a calculator to multiply 81 times 43, which is 3483, and then add 29.

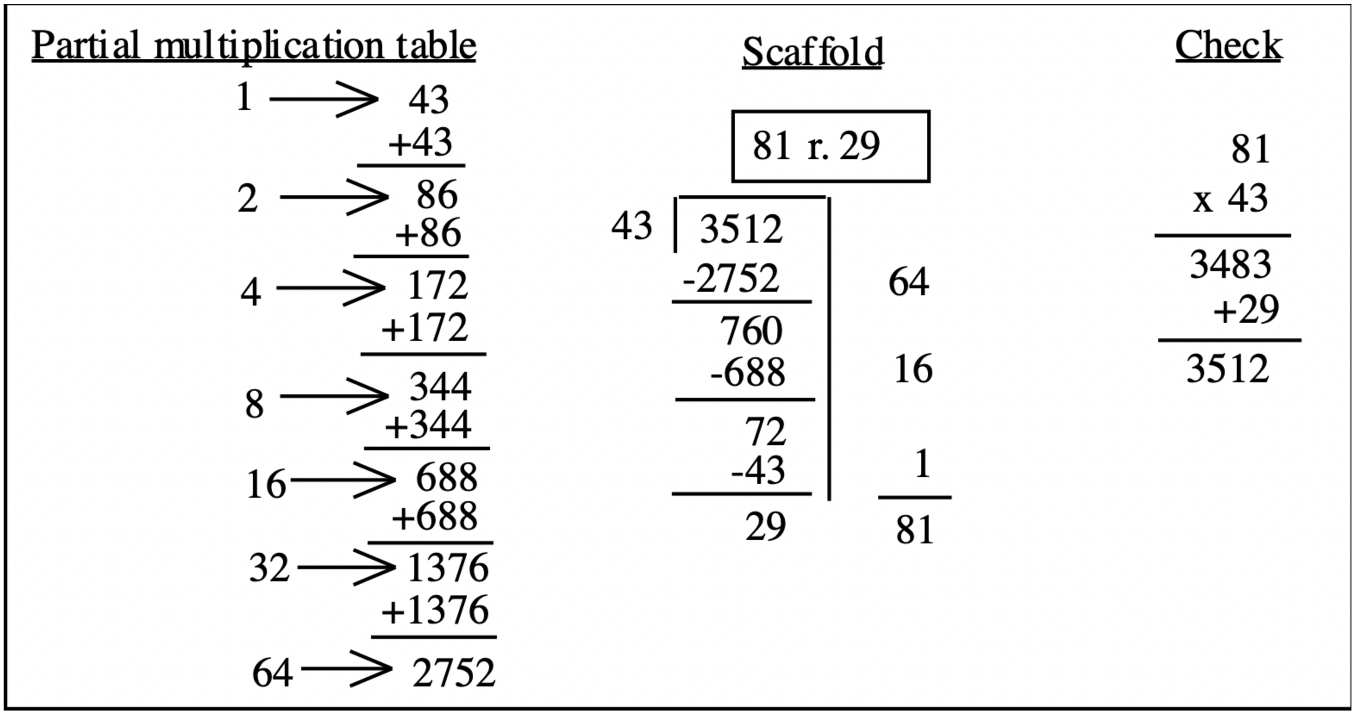
The last four steps are shown below.


Step 5
43 | 3512
-3440
72
-43
29
80
1
Step 6
43 | 3512
-3440
72
-43
29
80
1
81
Step 7
81 r. 29
43 | 3512
-3440
72
-43
29
80
1
81
Step 8
Check:
81
x 43
3483
+29
3512

If you were to show your work, this is what the whole problem would look like:


3512 ’ 43
Partial multiplication table
1 → 43
+43
2 → 86
+86
4 → 172
+172
8 → 344
Scaffold
81 r. 29
43 | 3512
-3440
72
-43
29
80
1
81
Check
81
x 43
3483
+29
3512

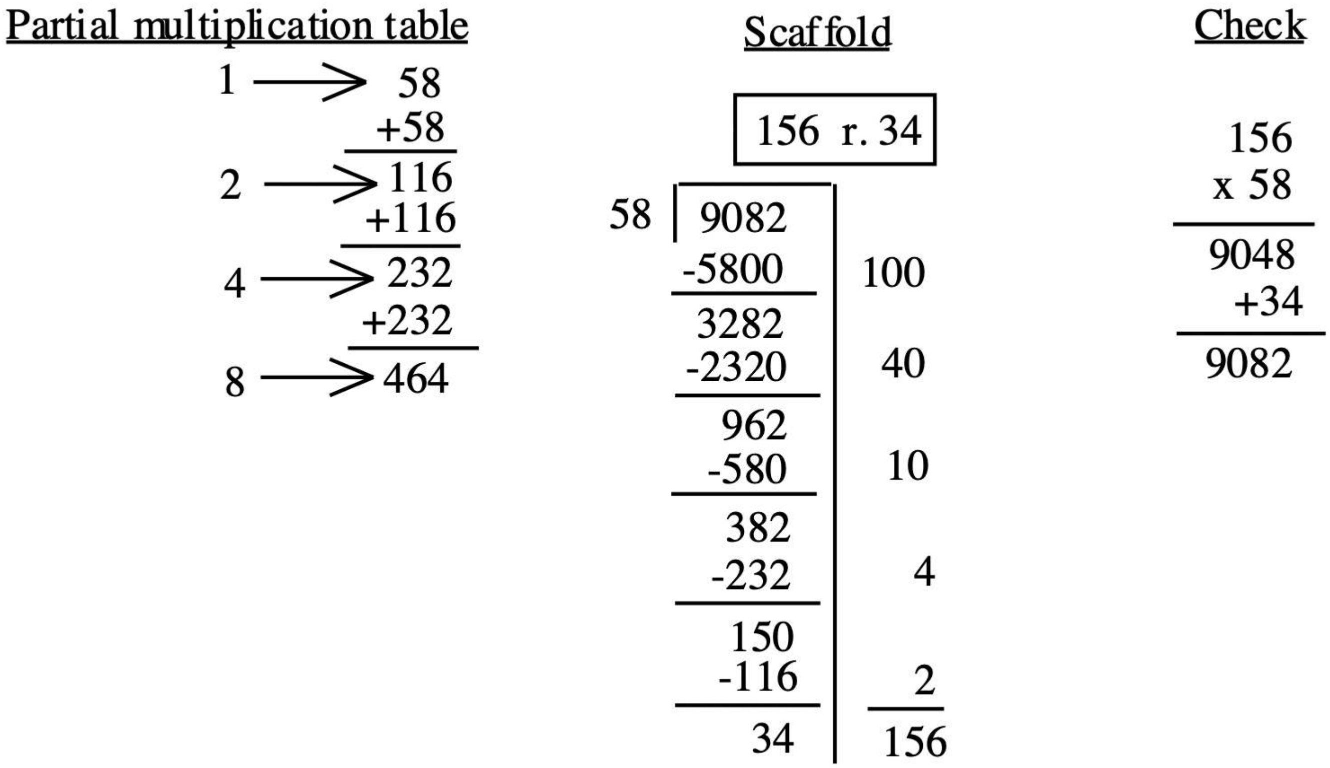
Here's what it would look like doing it the way you did it in exercise 2:

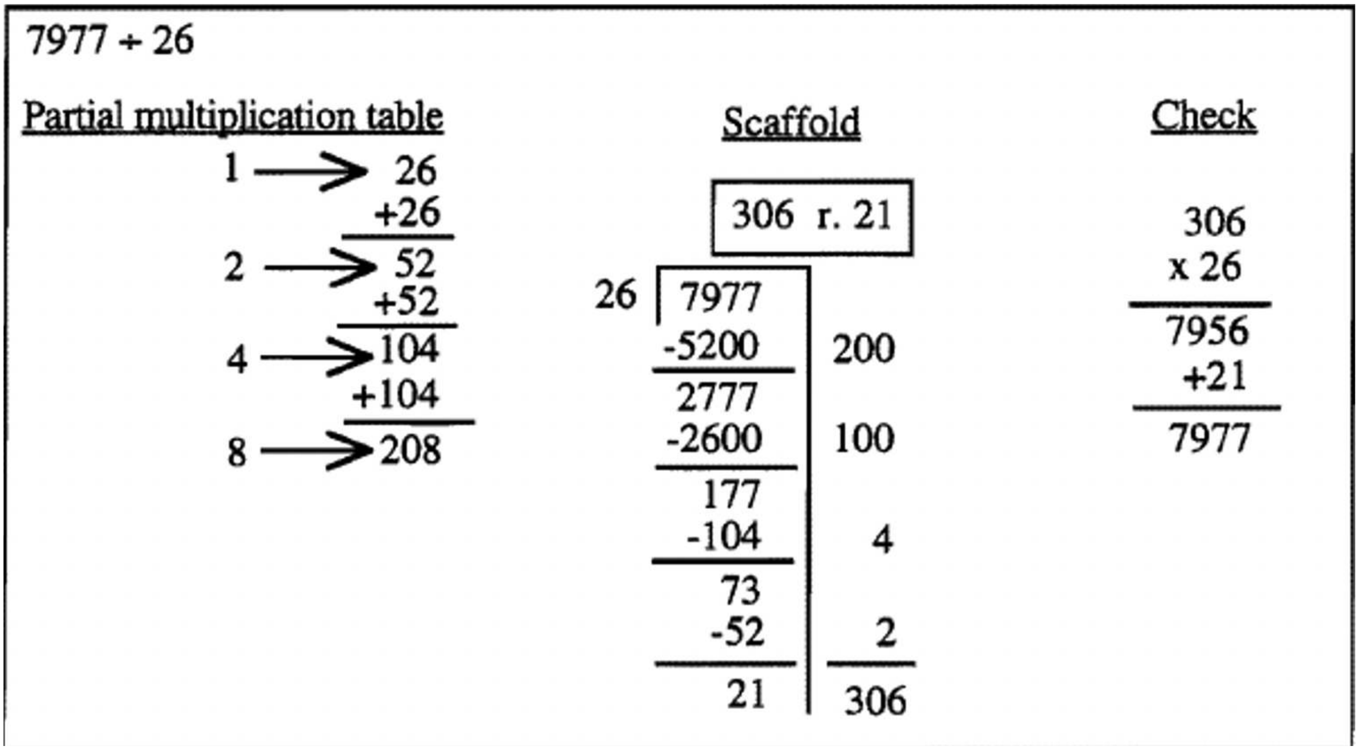

Partial multiplication table
1 → 43
+43
2 → 86
+86
4 → 172
+172
8 → 344
+344
16 → 688
+688
32 → 1376
+1376
64 → 2752
Scaffold
81 r. 29
43
3512
-2752
64
760
-688
16
72
-43
1
29
81
Check
81
x 43
3483
+29
3512

For both methods, when you add the numbers on the right of the scaffold to get the quotient, be careful to line up the place values! Both methods produce the same result and work equally well. What's important is that the procedure makes sense to you!

Two more examples using multiples of tens (putting in the extra zeroes) is shown below.

9082 ’ 58


Partial multiplication table
1 → 58
+58
2 → 116
+116
4 → 232
+232
8 → 464
Scaffold
156 r. 34
58
9082
-5800
100
3282
-2320
40
962
-580
10
382
-232
4
150
-116
2
34
156
Check
156
x 58
9048
+34
9082



In the above example, many people (children and adults) miss writing the zero in the tens place for the answer of 306 when they do it the short division way it is usually taught. A very common answer to this problem is 36 r. 21. In the first step above, writing the full number on the side really emphasizes the fact that you aren't just multiplying 26 by a 2 to get 52 you are multiplying 26 by **200** to get 5200. You know the answer to the problem will be more than 200 since you already subtracted 200 26's from 7977. When you have 177 left, you simply note you can only subtract a few more 26's. If one neatly writes the numbers on the right, there are no numbers in the tens place to add up. The confusion about the zero doesn't even come up!

Exercise 3

For each division problem, make a partial multiplication table up to 8 times the divisor. Then, draw a scaffold, and use repeated subtraction to do each division. Use multiples of tens (putting in extra zeroes where necessary) to do these problems. Then, check the answer. All work should be shown in the space provided. After checking, write the answer on the space provided at the beginning of the problem. A calculator may be used to check the answer, but write down the steps.

a. $1230 \div 27 =$ ________________

Partial multiplication table:	Scaffold:	Check:

b. $603 \div 48 =$ ________________

Partial multiplication table:	Scaffold:	Check:

c. $1346 \div 41 =$ ________________

Partial multiplication table:	Scaffold:	Check:

d. $3512 \div 43 =$ ________________

Partial multiplication table:	Scaffold:	Check:

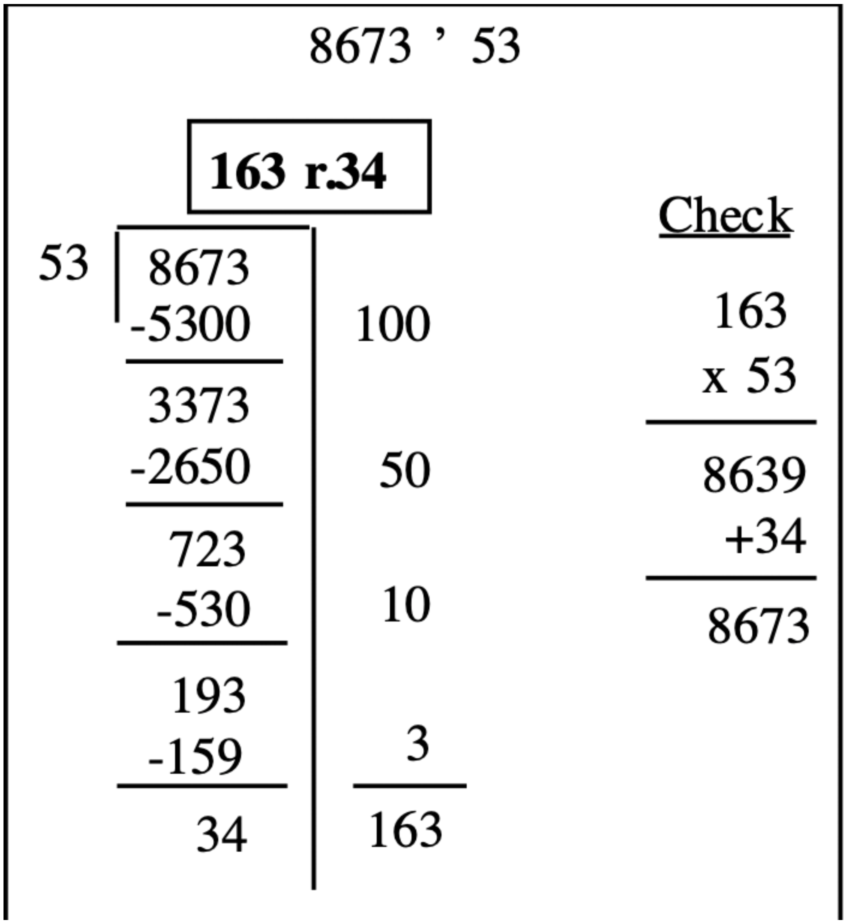
e. $66289 \div 325 =$ ______________

Partial multiplication table:	Scaffold:	Check:

f. $5222 \div 21 =$ _______________

Partial multiplication table:	Scaffold:	Check:

Doing the partial multiplication table on the side, in conjunction with the repeated subtraction method, is completely optional. But, if you make up the table first, you do not need to do any multiplying or estimating in your head. Another option is to skip doing the partial multiplication table altogether and still use the repeated subtraction method. In this case, if you aren't sure how many times the divisor can be subtracted, underestimate at each step that you aren't absolutely sure how many of the divisor to subtract. In other words, make sure you don't multiply something by the divisor and come out with a number too big to subtract from the dividend. Underestimating is okay with the repeated subtraction method because you just subtract some more at the next step. For instance, in the previous example, you could have subtracted 200 26's (5200) at the first step and then 100 more 26's at the second step, or you could have subtracted 300 26's at the first step by multiplying 26 by 3 in your head if you were *sure* that 26 times 3 wasn't too big. I like the partial multiplication table because it takes the guesswork out of the problem. In the example in the box below, the division is done using repeated subtraction without making up a table Instead, you estimate how many of the divisor can be subtracted at each step. In this case, the multiplication must be done in your head, or on the side, as you go along.



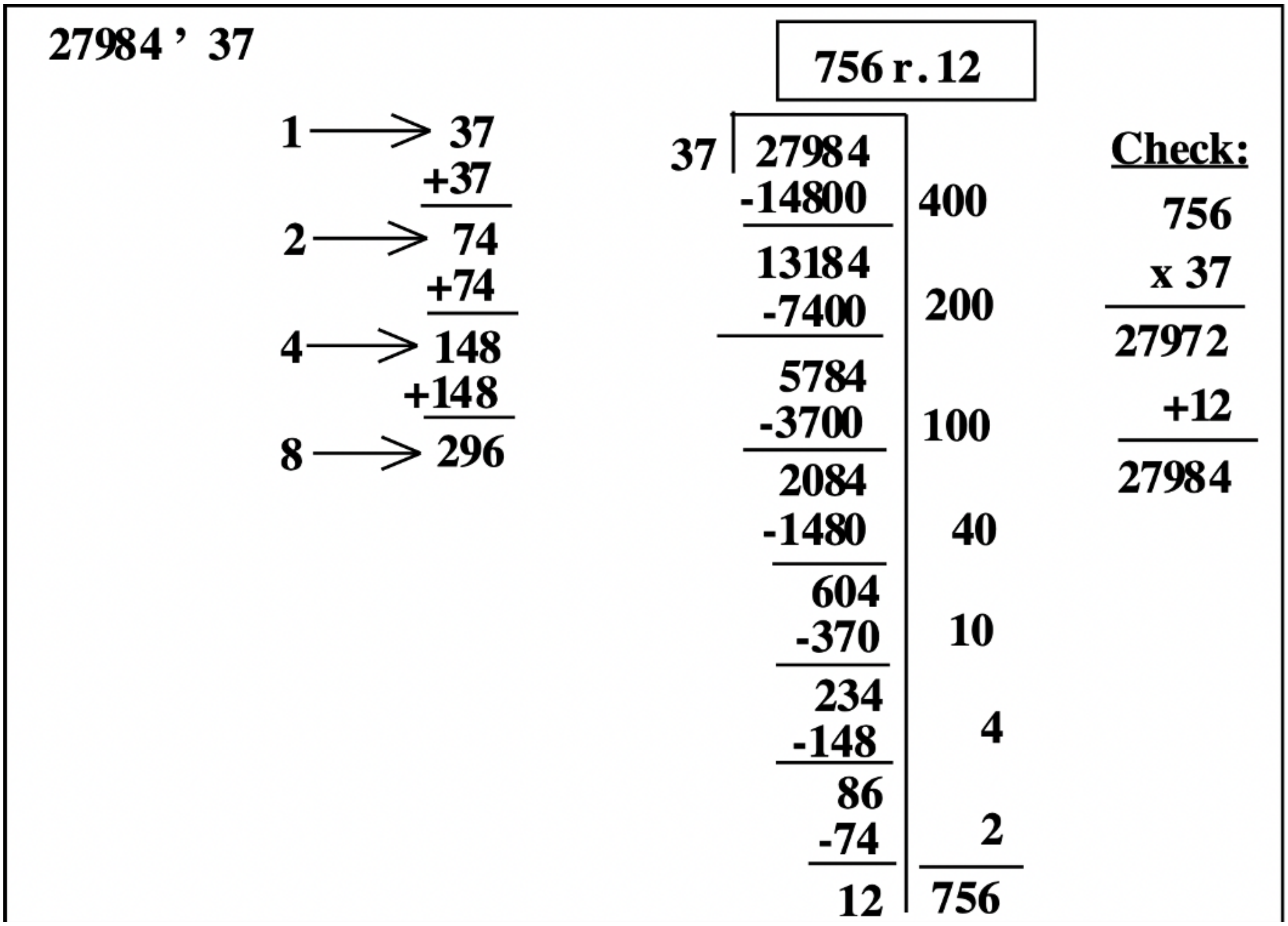
Here is a way to divide $8673 \div 53$ without the usual prep work of making up a partial multiplication table first. This means the multiplication must be done along the way. It's important to note that it is okay to underestimate when using the repeated subtraction method (unlike the traditional division method), because you can always subtract more at the next step. On the other hand, overestimating will cause you to go back and redo the problem. First I note that 53 goes into 86 one time, so I subtract 100

53's, or 5300, from 8673. Next, estimate $337 \div 53$. Since I'm not sure if it is 5 or 6 times, I play it safe and therefore, estimate 5. So, I subtract 50 times 53, or 2650, from 3373. (Note I have to do the multiplication in my head.) Well, I see another 53 can be subtracted from 72, so I subtract 10 times 53, or 530, from 723. Then, I estimate 53 goes into 193 at least 3 times, so I subtract 3 times 53 from 193. Since 34 is less than 53, 34 is the remainder. Add the numbers on the side to get the quotient of 163. Check by multiplying 53×163 and adding the remainder of 34 to get 8673.

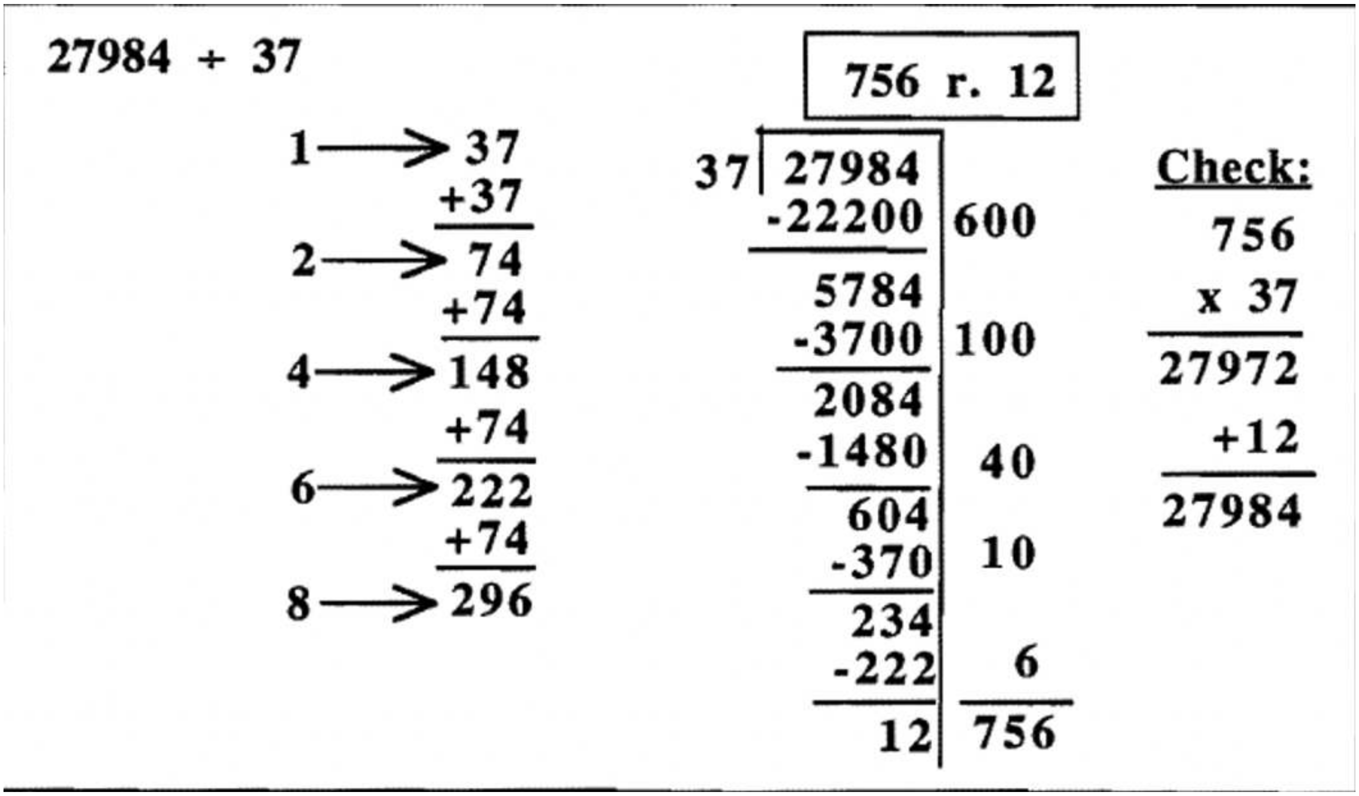
In the traditional division algorithm, there is no room for underestimating or overestimating. You have to go back and erase if you put the wrong digit in a particular place value. If you estimate incorrectly, the multiplication you do ends up being a waste of time and effort, and you have to start over. Also, most people don't think about what or why they are putting digits in a particular place. They don't think about the place values, or that division is really about repeated subtraction. I think the scaffold method, with or without making up a partial multiplication table, promotes a better understanding of division, and is less stressful than the traditional "guess" method where there is absolutely no room for error. The one advantage to the traditional algorithm is that it is a shortcut to the scaffold method, and it uses less space on paper.

In the end, it's really up to the individual doing the division to decide which way to do it.

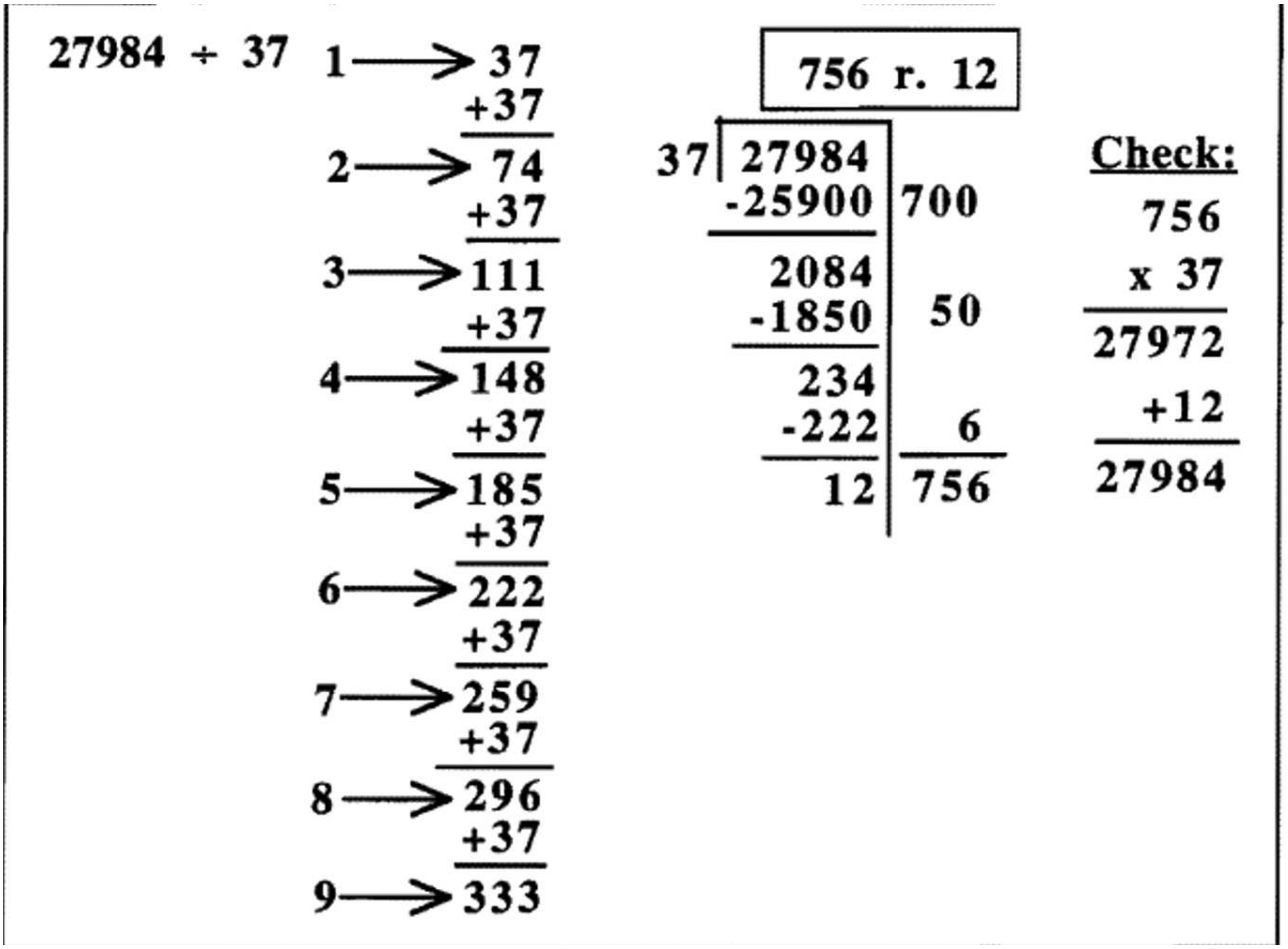
Here is an example of dividing $27984 \div 37$ using the doubling method.



When making up a multiplication table, it doesn't have to be partial. In base ten, doubling to get the divisor times 1, 2, 4 or 8 gives us close enough estimates most of the time. The worst possible case is that 7 is the correct number of times something can be subtracted, as in the above example. In that case, we would first subtract 4, then 2, then 1. That doesn't happen too often, but once in awhile it does. Another strategy is to add one more addition to the partial table so you also know what 6 times the divisor is. To do this, after figuring what 4 times the divisor is, add 2 more, as shown below (add 74 instead of doubling 148), then add 2 more again (add 74 again) to get 8 times the number. Look at the difference.


27984 ÷ 37
1 → 37
+37
2 → 74
+74
4 → 148
+74
6 → 222
+74
8 → 296
756 r. 12
37 | 27984
-22200 600
5784
-3700 100
2084
-1480 40
604
-370 10
234
-222 6
12 756
Check:
756
x 37
27972
+12
27984

In fact, we could have used repeated addition to find out the answers to the divisor being multiplied by each digit from 1 to 9, in which case neither underestimating nor overestimating would ever occur. Although this leaves absolutely no guesswork at all, it's more work than necessary in most cases. Here is an example of how someone could do the same problem done on the previous page by making a complete multiplication table first. It's easy to do the division if you do this first, but there is a lot of prep work at the beginning to make the table.



Look back at the three different ways the division $27984 \div 37$ was done. The first way, shown on the previous page, had a short partial multiplication table (only three additions for prep work), but the repeated subtraction method took a little longer. It required seven subtractions to get the answer (look to the right of the scaffold), because the 700 took three steps 400, 200, 100; the 50 took two steps 40 and 10; and the 6 took two steps 4 and 6. The second way had one more addition for the prep of the multiplication table, and then only required five subtractions to get the answer (look to the right of the scaffold), because the 700 took two steps 600, and 100; the 50 took two steps 40 and 10; and the 6 took one step. The third way, shown at the top, required eight additions of prep work to make the multiplication table, but the division was a breeze, taking only three steps. If you do it this way, you may as well write the 756 up at the top, as you do when you do it using the traditional algorithm because you can never underestimate or overestimate.

Depending on the particular problem, one method may be faster than another, but many times you wouldn't know until you've completed the problem. I usually just do the doubling to 8 to get good enough estimates, but if you'd like to make a more complete table, it works just as well. You have to decide if you prefer to do more prep work up front to possibly make the division faster.

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7.3: Division in other Bases

You will need: ***Base Blocks*** *(Material Cards 3 - 15)*

We will now work on dividing in other bases. First, we will divide by using repeated subtraction with the actual base blocks. Later, we'll employ the same algorithm we did in the previous exercise set to do the problems using repeated subtraction with the partial multiplication table, and the scaffold.

Get out your Base Three base blocks. We are going to use them with the repeated subtraction method to divide $221_{\text{three}} \div 21_{\text{three}}$. The dividend is 221_{three}, which consists of 2 flats, 2 longs and 1 unit. We need to find out how many times we can repeatedly subtract the divisor, 21_{three}, from the dividend. To do this, we can convert the dividend into only longs and units by making some exchanges, and then form as many subsets containing 1 long and 2 units as possible. The number of subsets formed is how many times the divisor could be subtracted from the dividend. Get out your blocks to do this problem with me.

Step 1: Form the dividend in base three. You should actually get out your base three blocks and do the problems with the blocks. I will be using the initials for the base blocks (U, L, F, B, etc.) to show what is going on. The dividend can be shown like this:

F F L L U

Step 2: Make exchanges so it is possible to form as many subsets as possible, each containing the divisor, which is 2 longs and 1 unit. Each flat can be exchanged for 3 longs to get:

L L L L L L L L U

Then, one of the longs can be exchanged for 3 units to get L L L L L L L U U U U. Let's see if we can subtract out some equal subsets of 2 longs and 1 unit from the dividend. As you take out each 2 longs and a unit, form them in a subset.

I can take out and form 3 equal subsets of 2 longs and 1 unit each from the dividend. In the dividend, one long and 1 unit is left. The subsets formed look like this:

LLU	LLU	LLU

The part left in the dividend, which is the remainder looks like this:

LU

Step 3: Each equal subset of 2 longs and 1 unit that was subtracted out of the dividend counts as 1 for the quotient. Give yourself a unit for each equal subset you formed this is the quotient. The long and unit left over are the remainder because it is less than the divisor.

Quotient: U U U	Remainder: L U

Step 4: In the quotient and/or remainder, make any exchanges necessary so the quotient and remainder can be written in base three.

Quotient: L	Remainder: L U

Step 5: Write the quotient and remainder in base three: $10_{\text{three}} r.11_{\text{three}}$.

Therefore, $221_{\text{three}} \div 21_{\text{three}} = 10_{\text{three}} r.11_{\text{three}}$

Step 6: Check the answer. This is shown to the right.

? Exercise 1

Following the example on the previous page, do the following division step by step. Use base three blocks to do this exercise. You can either use initials, or draw pictures of the blocks to show how you do the division.

Do the following division: $222_{\text{three}} \div 12_{\text{three}}$

a. Form the dividend in base three. Draw a picture of what the dividend looks like:

b. Make exchanges so it is possible to form as many subsets as possible, each containing the divisor, which is 1 long and 2 units. Subtract out some equal subsets of 1 long and 2 units from the dividend. As you take out each long and 2 units, form them in a subset.Draw a picture of the all the equal subsets you were able to form by subtracting from the dividend, then draw a separate picture of what was left in the dividend, which will be the remainder (this must be less than 1 long and 2 units).

i) Show all of the equal subsets formed:

ii) Remainder:

c. For each of the equal subsets of 1 long and 2 units that was subtracted out of the dividend and formed (shown in b.i.), give yourself a unit — this is the quotient. Draw a picture of the quotient in terms of units:

d. In the quotient (part c) and/or remainder (part b.ii), make any exchanges necessary so the quotient and remainder can be written in base three. Draw a picture of what the quotient and remainder looks like after making any and all exchanges possible.

Quotient:	Remainder:

$$\text{Check: } \begin{array}{r} 21_{\text{three}} \\ \underline{\times 10_{\text{three}}} \\ 210_{\text{three}} \\ \underline{+11_{\text{three}}} \\ 221_{\text{three}} \end{array}$$

e. Write the quotient and remainder in base three: ________________________

f. Check the answer.

? Exercise 2

Use base blocks in the bases indicated to find the quotient and remainder for each problem. Then check your answer in the base given. Show your work. You can multiply using any multiplication algorithm you prefer — you might like to try the lattice.

a. $1204_{\text{five}} \div 42_{\text{five}} =$ Check:	c. $1345_{\text{six}} \div 25_{\text{six}} =$ Check:
b. $323_{\text{four}} \div 23_{\text{four}} =$ Check:	d. $11111_{\text{two}} \div 101_{\text{two}} =$ Check:

You may have noticed in parts c and d that you had to make a lot of piles, and that the repeated subtraction was somewhat tedious when you could only subtract one subset of divisor from the dividend at a time. This is where it is useful to use the idea that a long times the divisor is similar to multiplying by 10 in base ten. Remember that 10b is a long for some base, b. For instance, when you multiply 25 times 10, you get 250. In base six, 25_{six} means 2 longs and 5 units. When you multiply this by a long (10_{six}), you get 250_{six} which is 2 flats and 5 longs — every piece shifts up a place value. We can use this in exercise 2c to make the subtraction go faster. We'll do exercise 2c using this idea on the next page.

Exercise 2c, $1345_{\text{six}} \div 25_{\text{six}}$, can be done using a more efficient method. Here are the steps:

Draw a picture of the dividend, 1345_{six}, using Base Six blocks.

B F F F L L L L U U U U U

You want to subtract as many subsets of the divisor, 25_{six}, from the dividend as you can. This would take a lot of steps. You'd have to break down the block and three flats into lots of longs and units. Now, 25_{six}, means 2 longs and 5 units. If you multiplied this by a long, you would have 250_{six}, which is 2 flats and 5 longs. If you subtracted 2 flats and 5 longs from the dividend at once, it would be the same as subtracting 2 longs and 5 units from the dividend six times! And it is possible to subtract 2 flats and 5 longs at once.

Let's trade some of the pieces in the dividend for smaller pieces, so it is possible to take subtract and make subsets of 2 flats and 5 longs. I started by trading the block in for 6 flats and two of the flats in for 12 longs:

F F F F F F F L L L L L L L L L L L L L L L L L U U U U U

Now, I'll form as many subsets of 2 flats and 5 longs as possible.

F F L L L L L	F F L L L L L	F F L L L L L

I still have F L U U U U U left in the dividend, and can't take any more subsets of 2 flats and 5 longs out. So now, I try to subtract subsets of the divisor from what's left in the dividend. Again, I need to make some trades. I'll trade the flat for 5 longs and 6 units, so I have:

L L L L L L U U U U U U U U U U U

Two subsets can be formed:

L L U U U U U	L L U U U U U

All that is left in the dividend is a unit and 2 longs, which is less than the divisor, and is the remainder. So the remainder, LLU, is 21_{six}.

Now, we have to figure out the quotient. The first 3 subsets I formed were each a long times the divisor, so for the quotient, I put in a long for each of the subsets formed. I put in a unit for each of the next two subsets I formed. So I have 3 longs and 2 units for the quotient, and no exchanges need to be made.

The quotient is L L L U U, or 32_{six}, and the remainder is 21_{six}.

Therefore, $1345_{\text{six}} \div 25_{\text{six}} = 32_{\text{six}}$r. 21_{six}

You should have gotten the same answer when you did exercise 2c, but you would have subtracted and formed 13 subsets of the divisor, and then traded 13 units in for the answer in base six.

(By the way, if the divisor, 25_{six}, was multiplied by a flat, you would have 2 blocks and 5 flats, which is bigger than the dividend, so we can't take that many of the divisor away at one time. So we start by subtracting subsets of a long times the divisor, one at a time.)

? Exercise 3

Divide the following using base blocks. You can use the method described on the previous page, or subtract one subset of the divisor at a time. **Explain or draw pictures with the blocks to show how you did the problem.** Check your answers using any multiplication algorithm.

a. $1046_{\text{seven}} \div 31_{\text{seven}}$ = ____
Check:

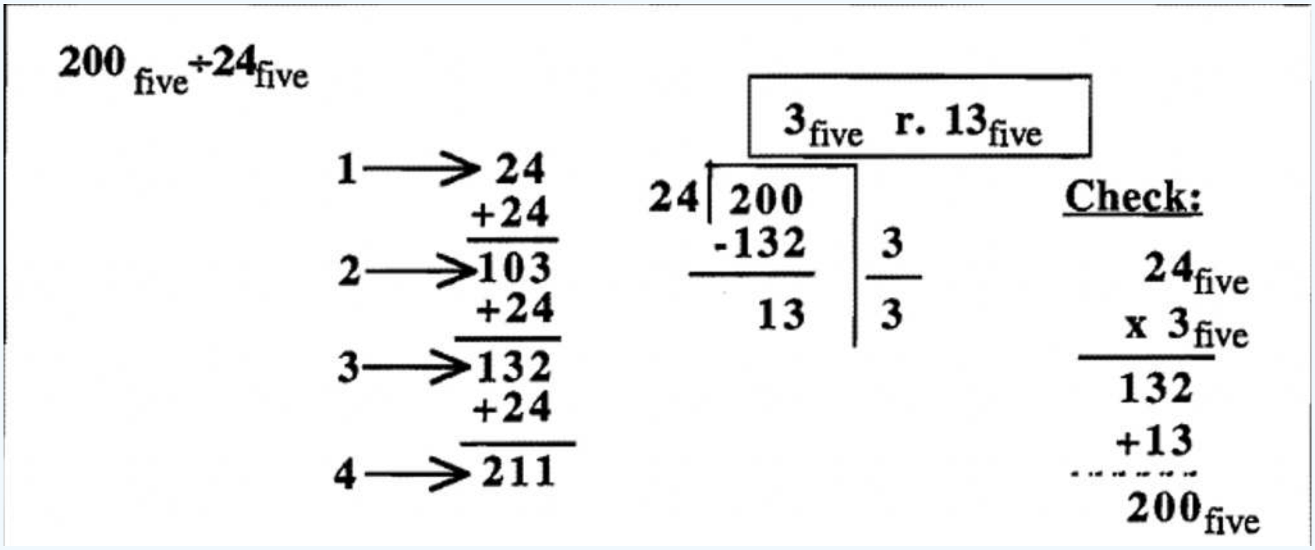
b. $10021_{\text{three}} \div 22_{\text{three}}$ = ____
Check:

We're now going to work on doing division in different bases using the same repeated subtraction algorithm we did in Exercise Set 2. Since most if not all of us aren't familiar enough with different bases to estimate in our head, making up a partial multiplication table becomes extremely useful for these problems.

Because each base has a different number of digits, we may opt to do the doubling method or some partial table for larger bases like base 11, 12 and 13. For smaller bases, you might consider making a complete table. In base five, you would only have to figure out 2, 3 and 4 times the divisor (since 4 is the highest digit in base five). Study the following examples of using the scaffold method in other bases.

Example 1

Division problem in base five: $200_{\text{five}} \div 24_{\text{five}}$. I leave off writing "five" except for the answer (boxed) and the check. You have to be very careful to remember you are in base five when you are adding (the prep work at the beginning, and the addition at the end of the check when you add the remainder), subtracting (doing the repeated subtraction), and multiplying (in the check). For the check, you could use the lattice method, or any other method you prefer.



$200_{\text{five}} \div 24_{\text{five}}$

1 → 24
+24
2 → 103
+24
3 → 132
+24
4 → 211

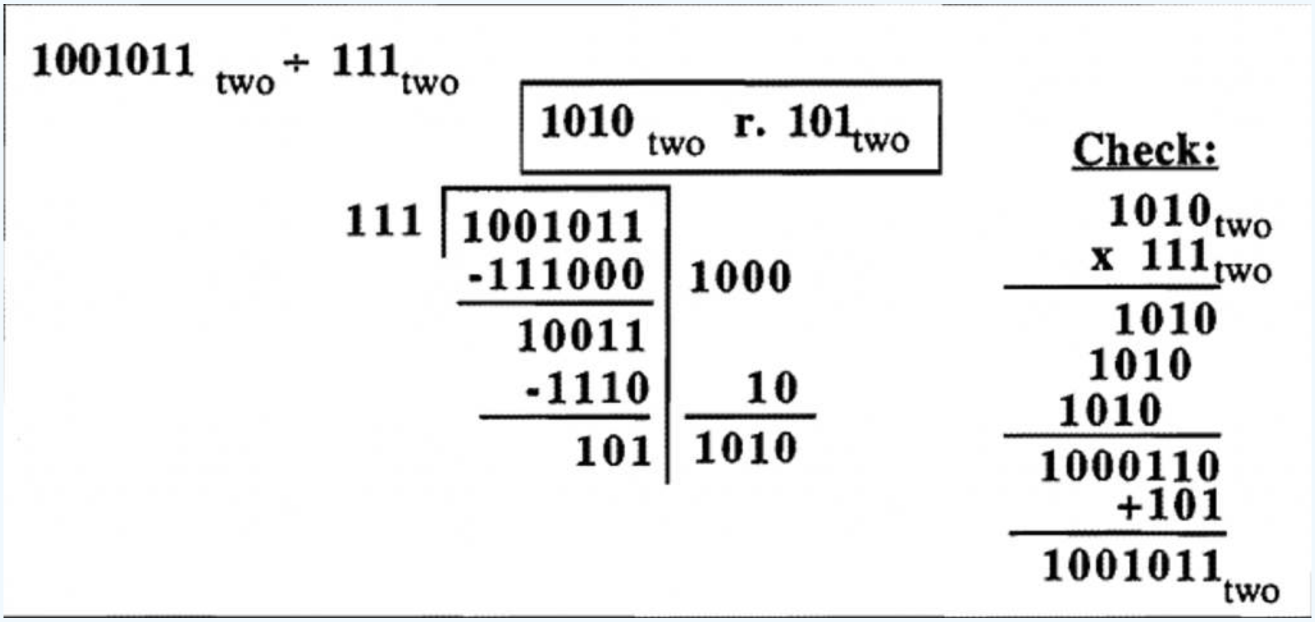
$\boxed{3_{\text{five}} \text{ r. } 13_{\text{five}}}$

24) 200
-132 | 3
13 | 3

Check:
24_{five}
x 3_{five}
132
+13
200_{five}

Example 2

Division problem in base two: $1001011_{\text{two}} \div 111_{\text{two}}$. There is really no prep work here, because the only digits in base two are 0 and 1.



$1001011_{\text{two}} \div 111_{\text{two}}$

$\boxed{1010_{\text{two}} \text{ r. } 101_{\text{two}}}$

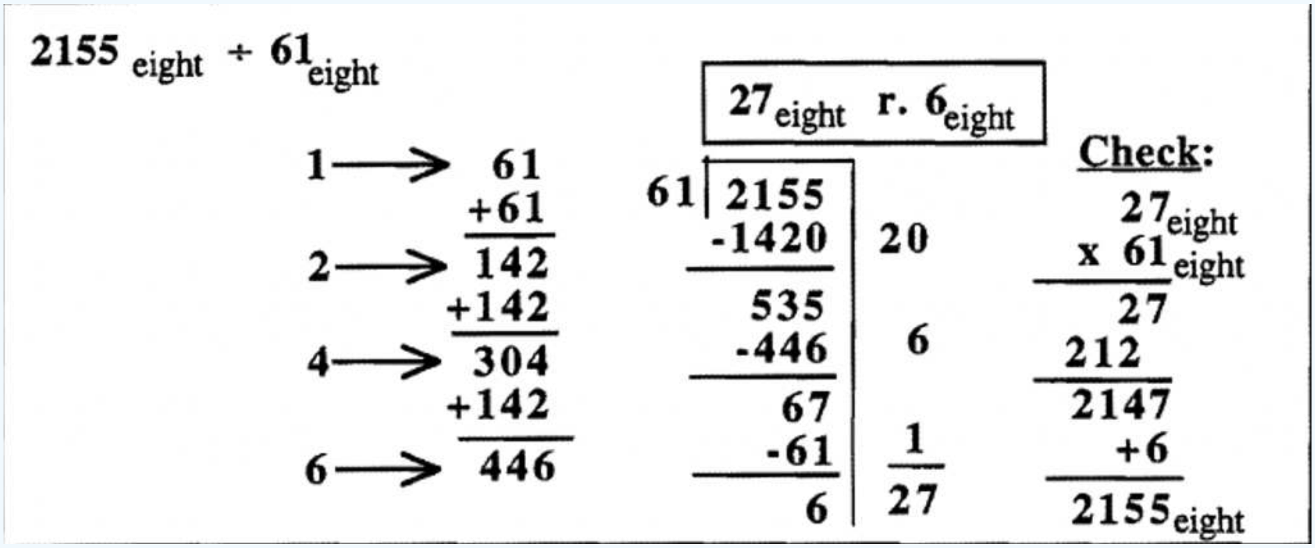
111) 1001011
-111000 | 1000
10011
-1110 | 10
101 | 1010

Check:
1010_{two}
x 111_{two}
1010
1010
1010
1000110
+101
1001011_{two}

For examples 3 and 4, I chose to do a different kind of partial multiplication table for each.

Example 3

Division problem in base eight: $2155_{\text{eight}} \div 61_{\text{eight}}$. Since this is a problem in base eight, I decided to make a partial table to figure out the divisor times 1, 2, 4, and 6. Another person might choose to do a table for only 1, 2, and 4. Remember that 7 is the highest digit in base eight. Yet another student might do a full table for all of the digits in base eight: 1, 2, 3, 4, 5, 6, and 7.



$2155_{\text{eight}} \div 61_{\text{eight}}$

1 → 61
+61
2 → 142
+142
4 → 304
+142
6 → 446

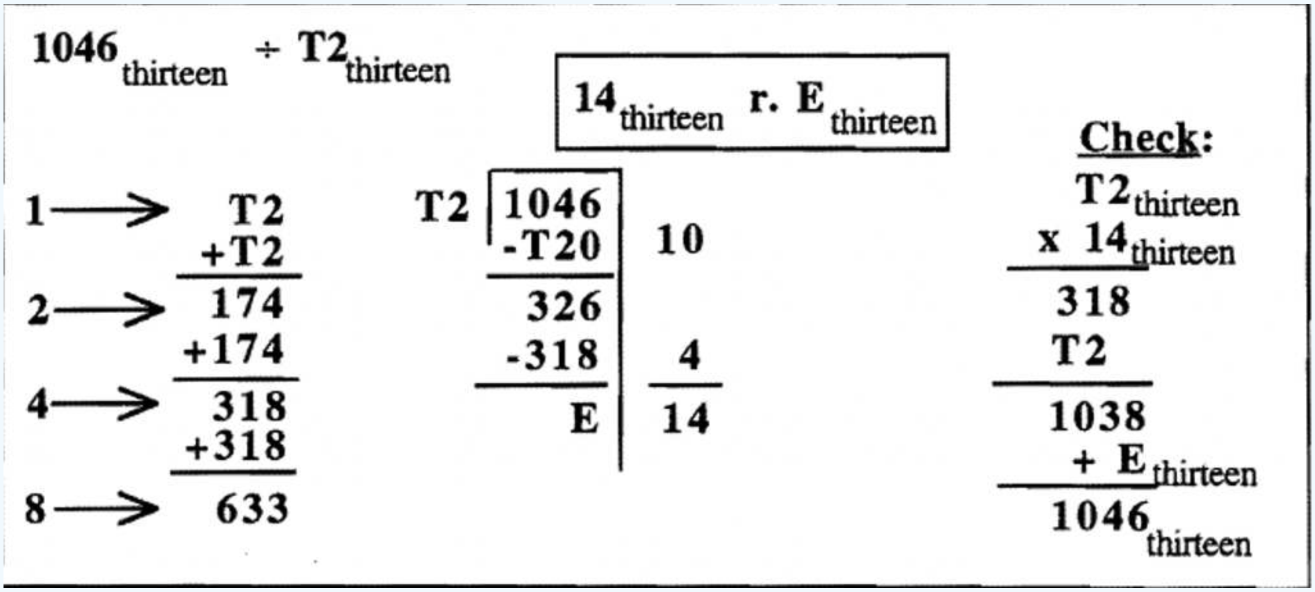
$\boxed{27_{\text{eight}} \text{ r. } 6_{\text{eight}}}$

61 | 2155
-1420 | 20
535
-446 | 6
67
-61 | 1
6 | 27

Check:
27_{eight}
x 61_{eight}
27
212
2147
+6
2155_{eight}

Example 4

Division problem in base thirteen: $1046_{\text{thirteen}} \div T2_{\text{thirteen}}$. Since this is a problem in base thirteen, I decided to make a partial table to figure out the divisor times 1, 2, 4, and 8. Remember that W is the highest digit in base eight. Another person might choose to do a table for 1, 2, 4, 6, 8, T, and W. Remember that W is the highest digit in base thirteen. Another student might make a full table for all of the digits in base thirteen, but that's a lot of prep work that usually doesn't pay off.



$1046_{\text{thirteen}} \div T2_{\text{thirteen}}$

1 → T2
+T2
2 → 174
+174
4 → 318
+318
8 → 633

$\boxed{14_{\text{thirteen}} \text{ r. } E_{\text{thirteen}}}$

T2 | 1046
-T20 | 10
326
-318 | 4
E | 14

Check:
$T2_{\text{thirteen}}$
x 14_{thirteen}
318
T2
1038
+ E_{thirteen}
1046_{thirteen}

Make sure you practice the previous four examples before doing the exercises that follow on the next pages.

For each of the division problems in the remainder of this section, show your work, using a model similar to the last four examples. First make up a partial or full multiplication table (prep for division using repeated subtraction). Then, divide using the scaffold method with the repeated subtraction method. Write your answer in a box above the scaffold. Then, check your answer by multiplying the divisor by the quotient and adding the remainder. Make sure you pay close attention to what base you are working in as you add, subtract and/or multiply in that base. Remember to write the base when you write the quotient and remainder.

? Exercise 4

$222_{\text{three}} \div 12_{\text{three}}$

? Exercise 5

$1204_{\text{five}} \div 42_{\text{five}}$

? Exercise 6

$323_{\text{four}} \div 23_{\text{four}}$

? Exercise 7

$1345_{\text{six}} \div 25_{\text{six}}$

? Exercise 8

$11111_{\text{two}} \div 101_{\text{two}}$

? Exercise 9

$1046_{\text{seven}} \div 31_{\text{seven}}$

? Exercise 10

$10021_{\text{three}} \div 22_{\text{three}}$

? Exercise 11

$200_{\text{five}} \div 3_{\text{five}}$

? Exercise 12

$1001011_{\text{two}} \div 1010_{\text{two}}$

? Exercise 13

$2155_{\text{eight}} \div 27_{\text{eight}}$

? Exercise 14

$1046_{\text{thirteen}} \div 14_{\text{thirteen}}$

? Exercise 15

$101\mathrm{E}_{\text{twelve}} \div 54_{\text{twelve}}$

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7.4: Homework

- **Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)**
- **Start a new module on the front side of a new page and write the module number on the top center of the page.**
- **Answers without supporting work will receive no credit.**
- **Some solutions are given in the solutions manual.**
- **You may work with classmates but do your own work.**

? HW #1

Mae had 72 lollipops and wanted to sell packages that had 4 lollipops in each package. How many packages could she make?

a. What division problem do you need to do to find the answer?

b. What interpretation of division does this word problem represent?

c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.

e. Write the multiplication problem shown by the picture:

f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem?

? HW #2

Tina had 72 marbles and wanted to divide them up evenly amongst the 9 children in her class. How many marbles should each child receive?

a. What division problem do you need to do to find the answer? :

b. What interpretation of division does this word problem represent?

c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.

e. Write the multiplication problem shown by the picture:

f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem?

? HW #3

Joan had 12 cups of sugar. A cookie recipe required 2/3 of a cup of sugar. How many recipes could she make?

a. What division problem do you need to do to find the answer? :

b. What interpretation of division does this word problem represent?

c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.

e. Write the multiplication problem shown by the picture:

f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem?

? HW #4

a. Make up a word problem that would require using the partitioning of subsets interpretation of division and the division problem: $20 \div 5$.

b. Draw a picture using this interpretation to find the answer.

c. Explain how you use the picture to find the answer to the problem. Be specific.

d. Write the multiplication problem shown by the picture.

e. What is the answer to this problem?

? HW #5

a. Make up a word problem that would require using the repeated subtraction interpretation of division and the division problem: $20 \div 5$.

b. Draw a picture using this interpretation to find the answer.

c. Explain how you use the picture to find the answer to the problem. Be specific.

d. Write the multiplication problem shown by the picture.

e. What is the answer to this problem?

? HW #6

a. Make up a word problem that would require using the repeated subtraction interpretation of division and the division problem: $6 \div 2/3$.

b. Draw a picture using this interpretation to find the answer.

c. Explain how you use the picture to find the answer to the problem. Be specific.

d. Write the multiplication problem shown by the picture.

e. What is the answer to this problem? ______

? HW #7

Draw a picture to do the division $30 \div 2$ using

a. partitioning into subsets

b. repeated subtraction.

c. Which interpretation is easiest to draw and why?

? HW #8

Draw a picture to do the division $32 \div 16$ using

a. partitioning into subsets

b. repeated subtraction.

c. Which interpretation is easiest to draw and why?

? HW #9

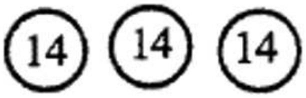
Fill in the blanks: $637 \div 14$ = 45 r.7 means _____ $\cdots$ _____ + _____ = ________

HW #10

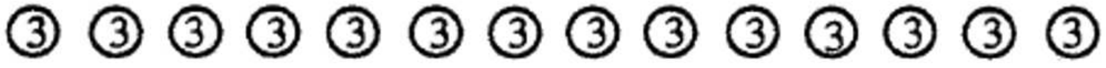
Someone was solving this problem: “Mark had 42 pencils and wanted to divide them up equally amongst his 3 children. How many pencils should he give each child?”

Which picture best represents a picture he might draw to find the solution?

Picture 1



Picture 2



HW #11

To use the division interpretation of “partitioning into subsets”

a. would the number you are dividing by be the number of subsets or the amount in each subset?

b. After doing the division, would the answer be represented by the number of subsets or the amount in each subset?

HW #12

To use the division interpretation of “repeated subtraction”,

a. would the number you are dividing by be the number of subsets or the amount in each subset?

b. After doing the division, would the answer be represented by the number of subsets or the amount in each subset?

For problems 13 – 17

a. Divide in the base given using repeated subtraction. All of your work must be shown. **Show ALL steps**.

b. Check each answer by multiplying the divisor by the quotient and then adding the remainder to see if this equals the dividend. **All of your work must be shown. Do not skip any steps.**

HW #13

$4320_{\text{seven}} \div 52_{\text{seven}}$

HW #14

$2316_{\text{nine}} \div 7_{\text{nine}}$

HW #15

$60382_{\text{eleven}} \div 21T_{\text{eleven}}$

HW #16

$43401_{\text{five}} \div 444_{\text{five}} =$ ________

HW #17

$2020100_{\text{three}} \div 220_{\text{three}} =$ ________

? HW #18

Decide which of the following sets are closed under division. If it is closed under division, write "closed" and provide at least one example and compelling reason why you are sure it is closed. If it is not closed, write "not closed" and provide a counterexample.

a. {-1}	b. {0}	c. {0, 1}
d. {-1, 1}	e. {1, 2, 3, 4, ...}	

? HW #19

Is division commutative?

Support your answer with either an example or a counterexample.

? HW #20

Is division associative?

Support your answer with either an example or a counterexample

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CHAPTER OVERVIEW

8: Number Theory

Many students are often amazed at the many formulas and discoveries in mathematics. They ask "Did someone just sit around and think that up?"

They often think mathematics is just a bunch of facts – that it only has to do with addition, subtraction, multiplication and division of numbers. In other words, they think it's all about arithmetic and being good at doing arithmetic in your head.

On the contrary, mathematics is about ideas. It is especially for those who are interested in asking deep questions, and working to find the answers. It is about thinking creatively, about trying things nobody has done before, about using your imagination.

There is a story about a mathematician, David Hilbert, who noticed that one of his students stopped attending class. When he was told that the student had decided to drop mathematics to become a poet, Hilbert replied, "Good – he did not have enough imagination to become a mathematician."

8.1: Digital Roots and Divisibility
8.2: Primes and GCF
8.3: LCM and other Topics
8.4: Homework

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8.1: Digital Roots and Divisibility

It's helpful to understand what is meant by the **DIGITAL ROOT** of a number because they are used in divisibility tests, and are also used for checking arithmetic problem. A **DIGITAL ROOT** of a number is *one* of these digits: **0, 1, 2, 3, 4, 5, 6, 7 or 8**.

Definition: The DIGITAL ROOT of a number is the remainder obtained when a number is divided by 9.

Exercise 1

Divide each of the following numbers by 9. Then, write the remainder.

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74
d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

Below is another easier way to find the **digital root** of a number.

Step 1: Add the individual digits of the number.

Step 2: If there is more than one digit after adding all the digits, repeat this process until you get a single digit. If the final sum is 9, write 0, because 9 and 0 are equivalent in digital roots (since the remainder is a number smaller than 9). That is why the digital root of a number is only one of these digits: 0, 1, 2, 3, 4, 5, 6, 7 or 8. Those are the only possible remainders that can be obtained when a number is divided by 9! It can't be 9. *The single digit you finally end up with is the **DIGITAL ROOT** of the number*.

Examples: Find the digital roots of the following numbers.

Example 1: 34

34: Add the digits: 3 + 4 = 7

*The digital root of 34 is **7***

Example 2: 321

321: Add the digits: 3 + 2 + 1 = 6

*The digital root of 321 is **6***

Example 3: 58

58: Add the digits: 5 + 8 = 13 Add the digits again: 1 + 3 = 4

*The digital root of 58 is **4***

Example 4: 97

97: Add the digits: 9 + 7 = 16 Add the digits again: 1 + 6 = 7

*The digital root of 97 is **7***

✓ Example 5: 72

72: Add the digits: 7 + 2 = 9 In digital roots, 9 is the same as 0

*The digital root of 72 is **0***

✓ Example 6: 346,721

346,721: Add the digits, we get 3 + 4 + 6 + 7 + 2 + 1= 23
Add the digits again: 2 + 3 = 5

*The digital root of 346,725 is **5***

Note that you **should not write 97 = 16 = 7.** 97 **IS NOT EQUAL** to 7!! The *digital root* of 97 is equal to 7. I use dashes, colons or arrows to record the digital root, so it looks something like this: 346,721 → 23 → 5. **Don't use equal signs!!!**

Another method for finding digital roots uses the fact that 9 and 0 are equivalent in digital roots. The process of finding digital roots is also called **Casting out Nines**. When you add the digits, you don't have to add the digit 9 or any combination of numbers that add up to 9 (like 2 and 7, or 5 and 4, or 2 and 3 and 4, etc.) – you can "cast out" all 9's. Cross them off; then add the remaining digits together. The following example illustrates how "casting out nines" simplifies the process of finding the digital root of a large number.

✓ Example

Find the digital root of 5,624,398.

Without casting out nines: 5 + 6 + 2 + 4 + 3 + 9 + 8 = 37. Add again: 3 + 7 = 10. Add again: 1 + 0 = **1**.

Casting out Nines: Since 5 and 4 add up to 9, cross them off: $\cancel{5}, 6, \cancel{2}4, \cancel{3}98$. Since 6 and 3 add up to 9, also cross them off: $\cancel{5}, \cancel{6}, \cancel{2}4, \cancel{3}98$. Also cross off the 9: $\cancel{5}, \cancel{6}, \cancel{2}4, \cancel{3}98$. The only digits to add are the 2 and 8, which is 10. The digital root of 10 is 1. So **1** is the digital root of 5,624,398 which is the same answer obtained without first casting out nines.

Below, the digital roots for examples 4, 5 and 6 on the previous page is computed again using casting out nines. Note that the digital root remains the same.

✓ Example 4: 97

97: Cross off the 9. Only the 7 remains. The digital root is **7**.

✓ Example 5: 72

72: 7 + 2 = 9, so cross them off. Therefore, the digital root is **0**.

✓ Example 6: 346,721

346,721: Cross off the 3 and 6, and also the 7 and 2. The only digits to add are the 4 and 1. Therefore, the digital root is **5**.

? Exercise 2

Find the digital roots of the following numbers, using either method. Remember that 9 is not a digital root. Write zero (0) if the sum is 9 (cast out the 9).

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74

d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

Exercise 3

Did you obtain the same answers for exercises 1 and 2?

Note: This method of finding the digital root only has to do with the number 9. You can't find the remainder of a number when dividing by 7 by adding the digits.

Using Digital Roots is to Check Addition and Multiplication Problems.

To check an addition problem, add the digital roots of the addends. Then, check to see if the digital root of that sum is the same as the digital root of the actual sum of the addends. This works whether there are only two addends, or several addends.

Below are some examples of how to check addition. The actual addition is shown to the left. Arrows are used to show the digital roots of the addends and sum. To check, the digital roots of the addends are added together, and then the digital root of that sum is computed. Compare it to the digital root of the actual sum. If they are equal, put a check to indicate the answer is ***probably*** correct. *NOTE: There is a* ***slight*** *chance that the digital roots match, but the answer is still not correct due to some other mistake, like transposing digits. For instance, in example 1 below, it's possible someone might write down 1153 for the answer. The digital roots in the check would match. The possibility of this happening is slight, so we generally assume the problem was done correctly if the digital roots check.* On the other hand, if the digital roots do not check, you know it is wrong.

Example 1

$$\begin{array}{r} 723 \to 3 \\ \underline{+412} \to 7 \\ 1135 \to 1 \end{array}$$

Ck: 3 + 7 = 10 → 1   Correct!

Explanation: To check, add the digital root of the addends (3 + 7 = 10); then find the digital root of 10 (1). Verify that this equals the digital root of the sum, 1135 (1). Since it does, the addition problem was probably done correctly.

Example 2

$$\begin{array}{r} 463 \to 4 \\ \underline{+529} \to 7 \\ 1630 \to 1 \end{array}$$

Ck: 4 + 7 = 11 → 2 Wrong!

Explanation: To check, add the digital root of the addends (4 + 7 = 11); then find the digital root of 11 (2). This should equal the digital root of the sum, 1630 (1). Since it doesn't , there is a mistake and the addition problem was done incorrectly.

Exercise 4

Someone did the following addition problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Note that the procedure is the same if there are more than two addends. Add the digital roots of all the addends. Show Work!

a. $\begin{array}{r} 4983 \\ \underline{+6829} \\ 10802 \end{array}$	b. $\begin{array}{r} 5567 \\ \underline{+4987} \\ 10554 \end{array}$	c. $\begin{array}{r} 3467 \\ \underline{+2541} \\ 5908 \end{array}$

d. $\begin{array}{r} 8972 \\ 8876 \\ \underline{+5873} \\ 23721 \end{array}$	e. $\begin{array}{r} 5873 \\ 3674 \\ \underline{+4763} \\ 13260 \end{array}$	f. $\begin{array}{r} 4789 \\ 9835 \\ \underline{+5301} \\ 19925 \end{array}$

For a multiplication problem, the check is similar, except to check, you **multiply** the digital roots of the numbers you are multiplying. To check addition, you ADD the digital roots of the addends and check against the digital root of the sum. To check multiplication, you MULTIPLY the digital root of the numbers being multiplied and check against the digital root of the product. Here are some examples that provide an answer someone might have written down after doing the multiplication problem on another piece of paper.

✓ Example 1

$$\begin{array}{r} 85 \rightarrow 13 \rightarrow 4 \\ \underline{\times 63} \rightarrow 0 \\ 5355 \rightarrow 18 \rightarrow 0 \end{array}$$

Ck: $4 \times 0 = 0$ Correct!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($4 \times 0 = 0$); then find the digital root of 0 (0). Verify that it equals the digital root of the product, 5355 (0). Since it does, the multiplication problem was probably done correctly.

✓ Example 2

$$\begin{array}{r} 43 \rightarrow 7 \\ \underline{\times 52} \rightarrow 7 \\ 2136 \rightarrow 3 \end{array}$$

Ck: $7 \times 7 = 49 \rightarrow 4$ Incorrect!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($7 \times 7 = 49$); then find the digital root of 49 (4). This should equal the digital root of the product, 2136 (3). Since it doesn't , there is a mistake. Therefore, the multiplication problem was done incorrectly.

? Exercise 5

Someone did the following multiplication problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 74 \\ \underline{\times 53} \\ 3922 \end{array}$	b. $\begin{array}{r} 29 \\ \underline{\times 36} \\ 944 \end{array}$	c. $\begin{array}{r} 68 \\ \underline{\times 28} \\ 1904 \end{array}$
d. $\begin{array}{r} 65 \\ \underline{\times 79} \\ 5125 \end{array}$	e. $\begin{array}{r} 81 \\ \underline{\times 55} \\ 4455 \end{array}$	f. $\begin{array}{r} 48 \\ \underline{\times 52} \\ 2596 \end{array}$
g. $\begin{array}{r} 569 \\ \underline{\times 61} \\ 34709 \end{array}$	h. $\begin{array}{r} 653 \\ \underline{\times 524} \\ 242172 \end{array}$	i. $\begin{array}{r} 2333 \\ \underline{\times 1203} \\ 2806599 \end{array}$

? Exercise 6

Add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ \underline{+5732} \end{array}$	b. $\begin{array}{r} 8308 \\ \underline{+956} \end{array}$

d. $\begin{array}{r} 6784 \\ \underline{+6835} \end{array}$	e. $\begin{array}{r} 9994 \\ \underline{+8721} \end{array}$
f. $\begin{array}{r} 57 \\ \underline{\times 8} \end{array}$	g. $\begin{array}{r} 34 \\ \underline{\times 7} \end{array}$
g. $\begin{array}{r} 87 \\ \underline{\times 52} \end{array}$	h. $\begin{array}{r} 825 \\ \underline{\times 13} \end{array}$

Using Digital Roots is to Check Subtraction Problems.

To check subtraction, you usually add the difference (the answer) to the subtrahend (the number that was subtracted) and see if you get the minuend (the number you subtracted from). In other words, to check that 215 – 134 = 81 was done correctly, simply add 81 and 134, which is 215. To check subtraction using digital roots, simply add the *digital root* of the difference to the *digital root* of the subtrahend and see if you get the *digital root* of the minuend. To check that 215 – 134 = 81 was done correctly, simply add the digital roots 81 and 134: 0 + 8 = 8. Verify that 8 is the digital root of the minuend, 215. It is! Here is another example. Let's say someone did the following subtraction problem: 5462 – 2873 = 2589. To check, add the digital roots of 2589 and 2873, or 6 + 2 to get 8. Since 8 is also the digital root of 5462, the answer is probably correct.

Here are some more examples of how to check subtraction using digital roots.

✓ Example 1

$$\begin{array}{r} 7362 \rightarrow 0 \\ \underline{-5732} \rightarrow 8 \\ 1630 \rightarrow 1 \end{array}$$

Ck: 1 + 8 = 9 → 0 √ Correct!

Explanation: To check, add the digital root of the difference and subtrahend (1 + 8 = 9). The digital root of 9 is 0. Verify that this equals the digital root of the minuend, 7632 (0). Since it does, the subtraction was probably done correctly.

✓ Example 2

$$\begin{array}{l} 5073 \rightarrow 6 \\ \underline{-878} \rightarrow 5 \\ 4205 \rightarrow 2 \end{array}$$

Ck: 2 + 5 = 7 WRONG!

Explanation: To check, add the digital root of the difference and subtrahend (2 + 5 = 7). This should equal the digital root of the minuend, 5073 (6). Since it doesn't, the subtraction was not done correctly.

? Exercise 7

Someone did the following subtractions problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 572 \\ \underline{-356} \\ 216 \end{array}$	b. $\begin{array}{r} 296 \\ \underline{-189} \\ 107 \end{array}$	c. $\begin{array}{r} 501 \\ \underline{-327} \\ 284 \end{array}$
d. $\begin{array}{r} 3217 \\ \underline{-2391} \\ 926 \end{array}$	e. $\begin{array}{r} 3546 \\ \underline{-1138} \\ 2408 \end{array}$	f. $\begin{array}{r} 7502 \\ \underline{-4729} \\ 2773 \end{array}$

Exercise 8

Subtract, add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r}7362\\ \underline{-5732}\end{array}$	b. $\begin{array}{r}8308\\ \underline{-956}\end{array}$
d. $\begin{array}{r}6784\\ \underline{-6335}\end{array}$	e. $\begin{array}{r}9994\\ \underline{-8721}\end{array}$
f. $\begin{array}{r}557\\ \underline{+348}\end{array}$	g. $\begin{array}{r}834\\ \underline{+767}\end{array}$
h. $\begin{array}{r}48\\ \underline{\times 6}\end{array}$	i. $\begin{array}{r}13\\ \underline{\times 29}\end{array}$

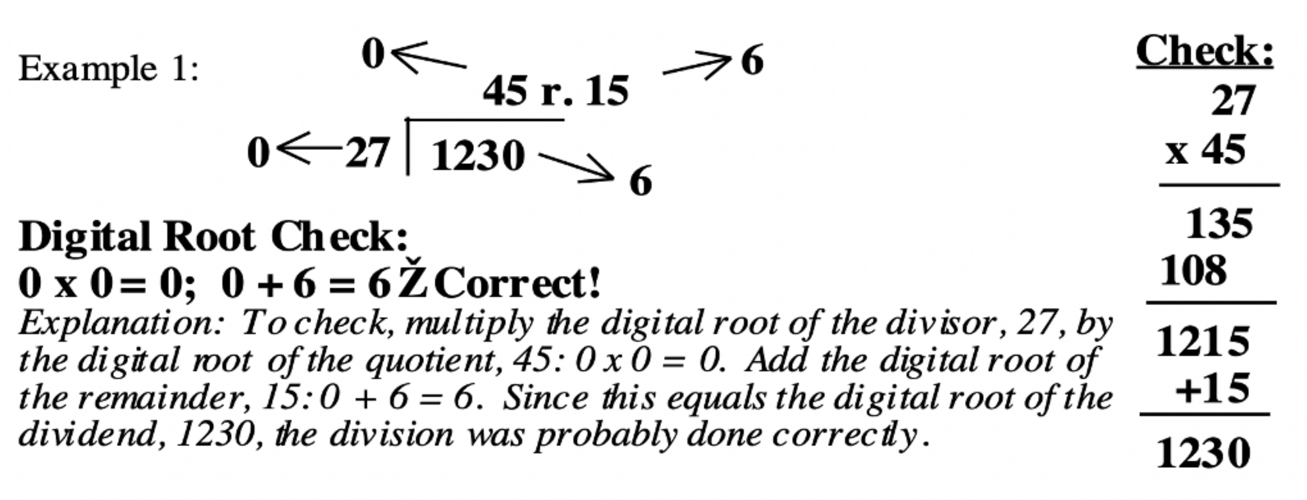
Using Digital Roots is to Check Division Problems.

To check division, you multiply the divisor (the number you are dividing by) by the quotient (the answer), and then add the remainder. It is correct if the answer obtained is equal to the dividend (the number you divided into). In other words, here is how to check this division, $431 \div 34 = 12$r. 23: Multiply 34 by 12 and add 23. Here are the steps: $34 \times 12 + 23 = 408 + 23 = 431$. Since 431 is the dividend, this problem was done correctly. To check division using digital roots, simply multiply the *digital root* of the divisor by the *digital root* of the quotient, and add the *digital root* of the remainder. If the digital root of that number equals the *digital root* of the dividend, it is probably correct.

Note: After the digital roots of the divisor and quotient are multiplied, you can first find the digital root of that product before adding the digital root of the remainder.

Here is how to check the division problem we did above by using digital roots. Multiply the digital root of the divisor, 34, by the digital root of the quotient, 12: $7 \times 3 = 21$. Then add the digital root of the remainder, 23: 21 + 5 = 26. Determine the digital root of that number, 26, which is 8. Check to see if 8 matches the digital root of the dividend, 431. It does. As stated in the note, you could determine the digital root of 21 before adding the digital root of the remainder. In this case, it would have looked like this: $7 \times 3 = 21 \rightarrow 3$. Then add the digital root of the remainder, 23: 3 + 5 = 8. Verify that this equals the digital root of the dividend, 431. It does. Therefore, the division was probably done correctly. More examples follow. The check without digital roots is shown to the right. The digital root check is shown under the division problem.

Example 1



Example 1:

$$\begin{array}{rl} 0 \leftarrow & 45 \text{ r. } 15 \rightarrow 6 \\ 0 \leftarrow 27 & \overline{)\,1230} \rightarrow 6 \end{array}$$

Digital Root Check:

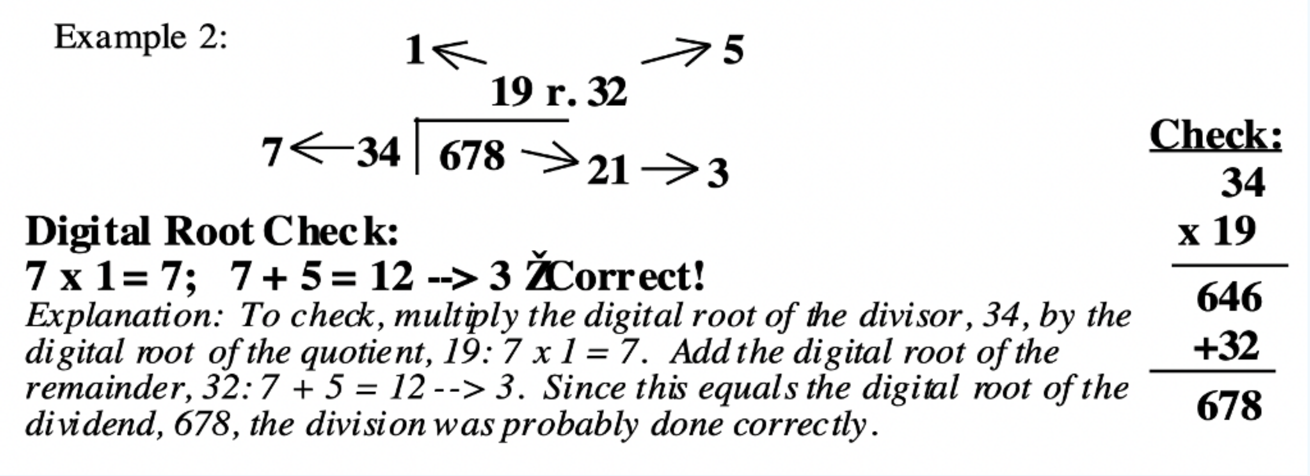
0 x 0= 0; 0 + 6 = 6 Ž Correct!

Explanation: To check, multiply the digital root of the divisor, 27, by the digital root of the quotient, 45: 0 x 0 = 0. Add the digital root of the remainder, 15: 0 + 6 = 6. Since this equals the digital root of the dividend, 1230, the division was probably done correctly.

Check:

$$\begin{array}{r} 27 \\ \underline{\times\ 45} \\ 135 \\ \underline{108} \\ 1215 \\ \underline{+15} \\ 1230 \end{array}$$

Example 2



Example 2:

1 ← 19 r. 32 → 5

7 ← 34 | 678 → 21 → 3

Digital Root Check:

7 x 1 = 7; 7 + 5 = 12 --> 3 ŽCorrect!

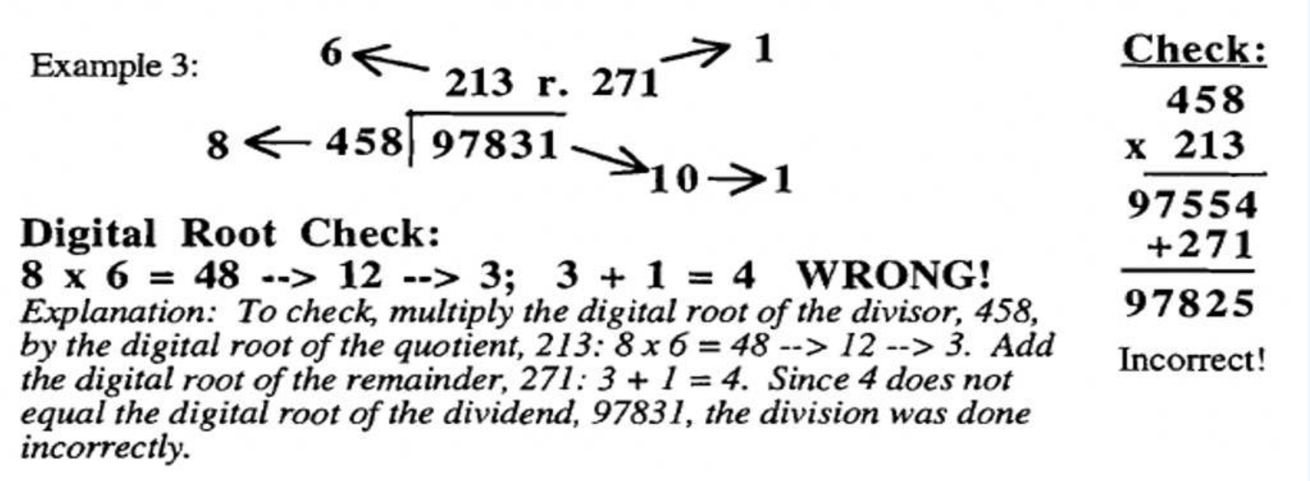
Explanation: To check, multiply the digital root of the divisor, 34, by the digital root of the quotient, 19: 7 x 1 = 7. Add the digital root of the remainder, 32: 7 + 5 = 12 --> 3. Since this equals the digital root of the dividend, 678, the division was probably done correctly.

Check:

```
  34
x 19
 646
+32
 678
```

Below is one more example for you to study.

Example 3



Example 3:

6 ← 213 r. 271 → 1

8 ← 458 | 97831 → 10 → 1

Digital Root Check:

8 x 6 = 48 --> 12 --> 3; 3 + 1 = 4 WRONG!

Explanation: To check, multiply the digital root of the divisor, 458, by the digital root of the quotient, 213: 8 x 6 = 48 --> 12 --> 3. Add the digital root of the remainder, 271: 3 + 1 = 4. Since 4 does not equal the digital root of the dividend, 97831, the division was done incorrectly.

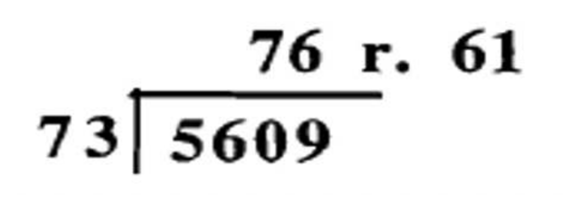
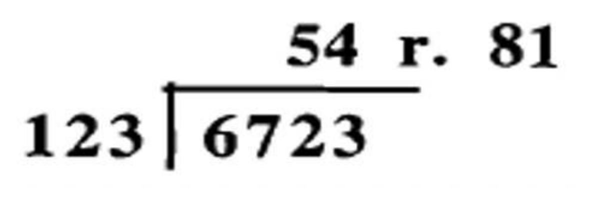
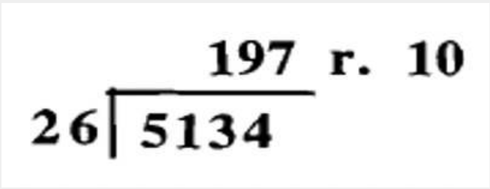
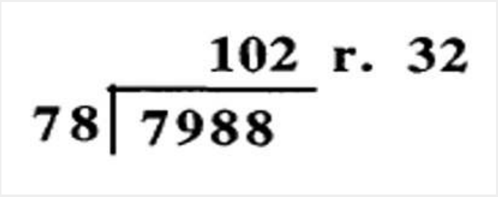
Check:

```
   458
x  213
 97554
  +271
 97825
```

Incorrect!

Exercise 9

Someone did the following division problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show work. You don't need to do the actual check as shown to the right of the last three examples. Show work!

a.  76 r. 61 73 | 5609	b.  54 r. 81 123 | 6723
c.  197 r. 10 26 | 5134	d.  102 r. 32 78 | 7988

One more note of caution about using digital roots:

In problem 9d, a common mistake some people make is to forget to put the zero in the quotient, 102. Instead, the answer might have been written as 12 r. 32. Using digital roots, the answer would check. So, you should also check the reasonableness of the answer by approximating. For instance, 78×12 is about 80 times 10 or 800, which is not even close to equaling the dividend, 7988. Another time digital roots might fail is when someone transposes the digits of a number. If the answer to a problem was 465, and it was written as 456, using digital roots wouldn't detect the mistake.

? Exercise 10

Do the following division problem, and check the answer using digital roots. Show work.

$23972 \div 156 =$ ______________

Later in this module, we will explore prime numbers, composite numbers, the greatest common factor of two or more numbers, and the least common multiple of two or more numbers. Factoring is the method used to find the prime factorization of a composite number, and it can also be used to find the greatest common factor and least common multiple of a set of numbers. One of the problems in factoring large numbers is that sometimes it isn't clear if it is prime or composite. In other words, it isn't clear whether it has any factors other than 1 or itself. Most of us know that if the last digit of a numeral is even, then 2 will divide into it; or if it ends in 0, 10 will divide into it; or if it ends in 0 or 5, that 5 will divide into it. Sometimes, people even have trouble determining if relatively small numbers are prime. For instance, many people think 91 is prime, but in fact it is not. Knowing some divisibility tests makes the task easier, so we'll soon take some time to discuss divisibility tests for several numbers. First, we need to go over some notation concerning divisibility.

When you see 12/3, this means 12 "**divided by**" 3. The slash that slants to the right is another way to write the division sign, $\div$. 12/3 (or $12 \div 3$) is a division problem, and the answer is 4.

Here is something altogether different. If I say "3 **divides** 12", I am making a *statement*. "3 divides 12" is not a division problem that needs to be done. It is a statement that happens to be true. The symbol used to represent the word "divides" is a vertical line. So, "3 **divides** 12" can be written "3|12". Again, this is a statement, a fact, not a division problem. The way to express "does not divide" is to put a slash through the symbol: $\nmid$

So, how do I know "3 ***divides*** 12" is a true statement? The definition of "divides" follows:

Definition: **"a divides b" if there exists a whole number, n, such that an = b.** This is simply saying that **"a divides b" means a is a factor of b!** In shorthand notation, this is written **"a|b if there exists a whole number, n, such that an = b, or a|b means that a is a factor of b."**

Okay, so what exactly does "there exists a whole number, n, such that an = b" mean? Well, it means that the first number times some whole number equals the second number. Or you can think "a divides b" is true if $b \div a$ is a whole number.

Let's get back to why "3 **divides** 12" is a true statement. Hmmm...you can be formal and ask yourself: Is there a whole number n such that 3n = 12? Or, you can simply ask yourself: Is 3 a factor (or divisor) of 12? In either case, the answer is yes, so the statement is true.

Let's work a little more on the difference between a statement using "divides", and an actual division problem.

Examples: Determine if each of the following is a statement or if it is a division problem. If it is a statement, state if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem.

✓ Example 1: 4|20

4|20

Solution

This is a true statement, because 4 is a factor of 20 (since $4 \cdot 5 = 20$)

✓ Example 2: 20/6

20/6

Solution

This is a division problem. The answer is 3 r. 2

✓ Example 3: 15|3

15|3

Solution

This is a false statement because 15 is not a factor of 3

✓ Example 4: 6 divides 20

6 divides 20

Solution

This is a false statement because 6 is not a factor of 20

✓ Example 5: 3 divides 21

3 divides 21

Solution

This is a true statement because 3 is a factor of 21 (since $3 \cdot 7 = 21$)

? Exercise 11

Determine if each of the following is a statement or if it is a division problem. If it is a statement, then decide if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem. Study the examples on the previous page if you need help getting started.

a. 35/7
b. 35|7
c. 7|35:
d. 40/7
e. 56|8
f. 7|40:
g. 12 divides 60
h. 80 divided by 30
i. 70 divided by 5
j. 42 divides 3
k. 6 divides 42
l. 80 divided by 10
m. 100/2
n. 4|100
o. 4|90:

p. 25|5

Let's go back to the formal definition of "divides":

Definition: **a|b if there exists a whole number, n, such that an = b.**

We need to use this formal notation in order to do some proofs. Consider the following:

✓ Example 1

Is the following statement true? If a|b and a|c, then a|(b+c).

a|b and a|c means that a is a factor of b and a is also a factor of c. We must determine if that necessarily implies that a is also a factor of the sum, b + c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample. If you find a counterexample, the answer is no and you are done. If you can't find a counterexample, maybe it is true. If you think it is true, you must PROVE it by being general and formal.

I would start by picking any number for a, like a = 3, and then choose numbers for b and c for which 3 is a factor, like 15 and 18. Plug them in: "If 3|15 and 3|18, does 3|(15+18)?" Since 3|33, the answer is yes. It looks like this statement might be true.

? Exercise 12

Choose different numbers for a, b and c in "If a|b and a|c, then a|(b+c)" to see if the statement seems to be true.

The statement, "If a|b and a|c, then a|(b+c)", happens to be true. You must write a proof to prove that it is always true. Here is one way to write a formal proof:

✓ Example 1

Is the following statement true?: If a|b and a|c, then a|(b+c).

Solution

If a|b, then an = b for some whole number, n. If a|c, then am = c for some whole number, m. Using these substitutions for b and c, we get that a|(b + c) is true if a|(an + am) which is true if a is a factor of an + am. Factor: an + am = a(n + m). This clearly shows that a is indeed a factor of an + am. Therefore, if a|b and a|c, then a|(b+c).

? Exercise 13

Write a formal proof to show that the following is true: If x|y and x|z, then x|(y + z).

✓ Example 2

Is the following statement true? If a|(b+c), then a|b and a|c.

a|(b + c) means that a is a factor of the sum, b + c. The question asks if that necessarily implies that a is a factor of b and also a factor of c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample, or if it looks like it is true. If you find a counterexample, the answer is no and you are done. If you think it is true, you must PROVE it by being general and formal.

Start by picking any number for a, like a = 3 and picking a number for the sum b + c, for which 3 is a factor, like 15 or 18, etc. Let b + c = 15. Note that there are many combinations of numbers that add up to 15: 1 + 14, 2 + 13, 3 + 12, 7 + 8, etc. You are being asked if 3 divides into the sum of any two whole numbers, will it necessarily divide into each individual addend as well? For instance, if you broke 15 up as the sum of 9 and 6, this would be the statement: "If 3|(9 + 6), then 3|9 and 3|6." Using these numbers, it is true and you haven't found a counterexample. Try splitting up 15 another way, perhaps as the sum of 10 and 5. Then, the statement becomes: "If 3|(10 + 5), then 3|10 and 3|5." The answer is no, and therefore these numbers may be used as a counterexample, which proves the statement is false. Remember: In order for a statement to be true, it must be for all values of a and b. On the other hand, only one counterexample showing it is not true is sufficient to prove a statement is false.

? Exercise 14

Consider the statement: "If a|(b+c), then a|b and a|c." Provide an example using numbers other than a = 3, b = 9 and c = 6 that makes it look like this statement might be true.

? Exercise 15

Consider the statement: "If a|(b+c), then a|b and a|c." Provide a counterexample using numbers other than a = 3, b = 10 and c = 5 to show this statement is false.

? Exercise 16

Consider the statement: "If a|c and b|c, then (a + b)|c." Provide a counterexample to show this statement is false.

? Exercise 17

Consider the statement: "If (a + b)|c, then a|c and b|c." Provide a counterexample to show this statement is false.

? Exercise 18

Prove that the following statement is true: "If a|b and a|c, then a|(bc)"

✓ Example

Prove that the following statement is true: "If a|b and a|(b + c), then a|c"

Solution

If a|b, then am = b for some whole number, m. If a|(b + c), then an = b + c for some whole number, n. Keep in mind that since b and c are positive, then (b + c) > b, which means n > m. We are trying to prove that a is a factor of c. Since am = b, we can substitute am for b into the equation an = b + c, which means an = am + c. Solving for c, this is equivalent to an – am = c. So if a is a factor of an – am, then a is a factor of c. Factor: an – am = a(n – m). This clearly shows a is a factor of an – am, which means a is a factor of c. Therefore, the statement "If a|b and a|(b + c), then a|c" is true. (Note: n – m must be a whole number since n and m are whole numbers and n > m.)

? Exercise 19

Prove that the following statement is true: "If c|a and c|(a + b), then c|b"

Okay, finally on to the divisibility tests. We will use our new notation for divides (|).

All counting numbers can be considered factors of zero. In other words, for all counting numbers, m, m|0 is always true since there is always some number times m that equals zero, namely zero itself.

Assume n is a positive whole number. The following are divisibility tests to determine what numbers divide into n, or what numbers are factors of n.

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

✓ Examples

Use the divisibility test for 2 to determine if the following is true or false

1. 2|97

This is false because the last digit (7) is not even

2. 2|356

This is true because the last digit (6) is even

Divisibility Test for 4: (think of 4 as 2^2)

4|n if 4|(the number represented by the last two digits of n)

✓ Examples

Use the divisibility test for 4 to determine if the following is true or false.

1. 4|527

This is false because 4 does not divide 27, (since 4 is not a factor of 27).

2. 4|25,356

This is true because 4|56.

3. 4|624

This is true because 4|24.

Divisibility Test for 8: (think of 8 as 2^3)

8|n if 8|(the number represented by the last three digits of n)

✓ Examples

Use the divisibility test for 8 to determine if the following is true or false.

1. 8|42,527

This is false because 8 does not divide 527

2. 8|25,336

This is true because 8|336

3. 8|7,624

This is true because 8 does divide 624, (since 8 is a factor of 624)

It should be clear to you that if a number is not divisible by 2, it is not divisible by 4 or 8; and if it is not divisible by 4, it is not divisible by 8. Conversely, if a number is divisible by 8, then it is divisible by both 2 and 4; and if it is divisible by 4, it is divisible by 2.

? Exercise 20

Use the divisibility tests for 2, 4 and 8 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 2|9,712 _____

b. 2|5,643 _____

c. 4|5,690 _____

d. 4|63,868 _____

e. 4|854,100 _____

f. 8|12,345,248 _____

g. 8| 54,094,422 _____

cc by nc

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

Examples

Use the divisibility test for 5 to determine if the following is true or false.

1. 5|527

This is false because 7 is the last digit of 527

2. 5|25,335

This is true because 5 is the last digit of 25,335

3. 5|7,620

This is true because 0 is the last digit of 7,620

Divisibility Test for 10: 10|n if the last digit of n is 0

Examples

Use the divisibility test for 10 to determine if the following is true or false.

1. 10|527

This is false because 7 is the last digit of 527

2. 10|25,335

This is false because 5 is the last digit of 25,335

3. 10|7,620

This is true because 0 is the last digit of 7,620

Exercise 21

Use the divisibility tests for 5 and 10 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 5|9,750 _____
b. 5|5,645 _____
c. 5|5,696 _____
d. 10|63,860 _____
e. 10|854,105 _____

Divisibility Test for 3: 3|n if 3|(the digital root of n)

Note: This is equivalent to saying 3|n if the digital root of n is 0, 3 or 6

Examples

Use the divisibility test for 3 to determine if the following is true or false.

1. 3|97

This is false because 3 does not divide 7, which is the digital root of 97

2. 3|356

This is false because 3 does not divide 5, which is the digital root of 356

3. 3|738

This is true because 3|0, where 0 is the digital root of 738

Exercise 22

Use the divisibility test for 3 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 3. Show work.

a. 3|9,750 ______

b. 3|5,645 ______

c. 3|5,696 ______

d. 3|63,860 ______

e. 3|854,115 ______

Divisibility Test for 9: 9|n if 9|(the digital root of n)

Note: **This is equivalent to saying that the digital root of n must equal zero.** Take a moment to think about this divisibility test for 9. In exercise 16, we discovered that the digital root of a number is the same as the remainder you obtain when you divide a number by 9. In order for a number to be divisible by 9, the remainder would be zero, which is exactly what the digital root would be.

Examples

Use the divisibility test for 9 to determine if the following is true or false.

1. 9|627

This is false because 9 does not divide 6, which is the digital root of 627.

2. 9|25,334

This is false since 9 does not divide 8, which is the digital root of 25,334.

3. 9|7,533

This is true because the digital root of 9 is 0.

Exercise 23

Use the divisibility test for 9 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 9. Show work.

a. 9|9,753 ______

b. 9|5,646 ______

c. 9|5,697 ______

d. 9|63,576 ______

e. 9|854,103 ______

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Important Note: A number is divisible by 6 only if it passes the divisibility test for **both** 2 and 3. If it doesn't pass one of the tests, then it is not divisible by 6. To show it is divisible by 6, you must show it passes both of the tests.

✓ Examples

Use the divisibility test for 6 to determine if the following is true or false.

1. 6|627

This is false because 2 does not divide 7 (the last digit, which is not even).

2. 6|25,334

This is false because 3 does not divide 25,334, since 3 does not divide the digital root, which is 8.

3. 6|7,620

This is true because 2|7,620 (since the last digit, 0, is even) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

? Exercise 24

Use the divisibility test for 6 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 6.

a. 6|9,753
b. 6|5,645
c. 6|5,696
d. 6|63,876
e. 6|854,103

Divisibility Test for 15: 15|n if 5|n AND 3|n. (Think of 15 as $5 \cdot 3$)

Important Note: A number is divisible by 15 only if it passes the divisibility test for both 5 and 3. If it doesn't pass one of the tests, then it is not divisible by 15. To show it is divisible by 15, you must show it passes both of the tests.

✓ Examples

Use the divisibility test for 15 to determine if the following is true or false.

1. 15|623: This is false because 5 does not divide 623 (since the last digit is not 0 or 5).

2. 15|24,335: This is false because 3 does not divide 24,335, since 3 does not divide the digital root, which is 8.

3. 15|7,620: This is true because 5|7,620 (since the last digit is 0) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

? Exercise 25

Use the divisibility test for 15 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 15.

a. 15|9,753
b. 15|6,645
c. 15|5,690
d. 15|63,872
e. 15|654,105

Divisibility Test for 7: This test isn't easy to describe. Below are the steps.

Step 1: Cross off the one's digit of the number to get a number with one less place value.

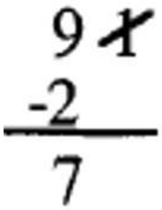
Step 2: Double the one's digit you crossed off and subtract from the new number obtained with the one's digit missing.

Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

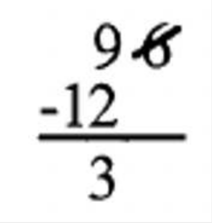
✓ Examples

Use the divisibility test for 7 to determine if the following is true or false. The way you would show the steps using the numbers is shown to the right of the explanation.

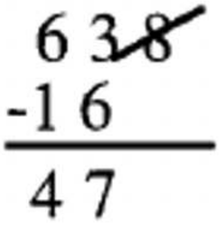
1. 7|91: Cross off the 1, double it (2), and subtract from what is left (9). The answer is 7 (9 – 2 = 7). Since 7|7, 7|91 is true.

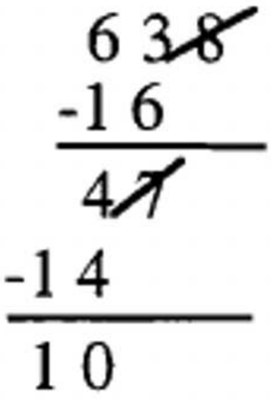


2. 7|96: Cross off the 6, double it (12), and subtract from what is left (9). When you subtract, ignore the sign (just do 12 – 9 = 3). Since 7 does not divide 3, 7|96 is false.

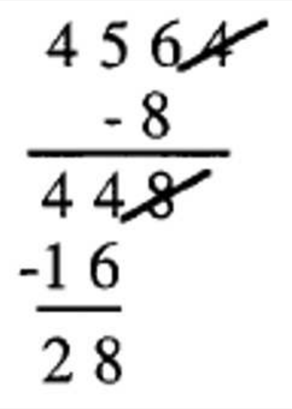


3. 7|638: Cross off the 8, double it (16) and subtract from what is left (63). The answer is 47 (63 – 16 = 47). Since 7 does not divide 47, then 7|638 is false. (If you weren't sure whether or not 7 divided 47, you can take it a step further; this is shown to the right.)

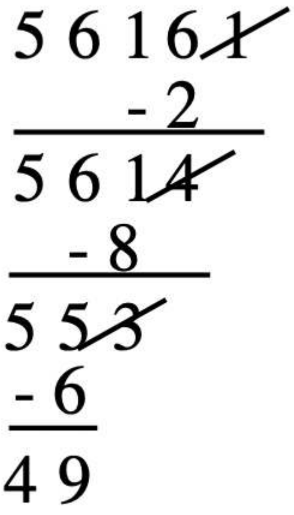




4. 7|4564: Cross off the 4, double it (8) and subtract from what is left (456). The answer is 448 (456 – 8 = 456). Cross off the 8, double it (16) and subtract from what is left (44). The answer is 28 (44 – 16 = 28). Since 7|28, then 7|4564 is true.



5. 7|56161: Cross off the 1, double it (2) and subtract from what is left (5616 – 2 = 5614). Cross off the 4, double it (8) and subtract from what is left (561 – 8 = 553). Cross off the 3, double it (6) and subtract from what is left (55 – 6 = 49). Since 7|49, then 7|56161 is true.



? Exercise 26

Use the divisibility test for 7 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 7. Show work.

a. 7|833

b. 7|5,645

c. 7|4,795

d. 7|14,763

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11.

This test is more confusing to describe than to do. What you do is add up every other digit. Then, add up the ones you skipped. Then, subtract these two numbers and see if 11 divides this number.

✓ Examples

Use the divisibility test for 11 to determine if the following is true or false.

a. 11|4,365: Add 4 + 6 = 10 Add 3 + 5 = 8 Subtract 10 – 8 = 2 Since 11 doesn't divide 2, then 11 doesn't divide 4365. Therefore, 11|4365 is false.

b. 11|540,879,216: Add 5 + 0 + 7 + 2 + 6 = 20. Add 4 + 8 + 9 + 1 = 22. Subtract 22 – 20 = 2 Since 11 doesn't divide 2, 11|540,879,216 is false.

c. 11|542,879,216: Add 5 + 2 + 7 + 2 + 6 = 22 Add 4 + 8 + 9 + 1 = 22 Subtract 22 – 22 = 0 Since 11|0, then 11|542,879,216 is true.

d. 11|4,052,631: Add 4 + 5 + 6 + 1 = 16 Add 0 + 2 + 3 = 5 Subtract 16 – 5 = 11 Since 11|11, then 11|4,052,631 is true.

? Exercise 27

Use the divisibility test for 11 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 11. Show work.

a. 11|9,053
b. 11|63,920,876
c. 11|568,696
d. 11|513,645
e. 11|803,003,808

Listed below are the divisibility tests that you should know. It's important to realize that each of the tests only work for the number specified. In other words, you can't use the divisibility test for 3 to determine is 7 divides a number. The divisibility test for 7 has nothing to do with digital roots!

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

Divisibility Test for 3: 3|n if 3|(the digital root of n); OR 3|n if the digital root of n is 0, 3 or 6

Divisibility Test for 4: 4|n if 4|(the number represented by the last two digits of n)

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Divisibility Test for 7: The steps of this test are described below.

- Step 1: Cross off the one's digit of the number to get a number with one less place value.
- Step 2: Double the one's digit you crossed off and subtract from the new number without the one's digit.
- Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

Divisibility Test for 8: 8|n if 8|(the number represented by the last three digits of n)

Divisibility Test for 9: 9|n if 9|(the digital root of n); OR 9|n if the digital root of n is zero

Divisibility Test for 10: 10|n if the last digit of n is 0

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11

Divisibility Test for 15: 15|n if 5|n AND 3|n

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8.2: Primes and GCF

You will need: ***Centimeter Strips*** *(Material Cards 16A-16L)*

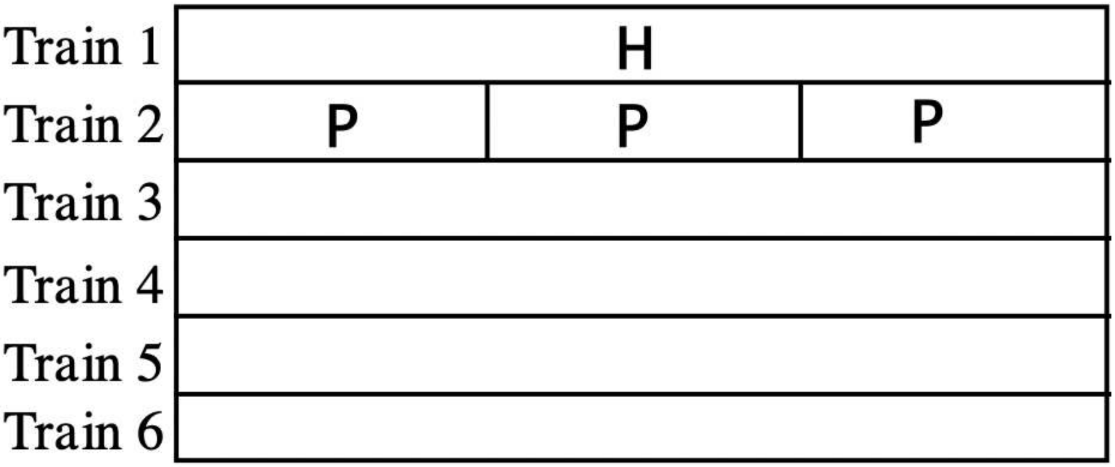
Prime Number Squares *(Material Cards 19A-19B)*

Composite Number Squares *(Material Cards 20A-20B)*

In these first few exercises, you'll be using C-strips to explore divisors, factors, prime numbers and composite numbers.

? Exercise 1

a. Take out the Hot Pink (H) C-strip. Use your C-strips (one color at a time) to see if you can form a train made up of C-strips of the same color equal in length to the hot pink C-strip in other words, see if you can do it with all whites (always possible for any train length), or all reds, or all light greens, etc. You should be able to make six different trains each train is made up of a single color. Draw a picture of each of these trains under the Hot Pink one shown. I've drawn two trains for you already one is simply the hot pink strip (a train consisting of just one C-strip), and a second is made up of three purple C-strips.



b. Take each train and form it into a rectangle. Then, find a C-strip that fits across the width of the rectangle, which is the top if the C-strips of the train made into a rectangle are placed vertically. From this, you should be able to tell from which multiplication problem each train was formed. From this information, write an equation using C-strips and then translate to an equation with numbers. For instance, for train 2, first I would make a rectangle out of the three purple C-strips. Next, I would try to find a C-strip to fit across the top, which would be light green. Therefore, train 2 was formed from the multiplication $L \times P$: remember that since the train is formed with purple C-strips, P is the *second* letter in the multiplication. So, the equation in C-strips is $L \times P = H$, and the numerical equivalent is $3 \times 4 = 12$. Follow this same procedure for the other four trains you made in part a.

Train 1 illustrates the multiplication $W \times H = H$, or $1 \times 12 = 12$. (Note that if the hot pink strip is placed vertically, the white C-strip fits across the top.)

Train 2 illustrates the multiplication $L \times P = H$, or $3 \times 4 = 12$.

Train 3 illustrates the multiplication ____ × ____ = H, or ____ × ____ = 12.

Train 4 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 5 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

Train 6 illustrates the multiplication ____ × ____ = H, or ____ × ____ = _____

In exercise part 1b, the set of numbers placed in the blanks before the equal sign in the equations with numbers are called **factors** or **divisors** of 12. Note, that because of the commutative property of multiplication, each factor or divisor was listed twice.

c. List the numbers that are factors of 12 (length of the hot pink C-strip). Only list each number once.

Exercise 2

Use the same procedure used in exercise 1 to make all possible rectangles from a train having each of the following lengths. Use the C-strip(s) shown in the parenthesis to make a train having the given length. Then, after discovering the possible rectangles that can be made, list the factors (the actual numbers) of each number. Every number greater than 1 has at least two factors.

a. Factors of 2 (red C-strip): ____
b. Factors of 3 (light green C-strip): ____
c. Factors of 4 (purple C-strip): ____
d. Factors of 5 (yellow C-strip): ____
e. Factors of 6 (dark green C-strip): ____
f. Factors of 7 (black C-strip): ____
g. Factors of 8 (brown C-strip): ____
h. Factors of 9 (blue C-strip): ____
i. Factors of 10 (orange C-strip): ____
j. Factors of 11 (silver C-strip): ____
k. Factors of 13 (brown + yellow): ____
l. Factors of 14 (orange + purple): ____
m. Factors of 15 (black + brown): ____
n. Factors of 16 (blue + black): ____

Exercise 3

a. List the numbers from exercise 2 that have **exactly** 2 factors: _____

b. What do you notice about the 2 factors each of these number have? Is there a pattern or anything they have in common with each other?

c. List numbers from exercise 2 that have an odd number (3 or 5) of factors: _____

d. What do all of these numbers have in common?

Definition: A whole number that has exactly two different factors is called a **prime** number.

If a number is prime, its only factors are 1 and itself. In other words, if a number, p, is prime, its only factors are 1 and p. If you are using the C-strips and try to make a rectangle out of a train that has a length that is a prime number, the only possibility is when the width is 1 and the length is the length of the original train. That is because those are the only factors, and no other number divides into it.

NOTE: **1 is NOT prime** since it doesn't have two different factors *it only has one 1*.

Definition: Any whole number larger than 1 that is not prime is called a **composite** number.

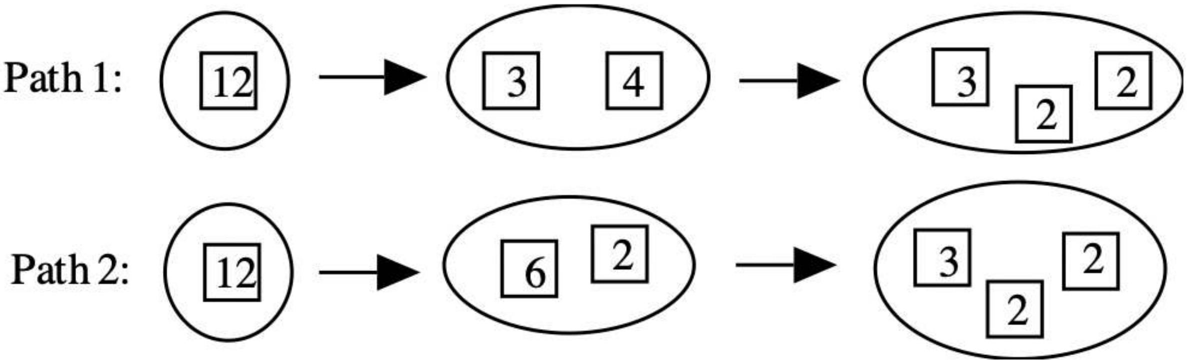
The whole numbers, 0 and 1, are neither prime nor composite. Any whole number larger than 1 is either prime or composite.

If a number is composite, it means that it has more than 2 factors, and can be written as a product of factors less than itself. For instance, 12 is not a prime number. It can be written as 2 · 6 or 3 · 4. This is called factoring, and $2 \cdots 6$ and $3 \cdots 4$ are only two ways of factoring 12.

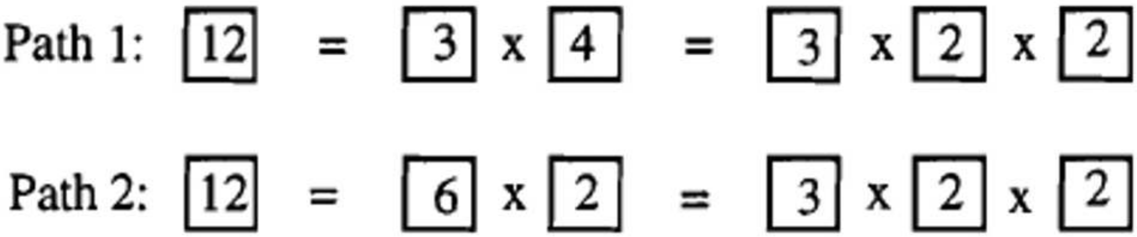
Get out your prime number squares and composite number squares. If you haven't already done so, color the prime number squares so that each prime number has a different color the 2's could be yellow, the 3's could be blue, the 5's could be green, the 7's could be purple, the 11's could be red, the 13's could be orange, the 17's could be pink, and so on. All of the composite number squares should remain white.

What we are going to do is factor numbers. Anytime we have a white number, we know it is composite and can be factored further. The ultimate goal will be to factor numbers into a product of primes, which means there will only be colored squares to represent each number.

Let's begin by factoring 12. Take out a white number square that says 12, and put it in front of you. Since it is white, it can be factored into a product of two smaller numbers. Someone might choose 3 and 4; another might choose 2 and 6. Replace 12 with the two squares you chose. For 12, it so happens that no matter which combination of two numbers you choose, one of the squares you choose will be colored, which means it is prime and can't be factored (or broken down) any further. But the other square will be white. Replace the white one with two other number squares by using smaller factors. In your pile, instead of a 12, you should have three squares, a 2, a 2 and a 3. There is no order it's just a group of three squares that if multiplied together represent 12. Below are two paths one might have taken to come up with the final solution. The arrows show one step after the other.



In the first path, the 12 was replaced with 3 and 4, and then the 4 was replaced with two 2's. In the second path, the 12 was replaced with a 6 and a 2, and then the 6 was replaced with a 2 and a 3. Using this model to prime factor, the numbers you end up should all be prime, and the understanding is that when the final (prime) factors are MULTIPLIED together, we obtain the original number we tried to factor. In other words, the prime factorization for 12 is $3 \cdot 2 \cdot 2$. Because of the commutative and associative properties of multiplication, the order of the factors is insignificant. Sometimes, for consistency, the factors are written in ascending or descending order, but this is not necessary unless you are instructed to write it in a particular order. If you are asked to write the prime factorization of 12, you might write $12=2 \cdot 2 \cdot 3$ (or $12=2^2 \cdot 3$). To check, multiply the prime factors to make sure that the product really is the number you set out to prime factor. For large numbers, you should use a calculator. Here is another way to show on paper the replacement process going on for factoring 12 using the number squares.



Example: Use the number squares to prime factor 210, and show the individual steps.

Note: Since there is not a number square for 210, write 210 on a blank square to begin.

Solution 1: $210=21 \cdots 10=3 \cdots 7 \cdots 2 \cdots 5$

Solution 2: $210=3 \cdots 70=2 \cdots 105=2 \cdots 3 \cdots 35=2 \cdots 3 \cdots 5 \cdots 7$

Solution 3: $210=30 \cdots 7=5 \cdots 6 \cdots 7=5 \cdots 2 \cdots 3 \cdots 7$

There are many other ways one might go about factoring 210, but in the end, there are 4 prime factors that when multiplied together equal 210. Because of the commutative and associative properties of multiplication, $3 \cdots 7 \cdots 2 \cdots 5 \ = 2 \cdots 3 \cdots 5 \cdots 7 = 5 \cdots 2 \cdots 3 \cdots 7$. Usually, the primes are written in ascending order (from the smallest factor to the largest factor. We write that the prime factorization of 210 is $2 \cdots 3 \cdots 5 \cdots 7$.

This leads to a very important theorem. You can think of the prime numbers as building blocks for all whole numbers greater than 1. Every whole number greater than 1 is prime, or it can be expressed as the product of prime factors (called the prime factorization). The fact that any composite number can be written as a **unique** product of primes is so important that it is called the **Fundamental Theorem of Arithmetic**.

The Fundamental Theorem of Arithmetic:

Every composite number has exactly one unique prime factorization (except for the order in which you write the factors.)

Note again that the order in which the factors are written doesn't matter. However, for consistency, they are usually written in ascending order (from smallest to largest). Also, exponents may be used if a factor is repeated in the prime factorization. For instance, the prime factorization of 12 is usually written as $2 \cdots 2 \cdots 3$ or $2^2 \cdots 3$

? Exercise 4

Use the number squares to prime factor each of the following composite numbers into a product of primes. Write the prime factorization so the factors are written in ascending order (from smallest to largest). None of these numbers is prime. **Show each of the individual steps one at a time, not just the final product.**

a. 45 = _____

b. 65 = _____

c. 200 = _____

d. 91 = _____

e. 76 = _____

f. 350 = _____

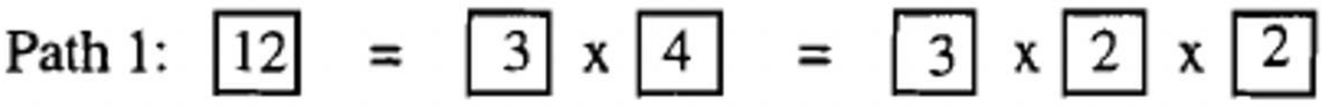
g. 189 = _____

h. 74 = _____

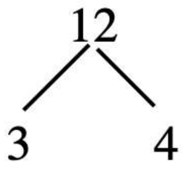
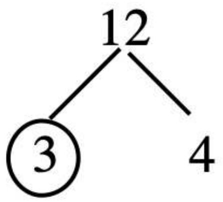
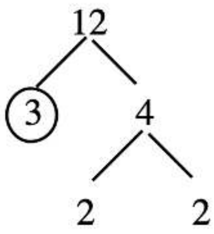
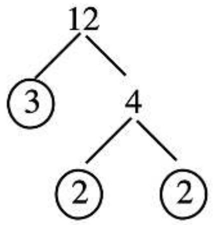
i. 512 = _____

j. 147 = _____

There are other methods commonly used to find the prime factorization. One uses a FACTOR TREE, which is similar to what you did with the prime and composite number squares. The difference is that it is done on paper, as opposed to using manipulatives. The number you are trying to factor is called the root, and is at the top. So, it's actually an upside down tree. If a number is not prime, you draw two branches down from that number and factor it as the product of any two factors. At the end of each branch is a smaller factor, which is called a leaf. If a leaf is prime, circle it, it is one of the factors in the prime factorization of the number. Otherwise, branch off again. When all of the leaves are circled prime numbers, you are done. The prime factorization of the root is the product of the leaves. Below is one way we factored 12, on the previous page, using squares.



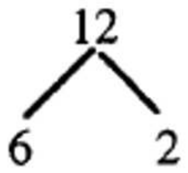
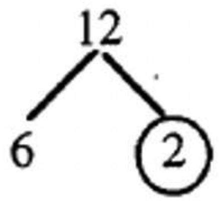
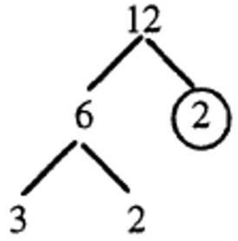
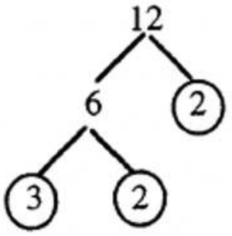
Here are the individual steps showing how one might use a factor tree to factor 12, similar to how it was factored using the squares, via Path 1. Note the similarity.

Step 1: Factor 12 as 3d4 	Step 2: Circle 3 since it is prime 	Step 3 Circle the 2s since they are prime 	Step 4: Circle the 2s since they are prime 	Step 5: The prime factorization of 12 is the product of the circled leaves 3 d2 d2

Below is the other way we factored 12 using squares.

Path 2: $\boxed{12} = \boxed{6} \text{ x } \boxed{2} = \boxed{3} \text{ x } \boxed{2} \text{ x } \boxed{2}$

Here are the individual steps showing how one might use a factor tree to factor 12, similar to how it was factored using the squares, via Path 2. Again, note the similarity.

Step 1: Facrtor 12 as 6×2 	Step 2: Circle 2 since it is prime 	Step 3: Factor 6 as 2×3 	Step 4: Circle 3 and 2 since they are both prime. 	Step 5: The prime factorization of 12 is the product of the circled leaves $3 \cdots 2 \cdots 2$

If you aren't sure whether a number is prime or composite and don't know how to start factoring, use the divisibility tests. See if it is possible to divide by 2, 3, 5, 7, 11, etc. Make **sure** there are no factors before you circle it and decide it's prime. You are going to repeat exercise 4 again, but this time, use a factor tree.

? Exercise 4

Use a factor tree to factor each of the following composite numbers into a product of primes. Write the prime factorization so the factors are written in ascending order (from smallest to largest). None of these numbers is prime. Show the individual steps. Show the factor tree under each problem.

a. 45 = _____	c. 200 = _____
b. 65 = _____	d. 91 = _____
e. 76 = _____	h. 74 = _____
f. 350 = _____	i. 512 = _____
g. 189 = _____	j. 147 = _____

The problem with trying to find the prime factorization is that sometimes it isn't obvious if a number you are trying to factor is prime or not. For instance, it is not immediately obvious whether or not 517 is prime or composite. This is where divisibility tests come in handy. Actually, if you want to find the prime factorization of 517, you only need to check if 517 has any prime numbers as factors. You'll need to try one prime at a time. Before going on, let's list the first several prime numbers starting with 2. Note: 2 is the only even prime number. To make a list, start with 2, 3, 5, and check to see if the next odd number is prime or not. It is not prime if one of the prime numbers listed earlier is a factor.

Exercise 5

List all of the prime numbers less than 100. You only need to use the divisibility tests for 2, 3, 5 and 7 at the most to check if any odd number less than 100 is prime. In other words, any composite number less than 100 has 2, 3, 5 or 7 as a factor. We'll discuss why after this exercise.

Consider the possible ways to factor 54 as a product of 2 factors:

$1 \cdots 54$, $2 \cdots 27$, $3 \cdots 18$, $6 \cdots 9$, $9 \cdots 6$, $18 \cdots 3$, $27 \cdots 2$, $54 \cdots 1$

Note that if you start with the smallest factor (1) as the left factor, you start repeating half-way through the list. This "half-way" mark happens after you get to the square root of the number you are factoring. The square root of 54 is between 7 and 8. So if you list a factor bigger than 7, it would have shown up earlier in the list as a factor less than 7. So that means if I am trying to find the prime factorization of 54, I only need to check prime numbers up to and including at most 7.

So why does any composite number less than 100 have 2, 3, 5 or 7 as a factor? The next prime number after 7 is 11. Since 11^2, or 121, is greater than 100, if there was a prime number greater than or equal to 11 that was a factor of a number less than 100, then the other factor would have to be less than 11. Think about it. If both factors were bigger than 11, the product would be more than 121! This realization makes finding the prime factorization of a number much easier. This is how it works.

Since $3^2 = 9$, any composite number < 9 has 2 as a factor.

Since $5^2 = 25$, any composite number < 25 has 2 or 3 as a factor.

Since $7^2 = 49$, any composite number < 49 has 2, 3 or 5 as a factor.

Since $11^2 = 121$, any composite number < 121 has 2, 3, 5 or 7 as a factor.

Since $13^2 = 169$, any composite number < 169 has 2, 3, 5, 7 or 11 as a factor.

Since $17^2 = 289$, any composite number < 289 has 2, 3, 5, 7, 11 or 13 as a factor.

Since $19^2 = 361$, any composite number < 361 has 2, 3, 5, 7, 11, 13 or 17 as a factor.

Since $23^2 = 529$, any composite number < 529 has 2, 3, 5, 7, 11, 13, 17 or 19 as a factor.

Now, we'll verify that some of the above statements are true.

Example

Since $3^2 = 9$, any composite number < 9 has 2 as a factor.

This can be verified by listing the composite numbers less than 9 and showing that 2 is a factor of each of those numbers: the only composite numbers less than 9 are 4, 6, and 8. Verify that 2 is a factor of each of these numbers: $4 = 2 \cdots 2, 6 = 2 \cdots 3, 8 = 2 \cdots 4$

Example

Since $5^2 = 25$, any composite number < 25 has 2 or 3 as a factor.

The previous example verified that the composite numbers less than 9 have 2 or 3 as a factor (since we've shown they all have 2 as a factor). So we only need to list and verify that the composite numbers less than 25 have 2 or 3 as a factor. The only composite numbers between 9 and 25 are 10, 12, 14, 15, 16, 18, 20, 21, 22, and 24. Mentally verify that 2 or 3 is a factor of each of these numbers. Note: each number only needs to have 2 OR 3 as a factor, although it may have both. For instance, 2 is a factor of 10, 3 is a factor of 15, and both 2 and 3 are factors of 18. You only have to make sure at least one of those factors is a factor of each composite number.

? Exercise 6

Since $7^2 = 49$, any composite number < 49 has 2, 3 or 5 as a factor. List the composite numbers between 25 and 49, and mentally verify that 2, 3, or 5 is a factor of each of these numbers.

? Exercise 7

Since $11^2 = 121$, any composite number < 121 has 2, 3, 5 or 7 as a factor. List the composite numbers between 49 and 121, and mentally verify that 2, 3, 5 or 7 is a factor of each of these numbers.

It gets tedious trying to verify for larger numbers, but the pattern continues. In other words, if you listed all of the composite numbers less than 361, which is 19^2, then you could actually verify that every one of them has 2, 3, 5, 7, 11 or 13 as a factor.

So, to find if 517 is prime or composite, since 517 is less than 23^2, I only need to check if any of the following are factors: 2, 3, 5, 7, 11, 13, 17 or 19. Instead of checking every number up to 517, only these eight need to be checked. You can use divisibility tests for the first five primes, then use a calculator for the last three. If you find a factor early on, there is no need to keep going.

? Exercise 8

Find the prime factorization of 517. Do this by using the divisibility tests up to 11, if necessary. If you still don't find a factor of 517, use a calculator to see if 13, 17 or 19 is a factor. Write the prime factorization here:

Because I wrote the squares of the first several prime numbers on the previous page, I knew I didn't have to check any primes higher than 19. One way to figure out the highest prime number you **might** have to check is to take the square root of the number you are trying to prime factor.

? Exercise 9

On your calculator, find the square root of 517 rounded to the nearest tenth:

You should have gotten 22.7. This tells us 517 is not a perfect square. Next, we know that if 517 is composite, it must have a prime factor less than 22.7. So, we need to determine the highest whole number that is prime that is less than 22.7. Start with 22 not prime; then 21 not prime; then 20 not prime; then 19 prime! Therefore, we at most need to check the primes up to 17: 2, 3, 5, 7, 11, 13, 17 and 19. When you actually prime factored 517, did you notice 11 was a factor? Did you use the divisibility test for 11?

So, $517 = 11 \cdots 47$. Then, you look at the other factor, 47, and note it is prime, so you are done! By the way, 5 is the highest prime you'd have to check to see if 47 is prime!

What we have just described is a theorem called the Prime Factor Test. This is the formal way of stating that theorem.

Prime Factor Test: To test for prime factors of a number n, one need only search for prime factors p of n, where $p^2 \leq n$ (or $p \leq \sqrt{n}$)

? Exercise 10

For each number, determine the highest prime that might need to be checked to find the prime factorization of the number. Then, find the prime factorization. If it is prime, simply write the number itself, since that ***is*** the prime factorization.

a. 149 highest prime to check: _____ Prime factorization for 149:
b. 273 highest prime to check: _____ Prime factorization for 149:
c. 381 highest prime to check: _____ Prime factorization for 149:

d. 437 highest prime to check: _____ Prime factorization for 149:
e. 509 highest prime to check: _____ Prime factorization for 149:
f. 613 highest prime to check: _____ Prime factorization for 149:
g. 787 highest prime to check: _____ Prime factorization for 149:

Now, let's look at a range of numbers and figure out how to determine which ones are prime. For instance, let's determine which numbers between 350 and 370 are prime. First of all, only odd numbers in this range are prime. So, begin by listing the odd numbers as possibilities: 351, 353, 355, 357, 359, 361, 363, 365, 367 and 369. Next, note 355 and 365 can't be prime since it is divisible by 5. Now, you might use the divisibility test for 3 to cross off 351, 357, 363 and 369 note you can cross of multiples of 3 by crossing off every third odd number if you start at a multiple of 3. Now, our list is down to these possibilities: 353, 359, 361, 367 and 369. The highest prime you'd have to check is the prime number that is less than the square root of 369, which is 19. So, simply check the rest of the primes (7, 11, 13, 17 and 19 at most) on each of these numbers to determine which, if any, are prime. 353 is prime; 359 is prime; 361 is 19, 367 is prime. Therefore, 353, 359 and 367 are the numbers between 350 and 370 that are prime. Furthermore, you should be able to write the prime factorization for all the numbers between 350 and 370 that are composite.

? Exercise 11

Find the prime factorization for all the numbers between 280 and 295. If a number is prime, simply write the number itself.

a. 280 = ___________________	i. 288 = ___________________
b. 281 = ___________________	j. 289 = ___________________
c. 282 = ___________________	k. 290 = ___________________
d. 283 = ___________________	l. 291 = ___________________
e. 284 = ___________________	m. 292 = ___________________
f. 285 = ___________________	n. 293 = ___________________
g. 286 = ___________________	o. 294 = ___________________
h. 287 = ___________________	p. 295 = ___________________

Twin primes are two consecutive odd numbers that are prime. For instance, 5 and 7 are twin primes, 11 and 13 are twin primes, 17 and 19 are twin primes. There is no pattern to determine how often twin primes come up. One unsolved question in mathematics is if there are a finite number of or infinitely many sets of twin primes. Nobody knows.

? Exercise 12

From your work in exercise 11, list any sets of twin primes: _____

? Exercise 13

Is it possible to have three odd numbers in a row that are prime? Why or why not?

One use of prime factoring a set of numbers is so you can find the **greatest common factor** (GCF) and **least common multiple** (LCM) of a set of numbers. Many people have trouble distinguishing between the greatest common factor and the least common multiple (LCM) because they don't think about what the words actually mean. The greatest common factor of a number is obviously a **factor**, but the adjective **common** describes that you want a factor that is common to all the numbers, and the adjective **greatest** describes that you want the very largest of the common factors of the numbers.

We are going to explore ways of finding the greatest common factor of two numbers, a and b. The notation to express this is GCF(a, b). It doesn't matter which number you list first in the parentheses – it's not an ordered pair. **The greatest common factor of a set of numbers is the largest number that is a factor of each number.**

We are going to explore different ways of finding the greatest common factor of two numbers.

First, we are going to explore one way to find the greatest common factor of 42 and 72.The notation to express this is GCF(42, 70). Remember: it doesn't matter which number you list first in the parentheses, *it's not an ordered pair*. GCF(42, 70) means the same thing as GCF(70, 42). **The greatest common factor of 42 and 70 is the largest number that is a factor of both 42 and 70.**

One way of doing this is to list every single factor of each number and then pick the biggest one that is a factor of each.

Exercise 14

a. List all of the factors of 42 in ascending order: ____

b. List all of the factors of 70 in ascending order: ____

c. List all of the factors that are common to both 42 and 70: ____

d. List the greatest common factor of 42 and 70: ____

e. Fill in the blank: GCF(42, 70) = ____

Listing all of the factors of a given number is sometimes a difficult task. For instance, it's easy to miss a factor. One remedy is to prime factor the number first. To list all of the factors of 42, one might first prime factor 42 like this: $2 \cdots 3 \cdots 7$. For a number to be a factor of 42, it must be composed of the prime factors listed. Of course, 1 is always a factor. Next, you'd check 2, and then 3, which are both factors. 4 is not a factor because if it were, $2 \cdots 2$ would be in the prime factorization! It's clear to see 5 is not a factor. 6 is a factor, since $2 \cdots 3$ is in the prime factorization. Continuing on, 7 is a factor, but 8 is not because $2 \cdots 2 \cdots 2$ is not in the prime factorization of 42. Neither is 9 $(3 \cdots 3), 10(2 \cdots 5), 11, 12(2 \cdots 2 \cdots 3)$, or 13. But 14 is a factor of 42 since $2 \cdots 7$ is in the prime factorization. You can use this strategy as you check every number up to 42, but that is still a lot of numbers to check. Eventually, you'd get this list: 1, 2, 3, 6, 7, 14, 21, 42.

Here's a way to shorten the process a little more. Starting with the smallest factor 1, immediately list the other factor you'd have to multiply by that factor to get 42. So, we start with 1, 42. We check the next number, 2, and note it is a factor. To get the other factor that pairs up with 2, either divide 2 into 42, or simply look at the prime factorization of 42, with the 2 missing. There is $3 \cdots 7$ left, which is the other factor. So the list is now 1, 42, 2, 21. Continuing, we note 3 is a factor. To get the other factor, either divide 42 by 3, or do it the easy way, which is to see what factors are left in the prime factorization of 42 with the 3 missing. Since there is a 2 and a 7, then the number that pairs up with 3 is 14.

The list is now 1, 42, 2, 21, 3, 14. Next, we note 4 and 5 are not factors. 6 is a factor since 2 and 3 $(2 \cdots 3)$ is in the prime factorization. 7 is the number that pairs up with 6. So, the list is now: 1, 42, 2, 21, 3, 14, 6, 7. If you were to continue, the next number to check would be 7. Since 7^2 > 42, you can stop. All the factors that are larger than 7 will already be in the list because there would have been a smaller factor that it paired up with already. Now, put the list of factors of 42 in ascending order: 1, 2, 3, 6, 7, 14, 21, 42.

The same procedure can be used to list all the factors of 70. First, write the prime factorization of 70: $2 \cdots 5 \cdots 7$. You would start off with 1 and 70: 1, 70. Next, it's clear 2 is a factor that pairs up with 35. The list is now: 1, 70, 2, 35. Next discard 3 and 4 as factors, and note 5 is a factor. The factors left are 2 and 7, which multiplied together is 14. So the list is 1, 70, 2, 35, 5, 14. Continuing on, note 6 is not a factor, and 7 is. 7 pairs up with 10. The list is now: 1, 70, 2, 35, 5, 14, 7, 10. Continuing on, note that 8 is not a factor. The next number to check would be 9. But 9^2 > 70, so all the factors higher are already in the list. Writing the list in ascending order, we get: 1, 2, 5, 7, 10, 14, 35, 70.

Make a note that in both of these examples, 42 and 70 each had exactly 3 prime numbers in the prime factorization. Consider the prime factorization of 220: $2 \cdots 2 \cdots 5 \cdots 11$. Note that as you list factors, there may be one, two, or three other factors to multiply together to get the pair. 2 is a factor; its pair is $2 \cdots 5 \cdots 11$, or 110. 4 $(2 \cdots 2)$ is a factor; its pair is 5 ··· 11, or 55. 5 is a factor; its pair is $2 \cdots 2 \cdots 11$, or 44. 10 $(2 \cdots 5)$ is a factor; its pair is $2 \cdot 11$, or 22. 11 is a factor; its pair is $2 \cdots 2 \cdots 5$, or 20. As I check numbers larger than 11, I stop at 15 since 152 > 220. So, the list is: 1, 220, 2, 110, 4, 55, 5, 44, 10, 22, 11, 20. Writing these in ascending order, we get: 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, 220.

By the way, it’s a lot more time-consuming for me to explain (and for you to read) how to list all the factors by prime factoring. But doing it is a lot easier! Now, do the next exercises. If you wish, use prime factorization to do parts a and b.

? Exercise 15

The goal of this problem is to find the greatest common factor of 92 and 115.

a. List all of the factors of 92 in ascending order:

b. List all of the factors of 115 in ascending order:

c. List all of the factors that are common to both 92 and 115:

d. List the **greatest** common factor of 92 and 115:

e. Fill in the blank: GCF(92, 115) =

? Exercise 16

The goal of this problem is to find the greatest common factor of 48, 54 and 63.

a. List all of the factors of 48 in ascending order:

b. List all of the factors of 54 in ascending order:

c. List all of the factors of 63 in ascending order:

d. List all of the factors that are common to 48, 54 and also 63:

e. List the **greatest** common factor of 48, 54 and 63:

f. Fill in the blank: GCF(48, 54 and 63) =

As you might have realized, listing factors can still be a very time-consuming way to find the greatest common factor, especially if the numbers are very large or have a lot of factors.

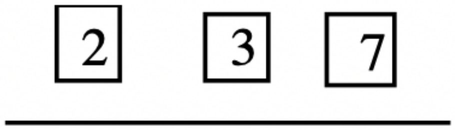
It's time to get out your colored prime number squares and white composite number squares again. We'll be using these to prime factor sets of numbers, which can then be used to find the greatest common factor of a set of numbers. This method is usually faster than the one we just used.

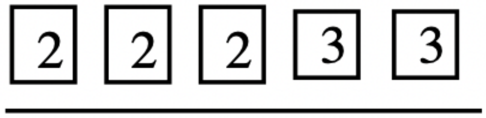
The goal of this next example is to find the greatest common factor of 42 and 72 using prime factorization with the prime and composite number squares.

Step 1: Use the squares to prime factor 42 and 72.

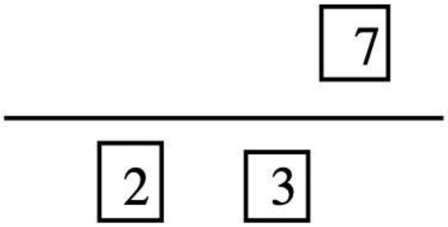
$42 = 2 \cdots 3 \cdots 7$ and $72 = 2 \cdots 2 \cdots 2 \cdots 3 \cdots 3$

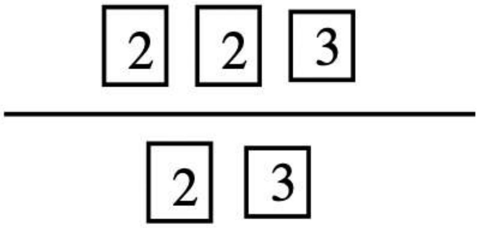
Step 2: Draw a line and put the prime factors of 42 above it. Next to that, draw a line and put the prime factors of 72 above it.





Step 3: Look at the prime factorization of the numbers. Move every factor they have in common under the line. They should have the same exact factors under the line, and they should have no common factors left above the line.





Step 4: The product of the factors under a line is the greatest common factor of the numbers. In this case, 2 ⋯ 3, or 6, is the greatest common factor of 42 and 72.

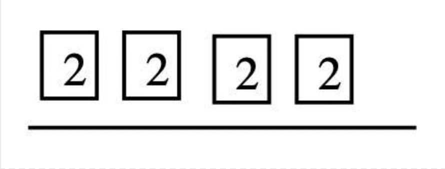
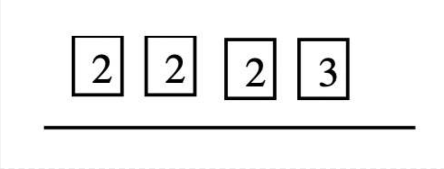
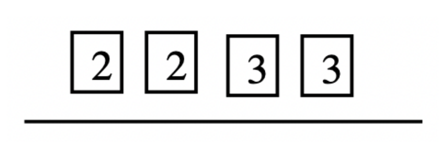
Step 5: Use the correct notation to write the answer: GCF(42, 70) = 6.

This next example used the same steps to find the greatest common factor of 16, 24 and 36.

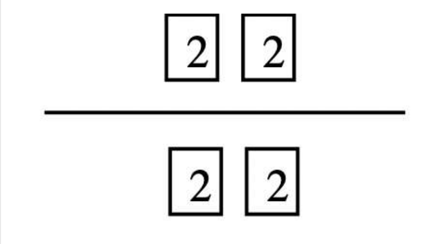
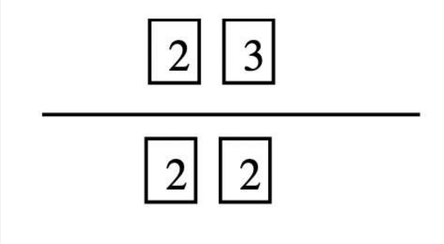
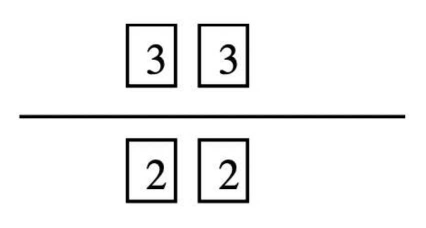
Step 1: Use the squares to prime factor 16, 24 and 36.

$16 = 2 \cdots 2 \cdots 2 \cdots 2, 24 = 2 \cdots 2 \cdots 2 \cdots 3$ and $36 = 2 \cdots 2 \cdots 3 \cdots 3$

Step 2: Draw a line for each number and put the prime factors of each number above it.

Step 3: Look at the prime factorization of the numbers. Move every factor that all three numbers have in common. There should be no factors left above the line that all three numbers have in common.

Step 4: The product of the factors under a line is the greatest common factor of the numbers. In this case, $2 \cdots 2$, or 4, is the greatest common factor of 16, 24 and 36.

Step 5: Use the correct notation to write the answer: GCF(16, 24, 36) = 4.

? Exercise 17

Use the steps shown in the previous two examples to find the greatest common factor of each problem. Show a picture of how the problem looks at step 3 where the prime factorization of the greatest common factor is shown under the line of each number you first prime factored.

a. GCF(42, 70) = _____ (This answer should agree with Exercise 14e.)
Show work below:

b. GCF(92, 115) = _____ (This answer should agree with Exercise 15e.)
Show work below:

c. GCF(48, 54, 63) = _____ (This answer should agree with Exercise 16f.)
Show work below:

d. GCF(306, 340) = _____
Show work below:

e. GCF(125, 275, 400) = _____
Show work below:

f. GCF(126, 168, 210) = _____
Show work below:

? Exercise 18

Let's say someone prime factored three large numbers, X, Y and Z like this:

$X = 2^5 \cdots 3^4 \cdots 7^2 \cdots 11^8 \cdots 13^3$	$Y = 2^4 \cdots 3^5 \cdots 5^3 \cdots 7^7 \cdots 13^2$	$Z = 2^6 \cdots 3 \cdots 5^5 \cdots 11^3 \cdots 13^4$

a. State the prime factorization (exponential notation is okay) of the greatest common factor of X, Y and Z. GCF(X, Y, Z) =

b. Explain how you did part a.

One way to do the previous exercise is to begin by listing the common prime factors (without exponents) of X, Y and Z. They are 2, 3 and 13. So, as a start, write 2 $\cdots$ 3 $\cdots$ 13. Next, determine how many of each prime factor is common to X, Y and Z. Since X has 5 factors of 2, Y has 4 factors of 2, and Z has 6 factors of 2, they only have 4 factors of 2 in common. Similarly, they only have one factor of 3 in common, and 2 factors of 13 in common. Put these exponents on the factors and you've got the GCF of the three numbers: GCF(X, Y, Z) = $2^4 \cdots 3 \cdots 13^2$

Did you notice that if you list the factors they have in common without exponents, you put on the smallest exponent they have in common for each prime?

? Exercise 19

Write the prime factorization of the greatest common factor of the set of numbers. For those that are factored with letters, assume each letter represents a different prime number.

a. If $X = 2^4 \cdots 3^2 \cdots 7^6 \cdots 11^3 \cdots 13^2$ and $Y = 2^5 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 13^3$
GCF(X, Y) = ______________________

b. If $X = 3^4 \cdots 5^2 \cdots 7^6$ and $Y = 2^5 \cdots 3^6 \cdots 5^6 \cdots 7^3$ and $Z = 2^4 \cdots 3^5 \cdots 5^4$
GCF(X, Y, Z) = ____________________

c. If $X = a^4 \cdots b^2 \cdots c^6 \cdots d^3 \cdots e^2$ and $Y = a^5 \cdots c^3 \cdots d^4 \cdots e^6 \cdots f^3$
GCF(X, Y) = ______________________

d. If $X = a^4 \cdots b \cdots c^4 \cdots d^3$ and $Y = a^5 \cdots b^3 \cdots d^4 \cdots e^6$ and $Z = a^2 \cdots c^3 \cdots d^7 \cdots f^3$
GCF(X, Y, Z) = ____________________

If two numbers have no factors in common, they are called **relatively prime**. In other words, **if GCF(a, b) = 1, then a and b are relatively prime**. The numbers, a and b, might both be prime, both be composite, or one might be prime and the other composite.

? Exercise 20

Give an example of two composite numbers that are relatively prime:

Exercise 21

Write a prime number and composite number that are relatively prime:

Exercise 22

Assume GCF(28, x) = 1

a. List any prime numbers that are not factors of x: _________________________

b. Give three examples (numbers) of what x could equal: ____________________

Exercise 23

If x and y are different prime numbers, GCF(x, y) = _____

Exercise 24

If m is a whole number, find the following:

a. GCF(2m, 3m) = _______	b. GCF (4m, 10m) = ________
c. GCF(m, m) = _______	d. GCF(m, 1) = _______
e. GCF(m, 0) = _______	

Trying to find the greatest common factor of two large numbers by prime factorization is sometimes quite time-consuming. There are two other algorithms you can use that we'll use. One is called **The Old Chinese Method**, and the other is **The Euclidean Algorithm**. Both of these methods use a fact we can prove using what we know about divisibility. First, let's look at some examples.

Exercise 25

Compute each of the following:

a. List the common factors of 42 and 72: _______
b. List the common factors of 42 and 30 (which is 72 – 42): _______
c. List the common factors of 30 and 12 (which is 42 – 30): _______
d. List the common factors of 12 and 18 (which is 30 – 12): _______
e. List the common factors of 12 and 6 (which is 18 – 12): _______
f. List the common factors of 6 and 6 (which is 12 – 6): _______
g. List the common factors of 6 and 0 (which is 6 – 6): _______

From your work in exercise 25, compute the following:

a. GCF(42, 72) = ____	b. GCF(42, 30) = ____
c. GCF(30, 12) = ____	d. GCF(12, 18) = ____
e. GCF(12, 6) = ____	f. GCF(6, 6) = ____

g. GCF(6, 0) = ____	

The previous two exercises illustrate that the greatest common factor of two numbers is equal to the greatest common factor of the smaller number, and the difference of the original two numbers; i.e., **if x ≥ y, then GCF(x, y) = GCF(y, x – y)**.

Prove the following theorem: Let a ≥ b. If c|a and c|b, then c|(a – b).

Solution: If c|a, then cn=a for some whole number, n. If c|b, then cm=b for some whole number, m. Using these substitutions for a and b, we get that c|(a – b) is true if c|(cn – cm) which is true if c is a factor of cn – cm. Factor: cn – cm = c(n – m). This clearly shows that c is indeed a factor of cn – cm. Therefore, if c|a and c|b, then c|(a – b).

This theorem can be used to show that **if a ≥ b, then GCF(a, b) = GCF(b, a – b)**. The above theorem states that if c is a factor of two numbers, then it is also a factor of their difference. Hence, if c is a common factor of a and b, where a ≥ b, then c is also a common factor of b and a – b. Since every common factor of a and b is also a common factor of ba and a – b, the pairs (a, b) and (b, a – b) have the same common factors. So, GCF(a, b) and GCF(b, a – b) must also be the same number.

The **Old Chinese Method** employs the fact that **GCF(a, b) = GCF(b, a – b)**.

Note three more properties:

GCF(x, x) = x: GCF(x, x) states that the greatest common factor of the same two numbers is itself. That should be clear since x is the greatest factor of each number, so each has x as the greatest common factor.

GCF(x, 0) = x: GCF(x, 0) = x, is true since every number is a factor of zero. So, since x is the greatest factor of x, then x is the greatest common factor of x and 0.

GCF(x, 1) = 1: GCF(x, 1) = 1 is true since 1 is a factor of every number, including x, and 1 is the only factor of 1. Therefore, 1 must be the greatest common factor of 1 and x.

Old Chinese Method of finding the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). Let that equal the GCF of the smaller of the two numbers, and the difference of the original two numbers. If the numbers in the parentheses are the same, that number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until the two numbers are the same, or 1 is one of the numbers. An example is shown below. On the right is an explanation of how I obtained the two new numbers in parentheses. The new numbers are underlined in the explanation. You do not need to write that out.

✓ Example

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, *156* = 546 – 390)
	= GCF(156, 234)	(*156* is smaller than 390, *234* = 390 – 156)
	= GCF(156, 78)	(*156* is smaller than 234, *78* = 234 – 156)
	= GCF(78, 78)	(*78* is smaller than 156, *78* = 156 – 78)
	= ***78***	78 is the GCF(78, 78); The answer is a number!

Therefore, GCF(546, 390) = 78.

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78 \cdots 7$ and $390 = 78 \cdots 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors).

Another example is shown below. On the right is an explanation of how I obtained the two new numbers (underlined) in parentheses. You do not need to write that out.

Example

GCF(1200, 504)	= GCF(504, 696)	(*504* is smaller than 1200, *696* = 1200 – 504)
	= GCF(504, 192)	(*504* is smaller than 696, *192* = 696 – 504)
	= GCF(192, 312)	(*192* is smaller than 504, *312* = 504 – 192)
	= GCF(192, 120)	(*192* is smaller than 312, *120* = 312 – 192)
	= GCF(120, 72)	(*120* is smaller than 192, *72* = 192 – 120)
	= GCF(72, 48)	(*72* is smaller than 120, *48* = 120 – 72)
	= GCF(48, 24)	(*48* is smaller than 72, *24* = 72 – 48)
	= GCF(24, 24)	(*24* is smaller than 48, *24* = 48 – 24)
	= ***24***	24 is the GCF(24, 24); The answer is a number!

Therefore, GCF(1200, 504) = 24.

Check: First, make sure 24 is a factor of 1200 and 504: 1200 = $24 \cdots 50$ and $502 = 24 \cdots 21$. Second, check to make sure the other factors of each (25 and 21) are relatively prime. If so, then 24 is not only a factor of 1200 and 504, but is in fact the greatest common factor of each since they have no other common factors (because 25 and 21 have no common factors.)

Here is one more example:

Example

GCF(667, 437)	= GCF(437, 230)	
	= GCF(230, 207)	
	= GCF(207, 23)	
	= GCF(23, 184)	
	= GCF(23, 161)	
	= GCF(23, 138)	
	= GCF(23, 115)	
	= GCF(23, 92)	
	= GCF(23, 69)	
	= GCF(23, 46)	
	= GCF(23, 23)	
	= ***23***	Remember: The answer is a number!

Therefore, GCF(667, 437) = 23.

Check: First, make sure 23 is a factor of 667 and 437: $667 = 23 \cdots 29$ and $437 = 23 \cdots 19$. Second, check to make sure the other factors of each (29 and 19) are relatively prime. If so, then 23 is not only a factor of 667 and 437, but is in fact the greatest common factor of each since they have no other common factors (because 29 and 19 have no common factors.)

A comment: You could have obtained the greatest common factor of the previous three examples by prime factoring. Often, this is a lengthy process for large numbers that look like they might be prime, as in the last example. Therefore, the Old Chinese provides an alternative way to obtain the greatest common factor. After doing a few using this method, we'll explore another alternate method, called the Euclidean Algorithm, which is related to, but usually takes less steps than, the Old Chinese Method.

? Exercise 27

Use the Old Chinese Method to compute the greatest common factor of the numbers given. Use correct notation, and show each step. State the answer. Then, show how you check your answer. Use the previous example as a model to do these problems.

a.

GCF(143, 91)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

b.

GCF(468, 378)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

c.

GCF(504, 180)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

Using the Chinese Method could be quite tedious. Take a look at the following example:

✓ Example

GCF(1200, 504)	= GCF(504, 696)	(*504* is smaller than 1200, *696* = 1200 – 504)
	= GCF(504, 192)	(*504* is smaller than 696, *192* = 696 – 504)
	= GCF(192, 312)	(*192* is smaller than 504, *312* = 504 – 192)
	= GCF(192, 120)	(*192* is smaller than 312, *120* = 312 – 192)
	= GCF(120, 72)	(*120* is smaller than 192, *72* = 192 – 120)
	= GCF(72, 48)	(*72* is smaller than 120, *48* = 120 – 72)
	= GCF(48, 24)	(*48* is smaller than 72, *24* = 72 – 48)
	= GCF(24, 24)	(*24* is smaller than 48, *24* = 48 – 24)
	= ***24***	24 is the GCF(24, 24); The answer is a number!

In the beginning of this example, GCF(1200, 504), we had to subtract 504 twice until we got GCF(504, 192). Then, notice once we wrote GCF(504, 192), we had to subtract 192 twice until we got GCF(192, 120). Basically, we do repeated subtraction until we get a number smaller than the one we are subtracting. Repeated subtraction is actually division. Note that $1200 \div 504 = 2$ remainder 192. On the second step, we see GCF(504, 192), which has the **smaller** of the numbers in the original parentheses (504), and the **remainder** after dividing the larger number (1200) by the smaller number (504). Note that $504 \div 192 = 2$ remainder 120. Look at the fourth step: we see GCF(192, 120), which has the **smaller** of the numbers in GCF(504, 192), and the **remainder** after dividing the larger number (504) by the smaller number (192). The Euclidean Algorithm uses division instead of repeated subtraction to shorten the steps.

How to use the Euclidean Algorithm to find the greatest common factor of two numbers:

Write the GCF of the two numbers in parentheses (remember the order of the numbers is irrelevant). The smaller of the two numbers will be one of the numbers in the next parentheses. To get the other number, divide the larger number by the smaller number, and put the **remainder** in parentheses. If the the smaller number is a factor of the larger number, that means it will divide evenly, so there will be no remainder. That means the remainder is 0. Remember to put the remainder in the parentheses, not the quotient! If one of the new numbers in the parentheses is zero, the other number is the GCF; if one the new numbers is 1, 1 is the GCF. Otherwise, repeat the process until one of the two numbers is 0 or 1. An example is shown below. This is the first example we did using the Old Chinese Method. You might want to look back and compare. On the right is an explanation of how I obtained the two new numbers in parentheses.

✓ Example

GCF(546, 390)	= GCF(390, 156)	(*390* is smaller than 546, $546 \div 390$ = 1 r *156*)
	= GCF(156, 78)	(*156* is smaller than 390, $390 \div 156$ = 2 r *78*)
	= GCF(78, 0)	(78 is smaller than 156, $156 \div 78$ = 2 r *0*)
	= ***78***	***Put the remainder (0), NOT 2, in the parentheses!***

		78 is the GCF(78, 78); The answer is a number!

Therefore, GCF(546, 390) = 78

Check: First, make sure 78 is a factor of 546 and 390: $546 = 78$\=$\cdots 7$ and $390 = 78 \cdots 5$. Second, check to make sure the other factors of each (7 and 5) are relatively prime. If so, then 78 is not only a factor of 546 and 390, but is in fact the greatest common factor of each since they have no other common factors (because 7 and 5 have no common factors

Below is another example – we did this earlier using the Old Chinese Method. On the right is an explanation of how I obtained the two new numbers in parentheses. You do not need to write that out.

✓ Example

GCF(667, 437)	= GCF(437, 230)	(*437* is smaller than 667, $667 \div 437 = 1$ r *230*)
	= GCF(230, 207)	(*230* is smaller than 437, $437 \div 230 = 1$ r *207*)
	= GCF(207, 23)	(*207* is smaller than 230, $230 \div 207$ = 1 r *23*)
	= GCF(23, 0)	(*23* is smaller than 207, $207 \div 23$ = 9 r *0*)
	= *23*	***Put the remainder (0), NOT 9, in the parentheses!***
		Remember: The answer is a number!

Therefore, GCF(667, 437) = 23

Check: First, make sure 23 is a factor of 667 and 437: 667 = $23 \cdots 29$ and $437 = 23 \cdots 19$. Second, check to make sure the other factors of each (29 and 19) are relatively prime. If so, then 23 is not only a factor of 667 and 437, but is in fact the greatest common factor of each since they have no other common factors (because 29 and 19 have no common factors.)

A comment: You could have obtained the greatest common factor of the previous three examples by prime factoring, the Old Chinese Method or the Euclidean Algorithm.

CAUTION: THE MOST COMMON MISTAKE PEOPLE MAKE WHEN USING THE EUCLIDEAN ALGORITHM IS ON THE LAST STEP, WHEN THE SMALLER NUMBER DIVIDES EVENLY INTO THE LARGER NUMBER, WHICH *MEANS THE REMAINDER IS ZERO.* IT DOESN'T MATTER WHAT THE QUOTIENT IS – IT'S THE *REMAINDER* THAT MATTERS – *0* GOES IN THE PARENTHESES!!! Also, zero is not a factor of any number except zero, so the GCF can't be zero. On the other hand, every number is a factor of zero. So, when zero is one of the numbers in parentheses, the other number is the GCF. **REMEMBER TO ALWAYS CHECK YOUR Answer!!!**

Before going on, I'm going to remind you of a quick and easy way to find out the quotient and remainder by using a simple calculator when doing these division problems, where you need to find the remainder. If you can already do it easily or your calculator figures it out for you, skip on down to Exercise 29. Let's say you wanted to divide $5263 \div 360$. When you do this on your calculator, it shows up something like 14.619444. This indicates that there are 14 360's in 5263, but the remainder isn't evident. At least, you know 14 is the quotient. To find the remainder on your calculator, key in 14×360 - 5263 and the number showing is the remainder if you ignore the negative sign! In this case, the remainder is 223. Remember that the remainder must be less than what you originally divided by – less than 360 in this case. Think about why this process works and try it on the next few problems.

Exercise 28

Use a calculator to find the quotient and remainder for these division problems.

a. $9876 \div 255 =$	b. $1509 \div 164 =$
c. $333 \div 46 =$	d. $4657 \div 579 =$

Exercise 29

Use the Euclidean Algorithm to compute the greatest common factor of the numbers given. Use correct notation, and show each step. State the answer. Then, show how you check your answer. Use the previous example as a model to do these problems. These are the same exercises you did using the Old Chinese Method in exercise 27. You might want to compare the two methods when you are done. Of course, the answer should be the same.

a.

GCF(143, 91)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

b.

GCF(468, 378)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= _____	

c.

GCF(504, 180)	= GCF (,)	Show the check here:
	= GCF (,)	
	= GCF (,)	
	= GCF (,)	
	= _____	

Exercise 30

Find GCF(418, 88) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(418, 88) = _______ Check your answer

? Exercise 31

Find GCF(527, 465) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(527, 465) = _______ Check your answer

? Exercise 32

Find GCF(353, 213) using the three methods discussed:

a) by prime factorization;
b) by the Old Chinese Method;
c) by the Euclidean Algorithm
d) Write the answer, and show how to check the answer.

a. Use prime factorization	c. Use the Euclidean Algorithm
b. Use the Old Chinese Method	d. Answer: GCF(353, 213) = _______ Check your answer

Now that we've covered a lot about prime numbers and factoring, we are going to revisit the divisibility tests one more time.

? Exercise 33

Note that the divisibility test for 6 utilized two other divisibility tests, the one for 2 and the one for 3. Also, note that the divisibility test for 15 utilized two other divisibility tests, the one for 5 and 3. Why do you think that works? What is the procedure for figuring out what tests to use?

Fact: If each prime in the prime factorization of a composite number, c, is listed only once, the divisibility test for that composite number is this: c|n if each of the prime factors of c divide n.

The prime factorization for 6 is $2 \cdot 3$. Since both primes are only listed once, the divisibility test for 6 is the union of the divisibility tests for 2 and 3.

The prime factorization for 15 is $5 \cdot 3$. Since both primes are only listed once, the divisibility test for 15 is the union of the divisibility tests for 5 and 3.

Exercise 34

Write the divisibility test for 14:

Exercise 35

Use the divisibility test for 14 to see which of the following is true. Show work.

a. 14|742
b. 14|968
c. 14|483

Exercise 36

The prime factorization for 20 is $2 \cdots 2 \cdots 5$. The divisibility test for 2, 2 and 5 does not work since 20|70 is false even though 2|70 and 2|70 and 5|70. Why doesn't it work?

Exercise 37

This is the divisibility test for 20: 20|n if 4|n and 5|n. Why do you think this works?

Exercise 38

Try to write the divisibility tests for each of the following numbers:

a. 12|n if
b. 18|n if

***Divisibility Test for a composite number:* Assume P_1, P_2, P_3, P_4, P_5,... are all different prime numbers. (You can think of P_1 as the first prime (2), P_2 as the second prime (3), P_3 as the third prime (5), P_4 as the fourth prime (7), P_5 as the fifth prime (11), etc. Assume the prime factorization of a number, c, is $(P_a)^x \cdots (P_b)^y \cdots \ldots \cdots (P_c)^z$ where $P_a \neq P_b \neq P_c$, etc. Then, c|n if $(P_a)^x$|n AND $(P_b)^y$|n AND $(P_c)^z$|n, etc.**

Example

The divisibility test for the following numbers is done by first factoring them.

1) Divisibility test for 26: Since 26 = $2 \cdots 13$: **26|n if 2|n and 13|n.**

2) Divisibility test for 12: Since 12 = $2^2 \cdots 3$: **12|n if 4|n and 3|n.**

3) Divisibility test for 24: Since 24 = $2^3 \cdots 3$: **24|n if 8|n and 3|n.**

4) Divisibility test for 45: Since 45 = $3^2 \cdots 5$: **45|n if 9|n and 5|n.**

5) Divisibility test for a number, c, whose prime factorization is $2^3 \cdots 3^2 \cdots 5^2 \cdots 11$: **c|n if 8|n AND 9|n and 25|n and 11|n**

7) Divisibility test for a number, b, whose prime factorization is $3^2 \cdots 5 \cdots 7^2 \cdots 11^2$: **b|n if 9|n AND 5|n and 49|n and 121|n**

Exercise 39

Write the divisibility test for the following numbers:

a. 35: ____
b. 28: ____

c. 75: ____

d. 56: ____

e. A number, c, whose prime factorization is $2^2 \cdots 3^3 \cdots 5^2 \cdots 11$: ____

f. A number, d, whose prime factorization is $2^4 \cdots 3 \cdots 5 \cdots 11$: ____

? Exercise 40

This exercise is for you to think about what you'd do if someone asked you to find the LEAST common factor of a set of numbers, instead of the greatest common factor.

a. Find the least common factor of 55 and 66: ____

b. Find the least common factor of 10 and 12: ____

c. Find the least common factor of 1 and 8: ____

d. Find the least common factor of 3, 6, and 9: ____

e. Find the least common factor of m and n: ____

f. Do you think it is a useful question to find the least common factor of a set of numbers? Why or why not? Explain your answer.

g. What is the least common factor of any set of numbers? ____

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8.3: LCM and other Topics

You will need: ***Prime Number Squares*** *(Material Cards 19A-19B)*

Composite Number Squares *(Material Cards 20A-20B)*

There are a ***finite*** *number of factors* for any given number. On the other hand, there are an ***infinite*** *number of nonzero multiples* of a number. For instance the list of multiples of 2 are all of the even numbers. It's impossible to list all of them, because this is an infinite set. A multiple of a number c is nc where n is a whole number. In other words, to find a multiple of c, multiply c by a whole number. Although 0 is a multiple of every number, we usually omit listing it as a multiple. The (nonzero) multiples of 6 are listed here: 6, 12, 18, 24, 30, 36, 42, ...

To list the multiples of a number, you begin with the number itself, then the number times 2, then the number times 3, etc. Since multiplication is repeated addition, you could repeatedly add the number to get the next multiple.

? Exercise 1

a. List the first 10 multiples of 8:

b. List the first 10 multiples of 12:

c. Of the list you produced in parts a and b, list the multiples that 8 and 12 have in common:

d. From part c, what is the *smallest multiple* that 8 and 12 have in common?

e. Is there a greatest common multiple that 8 and 12 have in common? Is so, what is it? Before you answer, remember that in a and b, you only listed the *first 10* multiples of 8 and 12.

Just like with the greatest common factor (GCF), many people don't really think about what **LEAST COMMON MULTIPLE *means***. Every number has an infinite number of multiples. Every set of numbers has an infinite number of multiples in common. The **LEAST COMMON MULTIPLE** is the ***smallest*** **multiple** they have in **common**.

In exercise 1, we write LCM(8, 12) = 24. The order of the numbers in the parentheses is irrelevant. So, LCM(8, 12) = LCM(12, 8). It is NOT an ordered pair. Also, you can find the least common multiple of a large set of numbers. Soon, you'll find how easy it is to find the least common multiple of several numbers; e.g., LCM(2, 3, 4, 5, 6, 7, 8, 9, 10). In this case, listing several multiples of each of these numbers until you find the smallest one all of them have in common is not the efficient way to find it, so you'll be exploring another way to find it using prime factorization.

? Exercise 2

a. List the first 15 multiples of 4:

b. List the first 15 multiples of 6:

c. List the first 15 multiples of 10:

d. Of the list you produced in parts a, b and c, list the multiples all three numbers (4, 6 and 10) have in common:

e. From part d, complete the following: LCM(4, 6, 10) =

The problem with trying to find the least common multiple this way is that you may have to write a lengthy list of multiples of each number until you finally find a multiple they all have in common. For instance, if you were asked to find the LCM(59,61), you wouldn't find a multiple they each have in common until you wrote 61 multiples of 59 and 59 multiples of 61. That is because these are prime numbers, so the least common multiple is their product: $59 \cdots 61$. This is the case with any numbers that are **relatively prime** as well. (Remember 4 and 15 are relatively prime, although neither is prime.)

If x and y have no factors in common, then GCF(x,y) = 1 and LCM(x,y) = xy.

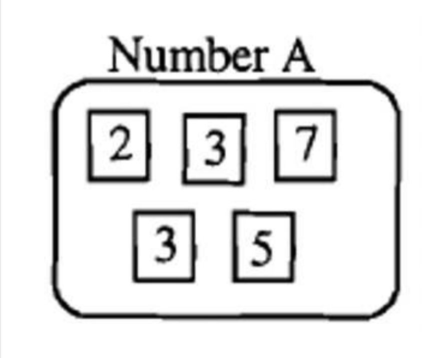
The following is also true:

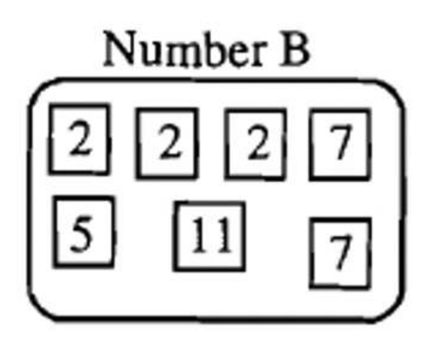
If GCF(x,y) = 1, then x and y have no factors in common, and LCM(x,y) = xy.

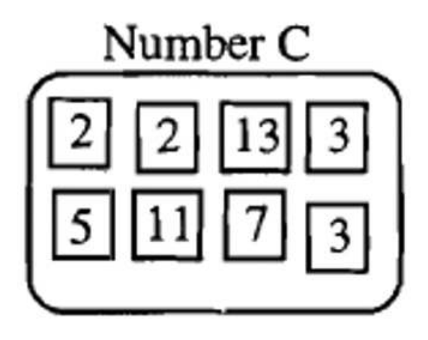
If LCM(x,y) = xy, then x and y have no factors in common, and GCF(x,y) = 1.

A more efficient way to find the least common multiple of a set of numbers is to find the prime factorization of each number, and then BUILD the least common multiple. Get out your prime number squares to do the following exercises.

Below is the prime factorization of three numbers, A, B and C.


Number A
2
3
7
3
5

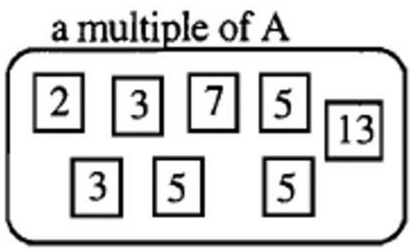

Number B
2
2
2
7
5
11
7

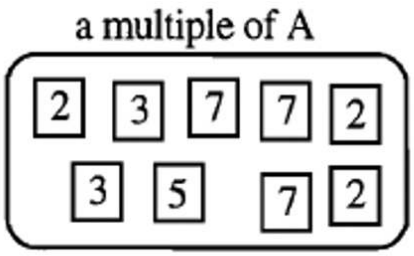

Number C
2
2
13
3
5
11
7
3

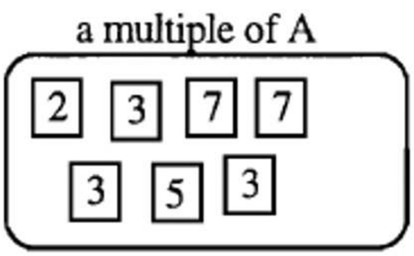
Below is how to write each of these numbers in prime factorization form:

$A=2\cdots3^2\cdots5\cdots7$	$B=2^3\cdots5\cdots7^2\cdots11$	$C=2^2\cdots3^2\cdots5\cdots7\cdots11\cdots13$

One way to make a multiple of A is to simply add extra factors to factors of A. *Note: A itself is a multiple of A, so you don't **have** to add any extra factors to get a multiple.* There are an infinite number of possibilities for multiples of A. Below are three multiples of A:


a multiple of A
2
3
7
5
13
3
5
5

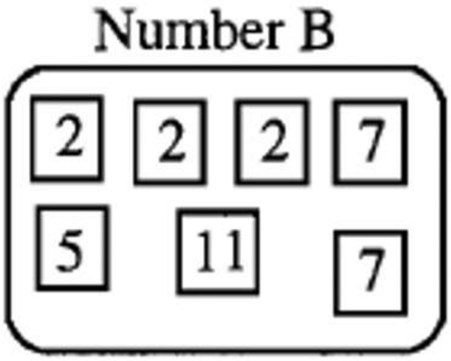

a multiple of A
2
3
7
7
2
3
5
7
2


a multiple of A
2
3
7
7
3
5
3

In prime factorization form, the first multiple of A above is written: $2\cdots3^2\cdots5^3\cdots7\cdots13$

Exercise 3

Below is the prime factorization of a number, B


Number B
2
2
2
7
5
11
7

a. Form the number B with your prime number squares. Form a multiple of B by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Show a picture of the multiple of B that you formed.

Write the prime factorization of this multiple: _____

b. Form the number B with your prime number squares. Form a multiple of B, *different* from the one you formed in A, by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Show a picture of the multiple of B that you formed:

Write the prime factorization of this multiple of B: _____

c. Form the number B with your prime number squares. Form another multiple of B by "throwing in" one or more prime number squares as factors to the original prime factorization of B shown above. Write the prime factorization of this multiple of B: _____

Number C is shown below toward the middle of the page. Use it to do exercise 4.

? Exercise 4

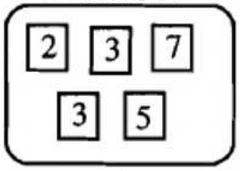
a. Form the number C with your prime number squares. Form a multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:

b. Form the number C with your prime number squares. Form a different multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:

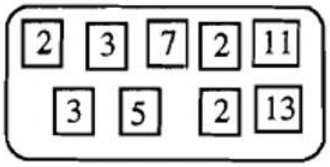
c. Form the number C with your prime number squares. Form another multiple of C by adding one or more prime number squares as factors. Write the prime factorization of the multiple of C you formed:

NOTE: X is a multiple of M if M is a factor of X. So, you have a multiple of a number if the prime factors of the number itself are factors in the multiple. It's as if you can "see" the number in a multiple. We are going to determine if any of the numbers shown on the right are multiples of A, B or C.

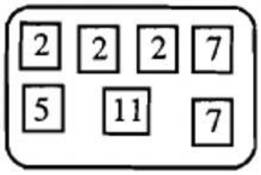
Number A



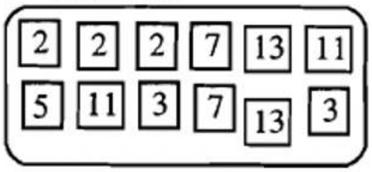
Number X



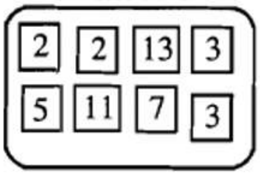
Number B



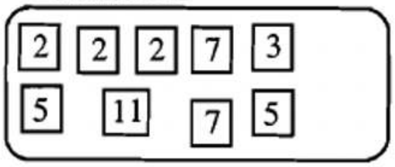
Number Y



Number C



Number Z

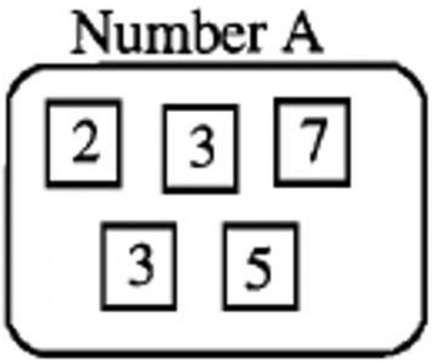
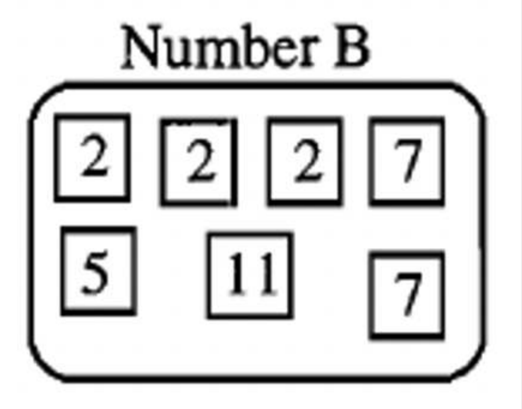
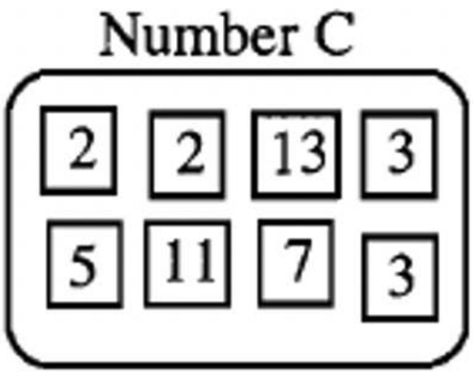


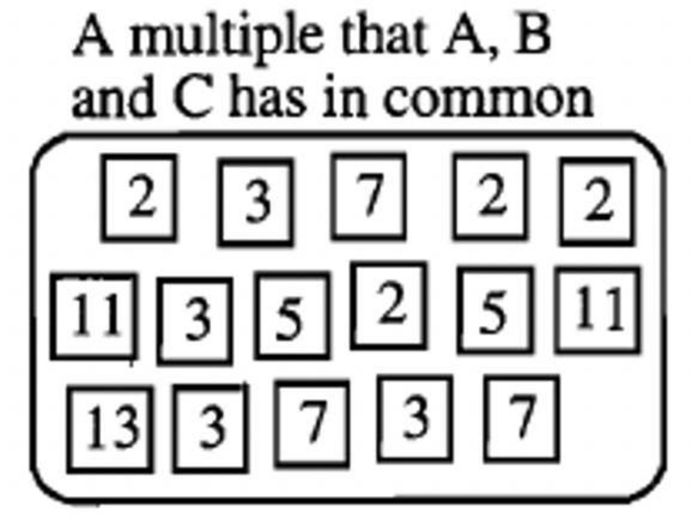
To decide if X, Y or Z is a multiple of A, see if each of X, Y or Z has the prime factors of A as a subset. In other words, A has one factor of 2, two factors of 3, one factor of 5 and one factor of 7. Any number containing those factors is a multiple of A. A multiple of A may have more factors of A, but can't be missing any factors of A.

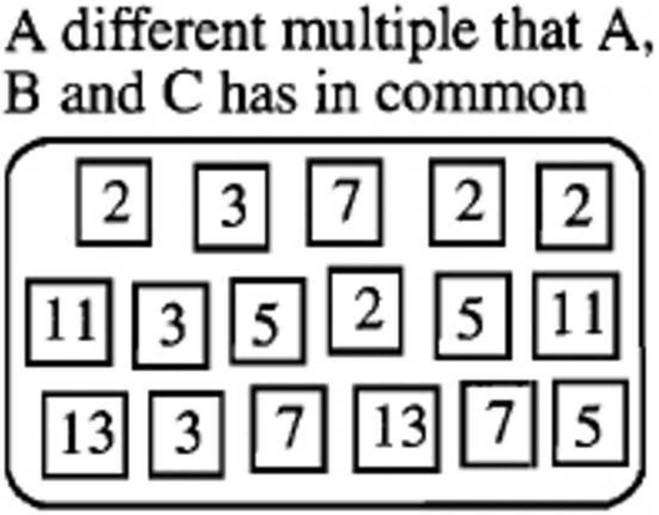
Exercise 5

a. List any number on the right (X, Y and/or Z) that is a multiple of A:
b. List any number on the right (X, Y and/or Z) that is a multiple of B:
c. List any number on the right (X, Y and/or Z) that is a multiple of C:

None of the numbers X, Y or Z was a multiple of all three of the numbers, A, B and C. Numbers A, B and C are shown below. Below that are only two examples of multiples that A, B and C have in common.

 Number A 2 3 7 3 5
 Number B 2 2 2 7 5 11 7
 Number C 2 2 13 3 5 11 7 3


A multiple that A, B and C has in common
2 3 7 2 2
11 3 5 2 5 11
13 3 7 3 7


A different multiple that A, B and C has in common
2 3 7 2 2
11 3 5 2 5 11
13 3 7 13 7 5

Exercise 6

Use prime number squares to form three other multiples that A, B and C have in common. Write the prime factorization of each of the multiples you formed.

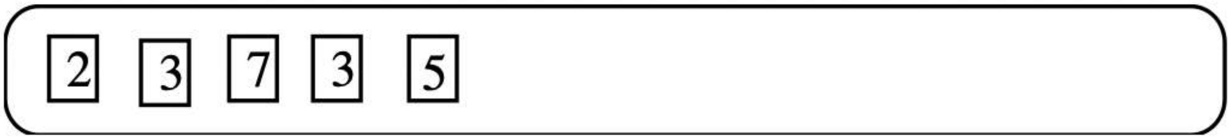
a. _____

b. _____

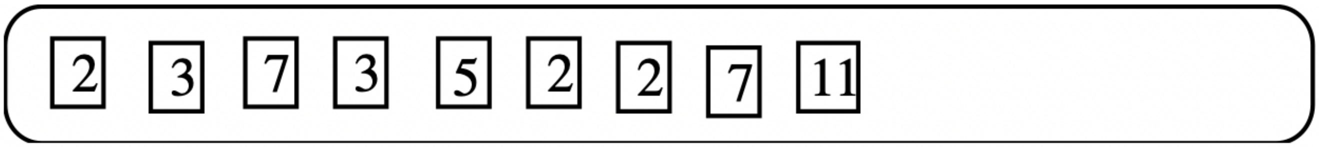
c. _____

Now, we are going to use the prime factorization to build the ***LEAST*** **COMMON MULTIPLE** of a set of numbers. The least common multiple is a multiple that the numbers have in common, but that has the least number of factors possible. The multiples formed above have 16 and 17 factors, respectively. To build a least common multiple, we only add in a factor if it is necessary.

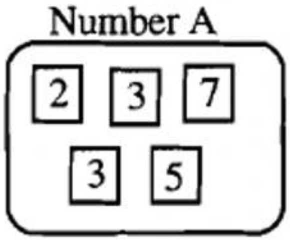
Let's say we want to build the least common multiple of A, B and C. In order to be a multiple of A, we need to have the factors of A. So, we start ***building*** the multiple by putting in the factors of A:

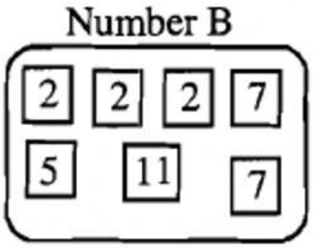


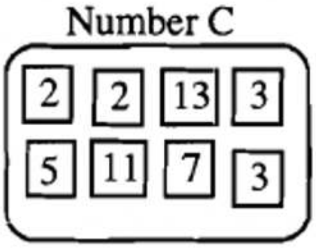
The factors of B must also be in the multiple. B has three factors of 2, but there is only one factor of 2 in the multiple so far. So, the common multiple will need two more factors of 2. B has two factors of 7, but there is only one factor of 7 in the multiple so far. So, the common multiple will need one more factor of 7. B has one factor of 5, which is already there. B also has one factor of 11. That is not there, so must be put in. Therefore, two factors of 2, one factor of 7 and a factor of 11 must be joined with the other factors in the common multiple. Once this is done, we have built the least common multiple of A and B:



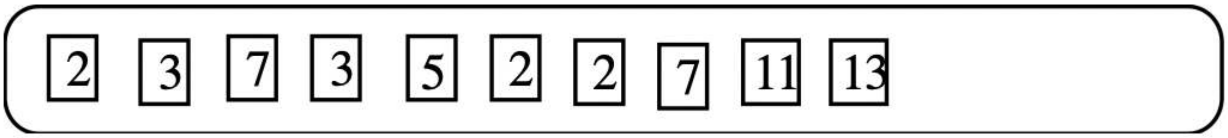
The prime factorizations of A, B and C are shown below for your convenience.







We have built the least common multiple of A and B. To find the least common multiple of A, B and C, we need to make sure the factors of C are also in the multiple. C has two factors of 2, which are already in the multiple, two factors of 3, which are already in the multiple, one factor each of 5, 7 and 11, each of which is already in the multiple, and a factor of 13, which is not in the multiple. So 13 is the only factor that needs to be joined with the factors already in the multiple we are building.



The above is a multiple of A, B and C. We only have we built a multiple, we actually built the **LEAST** COMMON MULTIPLE of A, B and C. Note it is a multiple of A, B *and* C; therefore it is a *common* multiple. It is the *least* common multiple because if any of the factors were removed, it wouldn't be a multiple of one of the numbers. For instance, if a factor of 2 was removed, it would not be a multiple of A. If a factor of 3 was removed, it would not be a multiple of A or C. If a factor of 7 was removed, it would not be a multiple of B, etc. If a factor of 11 was removed, it would not be a multiple of B or C. If a factor of 13 was removed, it would not be a multiple of C.

We write: LCM(A, B, C) = $2^3 \cdots 3^2 \cdots 5 \cdots 7^2 \cdots 11 \cdots 13$

? Exercise 7

Build the least common multiple of A and B using prime number squares. Then, write the prime factorization of the least common multiple of A and B. Let $A = 2^2 \cdots 3^5 \cdots 5 \cdots 13$ and $B = 2^2 \cdots 3^2 \cdots 7^3 \cdots 11^2 \cdots 13$

LCM(A, B) = ____

Exercise 8

Build the least common multiple of A, B and C using prime number squares. Then, write the prime factorization of the least common multiple of A and B.

Let A = $2^2 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^2 \cdots 7 \cdots 11^2$ and $C = 2^2 \cdots 3^4 \cdots 7^3 \cdots 13^2 \cdots 19$

LCM(A, B, C) = _____

Exercise 9

Build the least common multiple of A, B and C using prime number squares. Then, write the prime factorization of the least common multiple of A and B.

Let $A = 2 \cdots 5^3 \cdots 11 \cdots 19$ and $B = 2^4 \cdots 3^6 \cdots 5^2 \cdots 23^2$ and $C = 5^4 \cdots 7^6 \cdots 11^2 \cdots 23$

LCM(A, B, C) = _____

Look back at the answer from exercise 9.

Let $A = 2 \cdots 5^3 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^6 \cdots 5^2 \cdots 23^2$ and $C = 5^4 \cdots 7^6 \cdots 11^2 \cdots 23$

Answer: LCM(A, B, C) = $2 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 19 \cdots 23^2$

If the numbers are prime factored using exponents, then the least common multiple contains each of the prime factors shown in any of the numbers. The exponent on each of the prime numbers is the highest exponent found on that prime number factor in the prime factorization of the numbers.

For instance, the prime numbers in the prime factorization of A, B and C are 2, 3, 5, 7, 11, 19 and 23. So begin by writing the product of these prime numbers: $2 \cdots 3 \cdots 5 \cdots 7 \cdots 11 \cdots 19 \cdots 23$

2 is a factor in A and B. The highest power of 2 found in either prime factorization of A and B is 1. 3 is a factor found in B, as 3^6. So, the highest power of 3 is 6. 5 is a factor of all three numbers. In A, the exponent on 5 is 3; in B, the exponent on 5 is 2; in C, the exponent on 5 is 4. So, the highest power of 5 is 4. Similarly, do the same for all of the other prime factors. Write the highest exponent on the factor.

Therefore, LCM(A, B, C) = $2 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 19 \cdots 23^2$

Exercise 10

Find the greatest common factor and least common multiple for each set of numbers, written in prime factorization. Assume a, b, c, d, and e are different prime numbers.

a. Let $A = 2^2 \cdots 3^5 \cdots 5 \cdots 13$ and $B = 2^2 \cdots 3^2 \cdots 7^3 \cdots 11^2 \cdots 13$

GCF(A, B) = ____

LCM(A, B) = ____

b. Let $A = 2^2 \cdots 11 \cdots 19$ and $B = 2 \cdots 3^2 \cdots 7 \cdots 11^2$ and $C = 2^2 \cdots 3^4 \cdots 7^3 \cdots 13^2 \cdots 19$

GCF(A, B, C) = ____

LCM(A, B, C) = ____

c. Let $X = a^5 \cdots b^4 \cdots c^5 \cdots d$ and $Y = b^2 \cdots c^3 \cdots d^2 \cdots e^2$ and $Z = a^2 \cdots c^4 \cdots d^3 \cdots e^2$

GCF(X, Y, Z) = ____

LCM(X, Y, Z) = ____

d. Let $X = a^6 \cdots b^3 \cdots c \cdots d$ and $Y = b^4 \cdots c^4 \cdots d^3 \cdots e^7$ and $Z = a^2 \cdots c^4 \cdots d^2 \cdots e^3$

GCF(X, Y, Z) = ____

LCM(X, Y, Z) = ____

So far, the prime factorization of numbers has been given, and all you had to do was to build the prime factorization of the least common multiple. Now, your job will be to first prime factor numbers; then you can build the least common multiple from the prime factorizations. In the end, the least common multiple is a number. Multiply the factors in the least common multiple to find the one number that is the least common multiple.

✓ Example

Find the least common multiple of 15, 18 and 20

Solution

Prime factor each of these numbers. $15 = 3 \cdots 5, 18 = 2 \cdots 3 \cdots 3$, and $20 = 2 \cdots 2 \cdots 5$

Build the least common multiple by first "throwing in" the prime factors that make up the prime factorization of 15; then "throw in" any prime factors needed for 18; next, "throw in" any prime factors needed for 20. We get

LCM(15, 18, 20) = $3 \cdots 5 \cdots 2 \cdots 3 \cdots 2 = 2^2 \cdots 3^2 \cdots 5$ **= 180**

? Exercise 11

Find the least common multiples. Show the prime factorization of each number, and how you use it to build the least common multiple.

a. LCM (6, 8, 10) = ______

b. LCM (25, 35, 40) = ______

c. LCM (49, 91, 26) = ______

d. LCM(56, 24, 30) = ______

e. LCM(22, 34, 55) = _____

Note how easy it is to find the least common multiple of a larger set of numbers using prime factorization. We'll find: LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14)

We don't need to expressly write down the prime factorization of the 12 numbers because it's easy enough to do that in our head. For instance, the prime factorization of 4 is $2 \cdots 2$, the prime factorization of 10 is $2 \cdots 5$, and the prime factorization of 13 is 13.

We build up the LCM of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 by first making sure the prime factorization of 2 is there.

Step 1: Make sure the prime factorization 2 is there. **So far, LCM: 2**

Step 2: Make sure the prime factorization 3 is there. We need to "throw in" a 3 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3}$

Step 3: Make sure the prime factorization 4 is there. We need to "throw in" a 2 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2}$

Step 4: Make sure the prime factorization 5 is there. We need to "throw in" a 5 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5}$

Step 5: Make sure the prime factorization 6 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $2 \cdots 3 \cdots 2 \cdots 5$.

Step 6: Make sure the prime factorization 7 is there. We need to "throw in" a 7 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7}$

Step 7: Make sure the prime factorization 8 is there. We need to "throw in" a 2 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2}$

Step 8: Make sure the prime factorization 9 is there. We need to "throw in" a 3 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3}$

Step 9: Make sure the prime factorization 10 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3}$

Step 10: Make sure the prime factorization 11 is there. We need to "throw in" an 11 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11}$

Step 11: Make sure the prime factorization 12 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11}$

Step 12: Make sure the prime factorization 13 is there. We need to "throw in" a 13 to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11 \cdots 13}$

Step 13: Make sure the prime factorization 14 is there. We don't need to "throw in" any factor(s) to the LCM, so there is no change to the LCM. **So far, LCM:** $\mathbf{2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11 \cdots 13}$

That's it! You have now built up the LEAST common multiple of all 12 numbers! You can check to make sure that the prime factorization of each of the 12 numbers is indeed in the prime factorization of the least common multiple you built. Furthermore, if any of the prime factors were removed, it wouldn't be a multiple of all 12 numbers. Therefore, the prime factorization of the LCM is as shown below:

LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) = $2 \cdots 3 \cdots 2 \cdots 5 \cdots 7 \cdots 2 \cdots 3 \cdots 11 \cdots 13$.

If these factors are multiplied together, LCM(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14) = 360, 360.

The prime factorization of the LCM may have been written with the factors in ascending order using exponents: $2^3 \cdots 3^2 \cdots 5 \cdots 7 \cdots 11 \cdots 13$

? Exercise 12

Use prime factorization to build the least common multiple. Show the prime factorization as the number is built. Then, multiply the factors to find the answer.

a. LCM(3, 4, 6, 7, 9, 10, 12, 14, 15, 18, 20) = _______ = _______

b. LCM(2, 3, 9, 11, 14, 15, 16, 17, 18, 22) = _______ = ________

c. LCM(5, 6, 7, 8, 12, 14, 15, 17, 18, 25) = _______ = _______

d. LCM(15, 18, 20, 25, 30, 35, 42, 45) = _______ = _______

Let's say you know the greatest common factor of 165 and some other number was 3, and the least common multiple of the same two numbers was 15,015. How would you figure out what the other number was?

First, it's a good idea to write down what you know. Let N be the other number.

Then, GCF(165, N) = 3 and LCM (165, N) = 15,015.

Since 3 is a factor, and in fact the greatest common factor, of both 165 and N, then each number can be written as 3 times something, and the other factor you obtain for 165 should be relatively prime to the other factor you obtain for N, *since 3 is the **greatest** common factor of 165 and N.*

So, let's rewrite, LCM (165,N) = 15,015 like this: LCM (3×55, $3 \times$ _____) = 15,015

In other words, I know N = 3 $\times$ _____, but I have to figure out what goes in the blank to figure out what N is. If you want to introduce another variable, like M, instead of writing a blank, that works just as well. It's up to you.

To find the LCM of 3×55 and 3 $\times$ ______ , where the 55 and the number on the blank have no common factors, you would multiply $3 \times 55 \times$ _______. But we know the product should be 15,015. So, the number that must go on the blank must be 91, since $3 \times 55 \times 91 = 15,015$. So, now we can figure out what N is: $N = 3 \times 91 = 273$.

Let's see if this makes sense by first rewriting 165 and 273 either in prime factored form or as the GCF(165,273) times something; then we'll figure out the GCF and LCM from the factored form, and see if it agrees with our original problem.

$165 = 3 \times 55$ and 273 = 3×91. First, make sure 3 is really a factor of each number!

Now, convince yourself that 3 really is the **greatest** common factor of 165 and 273 by checking to see that the other factor of one number is relatively prime (no factors in common) with the other factor of the other number. So, ask yourself if 55 and 91 have any factors in common (besides 1) if you aren't sure, then prime factor each of these numbers first ($55 = 5 \times 11$ and $91 = 7 \times 13$); note that they have no factors in common.

Since 3 is the greatest common factor, then the least common multiple is obtained by multiplying $3 \times 55 \times 91$, which is 15,015 which is what it should be according to the original information given. Therefore, 273 really was the number we were looking for.

N = ***273***

There is another way to do the above problem using the following property:

For any *two* numbers, m and n, the following is always true:

$m \times n = GCF(m, n) \times LCM(m, n)$

This equation is *not true* for more than two numbers $r \times s \times t \neq GCF(r, s, t) \times LCM(r, s, t)$

Example

Verify the above property for numbers 15 and 18.

Since GCF(15, 18) = 3 and LCM(15, 18) = 90, we want to verify that the product of the two numbers, 15 and 18, equals the product of the GCF and LCM of the two numbers.

$15 \cdots 18 = 270$; GCF(15, 18) $\cdots$ LCM(15, 18) = $3 \cdots 90 = 270$

Therefore, $15 \cdots 18 =$ GCF(15, 18) $\cdots$ LCM(15, 18)

Example

Show that the above property does not work for three numbers.

Counterexample: Use the numbers 4, 6 and 8. GCF(4, 6, 8) = 2 and LCM(4, 6, 8) = 24

$4 \cdots 6 \cdots 8 = 192$; GCF(4, 6, 8) $\cdots$ LCM(4, 6, 8) = $2 \cdots 24 = 48$

Clearly, $4 \cdots 6 \cdots 8 \neq$ GCF(4, 6, 8) $\cdots$ LCM(4, 6, 8)

Example

Find N if GCF(165, N) = 3 and LCM (165, N) = 15,015

In this case, the two numbers are 165 and N, the GCF is 3 and the LCM is 15,015. Plug these values into the equation shown above in bold.

$165N = 3 \times 15,015$. Divide both sides by 165 to find N: N = $\frac{3 \times 15015}{165} = 273$

The property can also be used to find the LCM of two numbers if you know the GCF. For instance, if you were asked to find the GCF and LCM of 24 and 30. You can easily find the GCF, which is 6. To find the LCM, multiply the two numbers together, and divide by the GCF. (This should make sense to you intuitively if you think about it: You wouldn't list the GCF twice as you build the LCM. Also, the GCF will cancel into either of the two numbers since it is a factor of each.) So, the LCM(24, 30) = $(24 \cdot 30) \div 6 = 120$.

Exercise 13

If GCF(1176, 288) = 24, find the LCM(1176, 288)

Exercise 14

Find X if GCF(2940, X) = 105 and LCM(2940, X) = 79,380

Exercise 15

If the greatest common factor of 3,211 and another number is 247 and the least common multiple of the same two numbers is 48,165, then what is the other number?

Exercise 16

If you aren't given at least one of the numbers, there might be more that one possible solution. Figure out the possibilities for a and b if all you know is GCF(a, b) = 2 and LCM(a, b) = 20

The next few problems are a review of GCF and LCM. You now have several methods you can use to find the GCF and LCM.

17-20 Find the greatest common factor of each of the following pairs of numbers using prime factorization, the Old Chinese Method or the Euclidean Algorithm. Then, find the LCM of each pair. Show all work.

Exercise 17

a. GCF (693, 546) = _______

b. LCM (693, 546) = _______

Exercise 18

a. GCF (2117, 2555) = _______

b. LCM (2117, 2555) =

Exercise 19

a. GCF (1369, 10693) = ______

b. LCM (1369, 10693) =

Exercise 20

a. GCF (24300, 14406) = _______

b. LCM (24300, 14406) =

Exercise 21

Make up a problem where the GCF of 2 different 3-digit numbers is 32. For instance, GCF (x, y) = 32. Find an x and y that will work.

Exercise 22

Make up a problem where the GCF of two different 4-digit numbers is 32. In other words, find 2 numbers, a and b, such that GCF (a, b) = 32.

Exercise 23

Make up a problem where the GCF of 2 different 3-digit numbers is 35. For instance, GCF (x, y) = 35. Find an x and y that will work.

? Exercise 24

Make up a problem where the GCF of two different 4-digit numbers is 28. In other words, find 2 numbers, a and b, such that GCF (a, b) = 28.

All even numbers are multiples of 2. Therefore, every even number can be written as the product of 2 and an integer. Symbolically, we can write every even number can be written in the form: 2k, where k is an integer. For example, 12 = 2(6), 20 = 2(10), 58 = 2(29), etc.

All odd numbers can be written as one more than an even number. Since an even number can be written as 2k, then symbolically, we write that every odd number can be written in the form: 2k + 1, where k is an integer. For example, 15 = 2(7) + 1, 41 = 2(20) + 1, etc.

Any integer that can be written in the form 2k is even and any integer that cannot be written in the form 2k is not even. Any integer that can be written in the form 2k + 1 is odd and any integer that cannot be written in the form 2k + 1 is not odd.

We'll be doing some proofs about even and odd numbers. The only way to express an even number, in general, is by 2k. The variable can be any letter. If you want to express more than one even number, you must use a new variable like 2m or 2n. The same is true for odd numbers.

For the examples and exercises that follow, assume all variables are integers.

EXAMPLES: State which of the following always represents an even number, which of the following always represents an odd number, and which are sometimes even and sometimes odd.

✓ Example 1: 6n+14

6n + 14

Solution

6n + 14 = 2(3n + 7), which is in the form of an even number. Therefore, 6n + 14 will always represent an even number (since n is an integer).

✓ Example 2: 4n+23

4n + 23

Solution

4n + 22 + 1 = 2(2n + 11) + 1, which is in the form of an odd number. Therefore, 6n + 14 will always represent an odd number.

✓ Example 3: 5n+2

5n + 2

Solution

It is impossible to write 5n + 2 in the form 2k or 2k + 1. Therefore, it can't be determined. It is sometimes even and sometimes odd.

Note: You can verify these answers by plugging in an even number for n and then an odd number for n, and see if the conclusion makes sense. In example 1, if n = 2, then 6n + 14 = 6(2) + 14 = 28 (even). If n = 3, then 6(3) + 14 = 32 (even). So, the result was even when n was replaced by either an even or an odd number. Do the same verification for example 2 and 3 in the space below:

? Exercise 25

State which of the following always represents an even number, which of the following always represents an odd number, and which are sometimes even and sometimes odd. Justify your answer.

a. 8n + 20

b. 10k + 9

c. 5x + 2

✓ Example 1

Formally prove that the sum of two odd numbers is even.

IMPORTANT Note: Don't define the two odd numbers to be the same odd number. You must be general, and assume they may be two different odd numbers! Use different variables to be the most general.

Solution

Let 2n+1 = one odd number, and let 2m+1 = another odd number.

The sum is: 2n+1 + 2m+1= 2n+2m+2 = 2(n+m+1), which is in the form of an even number. Therefore, the sum of 2 odd numbers is even.

✓ Example 2

Formally prove that the product of two odd numbers is odd.

Solution

Let 2n+1 = one odd number, and let 2m+1 = another odd number.

The sum is: (2n+1) (2m+1)= 4nm + 2m + 2n + 1 = 2(2nm + n + m) +1, which is in the form of an odd number. Therefore, the product of 2 odd numbers is odd.

? Exercise 26

Formally prove that the sum of two even numbers is even.

? Exercise 27

Formally prove that the sum of two odd numbers is even.

? Exercise 28

Formally prove that the sum of an even number and an odd number is odd.

? Exercise 29

Formally prove that the product of two even numbers is even.

? Exercise 30

Formally prove that the product of two odd numbers is odd.

? Exercise 31

Formally prove that the product of an even number and an odd number is even.

We are going to explore a way to find the sum of several consecutive whole numbers.

✓ Example 1

1 + 2 + 3 + 4 + 5 + 6

✓ Example 2

The sum of the first 100 counting numbers would take a long time to write out and compute. We write it in the following way: 1 + 2 + 3 + ... + 98 + 99 + 100

✓ Example 3

55 + 56 + 57 + ... + 128 + 129 + 130

The first example is fairly easy to compute. But as more and more numbers are added, the computation becomes cumbersome. The first two examples start with the number, 1. Our first goal will be to find a pattern, and then a formula for adding any set of consecutive counting numbers starting with 1.

Even though we know the answer to example 1, we'll use that to find a pattern for the sum for other sums. Let X = the sum we are looking for (1 + 2 + 3 + 4 + 5 + 6). Notice what happens if X is written down twice – first in ascending order, then in descending order – and then the two rows are added by adding lined-up columns:

$$\begin{array}{r}X=1+2+3+4+5+6\\ +\underline{X}=\underline{6}+\underline{5}+\underline{4}+\underline{3}+\underline{2}+\underline{1}\\ 2X=7+7+7+7+7+7\end{array}$$

The left side equals 2X, which is twice the actual sum we want. How many columns of numbers are on the right hand side of the equals sign? _____ Note: the number of columns is the same as the amount of numbers in the actual sum we are trying to find. What does each column on the right side of the equals sign add up to? _____ Since the right hand side of the equal sign is repeated addition, you can obtain the answer by using multiplication. What multiplication problem is this? _________ Since the left side equals the right side (2X = 42), then X = 21. Even if you didn't represent the sum using a variable, you would divide the right side by 2 because the sum is twice as large since the numbers in the sum were added twice.

Let's use this same technique to add the first 100 consecutive counting numbers. Let N = the actual sum. You fill in the third row by adding the left side, and then adding all of the columns on the right side.

$$\begin{array}{rccccccccccc}N= & 1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99+ & 100\\ +\underline{N}= & \underline{100} & + & \underline{99} & + & \underline{98} & + & \ldots & + & \underline{3} & + & \underline{2}+ & \underline{1}\end{array}$$

How many columns (it's the same answer as the amount of numbers in the sum) are there on the right? ______ Since each column adds up to the same number, this is repeated addition. State the number that the right side adds up to: ____________ That is twice as big as the actual sum, so what does the sum (1 + 2 + 3 + ... + 98 + 99 + 100) equal? ______

Hopefully, you got the answer of 5,050!

? Exercise 32

Use the previous technique to find the sum of the first 80 counting numbers. Show work

Let's try to find a formula for finding the first n counting numbers using the same technique. Before writing the sum, what is the counting number preceding n? _______ What is the counting number preceding that number? ______

Let X = the sum of the first n counting numbers. Below, we'll use the same techique to find the sum. Fill in the sum on the left side of the equals side and the sum of each column on the right.

What does each column on the right add up to? ________ Again, the right side is a repeated addition again. Multiply the number of columns (which is equal to the amount of numbers in the sum of the first n counting numbers) by the number each column adds up to.

What does the right side equal? _____________ This is twice as large as the actual sum we want.

What does the actual sum of the first n counting numbers equal? _____________

Hopefully, you got this answer: $\frac{n(n+1)}{2}$

Of course, the answer depends on n. Let's using this formula to compute the sum of the first six counting numbers, which was the first example we did. In this example, n = 6. So, plugging n = 6 into the formula, we get $(6 \cdot 7)$ / 2 = 21. Same answer!

Use the formula to find the sum of the first 80 counting numbers.

? Exercise 34

Use the formula to find this sum: 1 + 2 + 3 + ... + 248 + 249 + 250.

What if you wanted to find this sum? 55 + 56 + 57 + ... + 128 + 129 + 130

If you used the formula, then you would have the sum of all the numbers from 1 to 130 instead of only the ones from 55 on. One strategy is to use the formula and then subtract off the extra numbers you added. For instance, if you add all the numbers from 1 - 130, the extra numbers added that are not part of the sum are: 1 + 2 + 3 + ... + 52 + 53 + 54. But this sum is easy to compute since we can use the formula to get this sum! So here is the strategy:

55 + 56 + 57 + ... + 128 + 129 + 130

= (the sum of the first 130 counting numbers) – (the sum of the first 54 counting numbers)

= (1 + 2 + 3 + ... + 128 + 129 + 130) – (1 + 2 + 3 + ... + 52 + 53 + 54)

= $\frac{130(131)}{2} - \frac{54(55)}{2}$ = 8,515 - 1,485 = 7,030

✓ Example

Find the sum: 81 + 82 + 83 + ... + 198 + 199 + 200

Solution

Subtract the sum of the first 80 counting numbers from the sum of the first 200 counting numbers: 200(201)/2 – 80(81)/2 = 20,100 – 3,240 = 16,860.

? Exercise 35

Find the following sums using the strategy just shown. Show work.

a. 51 + 52 + 53 + ... + 98 + 99 + 100

b. 146 + 147 + 148 + ... + 561 + 562 + 563

c. 500 + 501 + 502 + ... + 798 + 799 + 800

There is a different strategy we can use to find the sum of consecutive whole numbers that do not begin with the number, 1. Let's look at another way to compute the sum of these whole numbers: 55 + 56 + 57 + ... + 128 + 129 + 130. We'll begin by using the same type of strategy we used at the beginning of this topic. First, let X = the sum. Write the sum down twice – first in ascending order, then in descending order – and then add the two rows by adding the individual lined-up columns. This is shown on the next page.

$$\begin{array}{rlllllll}
X = & 55+ & 56+ & 57+ & \ldots+ & 128+ & 129+ & 130 \\
\underline{X} = & \underline{130+} & \underline{129+} & \underline{128+} & \ldots+ & \underline{57+} & \underline{56} + \underline{55} & \\
2X = & 185+ & 185+ & 185+ & \ldots+ & 185+ & 185+ & 185
\end{array}$$

At this point, we can see the similarity to how we derived the sum of the first n counting numbers, but there is one big difference here. It's not clear exactly how many addends of 185 are on the right-hand side of the equals sign. Most people will say that there are 130 – 55, or 75 of them, but actually that reasoning (subtracting the first number from the last number) is not correct. For instance, if you were adding the numbers from 1 to 130, most people will agree that there are 130 numbers being added, which is NOT the same answer you would get if you subtracted the first number from the last number since 130 – 1 only equals 129. Look at simpler example: 17 + 18 + 19 + 20. It's clear there are 4 numbers in the sum, but if you did the subtraction 20 – 17, you would get 3, which is the incorrect answer. One way to figure out how many numbers are in the sum is to decide how many numbers are missing from the sum if it did begin with 1. Look at the sum again: 17 + 18 + 19 + 20 has the first 16 numbers missing, so instead of 20 numbers in the sum (which is how many there would be if we started at 1), there are 20 – 16 numbers in the sum. Okay, back to figuring out how many numbers are actually in the sum above.

How many numbers are in the sum: 55 + 56 + 57 + ... + 128 + 129 + 130? ________

So, the right side of the equation has 76 185's added together, which is 76 ⋯ 185, which equals 14,060. But, this is twice as big as the actual sum, so after dividing by 2, we get the actual sum of 7,030. This is the same answer we got when we did this problem on the previous page using a different strategy.

Exercise 36

Find the following sums using the strategy just shown. Show work.

a. 51 + 52 + 53 + ... + 98 + 99 + 100

b. 146 + 147 + 148 + ... + 561 + 562 + 563

c. 500 + 501 + 502 + ... + 798 + 799 + 800

d. Challenge: k + (k + 1) + (k + 2) + ... + (n - 2) + (n - 1) + n

So far, you have learned how to easily find the sum of several consecutive whole numbers. Let's take it one step further. What if the sum you want to find are numbers that are not consecutive. Depending on the the sequence of numbers, the sum may or may not be easy to find. We are going to look at sums that have consecutive numbers added together in disguise. For instance, look at the following sum:

7 + 14 + 21 + ... + 693 + 700

Exercise 37

What do you notice about the numbers added together?

Exercise 38

Hopefully, you noticed that all of the numbers were multiples of seven, or perhaps you noticed you add seven to each number to get the next number. Take the sum and rewrite the problem by factoring out a 7; just fill in the blanks below:

7 + 14 + 21 + ...+ 693 + 700 = 7 (____ + ____ + ____ + ... + ____ + ____)

Exercise 39

Now, you should be able to figure out the sum of the numbers in parentheses. ***Show your work to figure out the sum***. Then answer a, b and c.

a. What is the sum of the numbers in parentheses? ___________

b. So, the sum 7 + 14 + 21 + 28 + ... + 700 becomes 7 __________

c. Therefore, 7 + 14 + 21 + 28 + ... + 700 = ________________

Exercise 40

Find the following sums. ***Show your work.***

a. 8 + 16 + 24 + ... + 992 + 1000 = ____________

b. 11 + 22 + 33 + ... + 935+ 946 = _____________

c. 20 + 40 + 60 + 80 + ... + 2980 + 3000 = _____________

Okay, just one more twist...this puts it all together. Consider this sum:

112 + 116 + 120 + ... + 524 + 528

Exercise 41

This time, what do you notice about the numbers added together?

Exercise 42

Hopefully, you noticed all the numbers were multiples of four, or that you added four to each number to get the next number. Take the sum and rewrite the problem by factoring out a 4; just fill in the blanks below:

112 + 116 + 120 + ... + 524 + 528 = 4 (____ + ____ + ____ + ... + _____ + _____)

Exercise 43

Now, you should be able to figure out the sum of the numbers in parentheses. Note that the sum does not start with a 1. ***Show your work*** to figure out the sum. Then answer a, b and c.

a. What is the sum of the numbers in parentheses? __________

b. So, the sum 112 + 116 + 120 + ... + 524 + 528 becomes 4 _________

c. Therefore, 112 + 116 + 120 + ... + 524 + 528 = _______________

Exercise 44

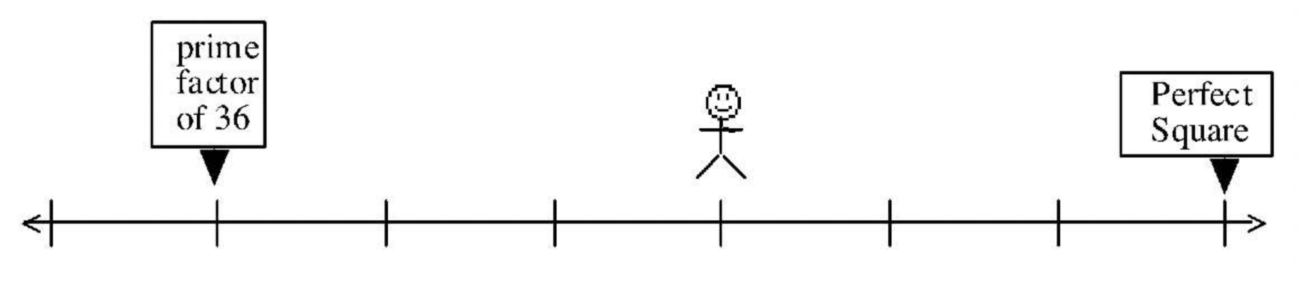
Find the following sums. ***Show your work.***

a. 85 + 90 + 95 + ... + 735 + 740 = ____________

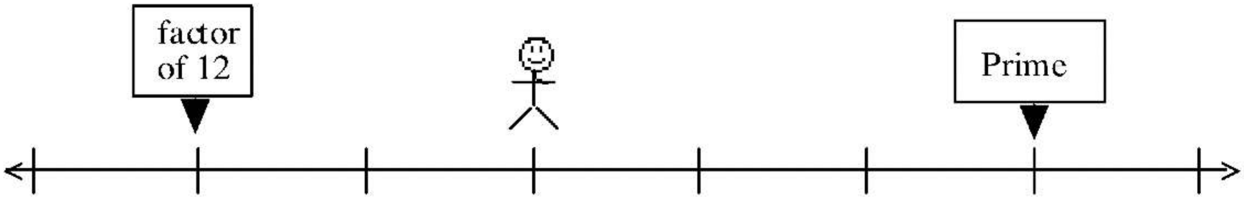
b. 430 + 473 + 516 + ... + 2838+ 2881 = _____________

Here are some problems for you to figure out what numbers are on the number line. For each problem, figure out where on the number line (what number) the man might be standing. Where there is more than one possibility, only list numbers between 1 and 1000.

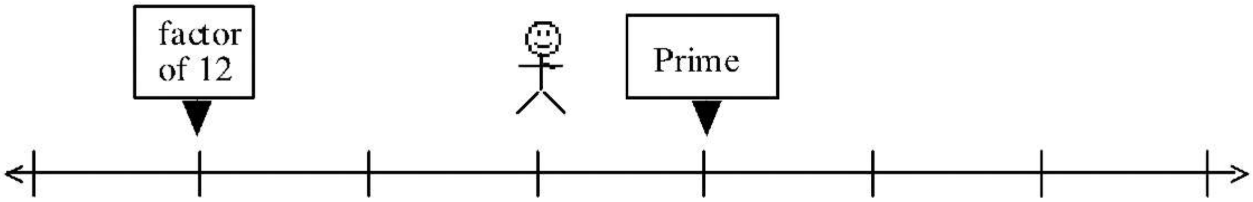
Exercise 45



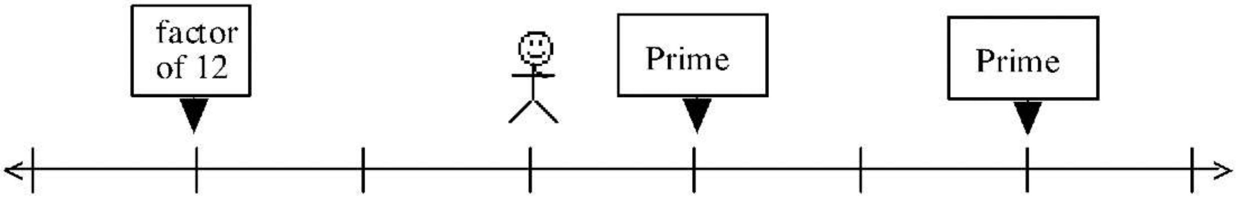
Exercise 46


factor of 12
Prime

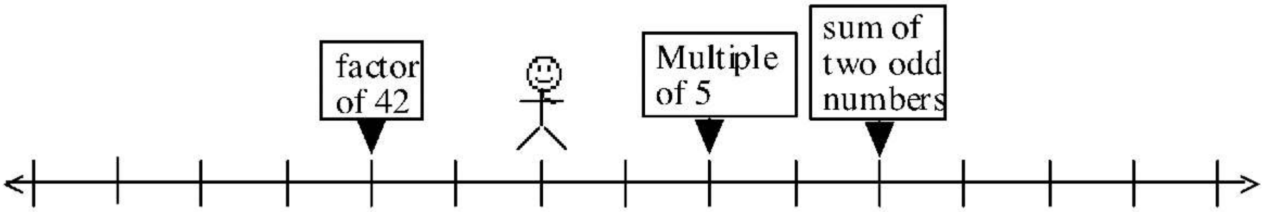
Exercise 47


factor of 12
Prime

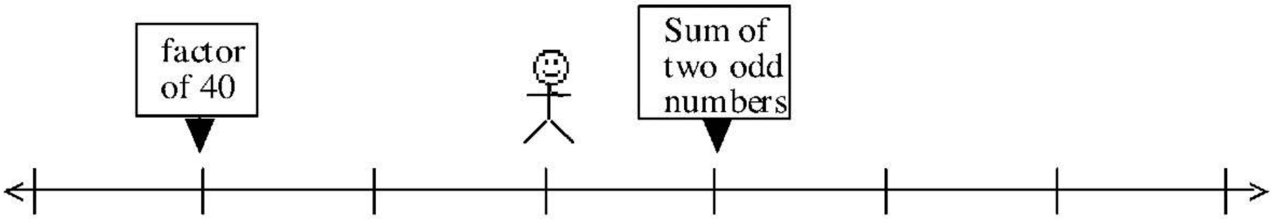
Exercise 48


factor of 12
Prime
Prime

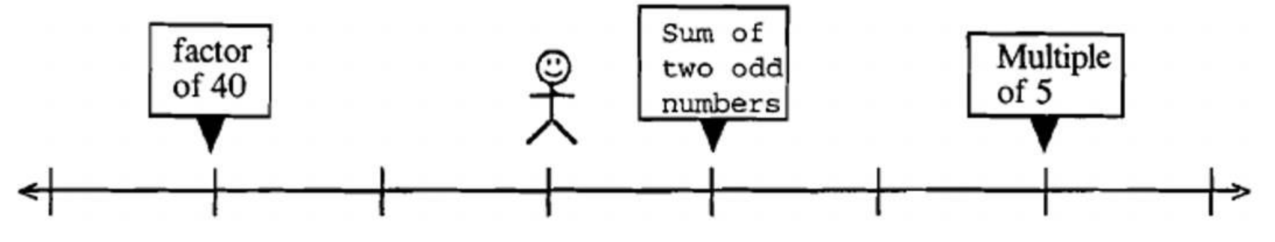
Exercise 49


factor of 42
Multiple of 5
sum of two odd numbers

Exercise 50

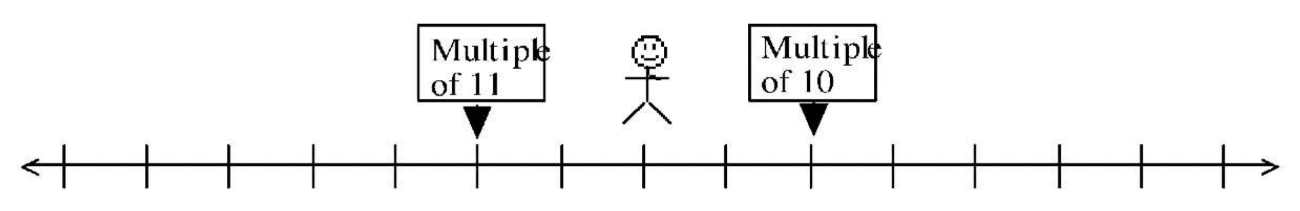

factor of 40
Sum of two odd numbers

Exercise 51

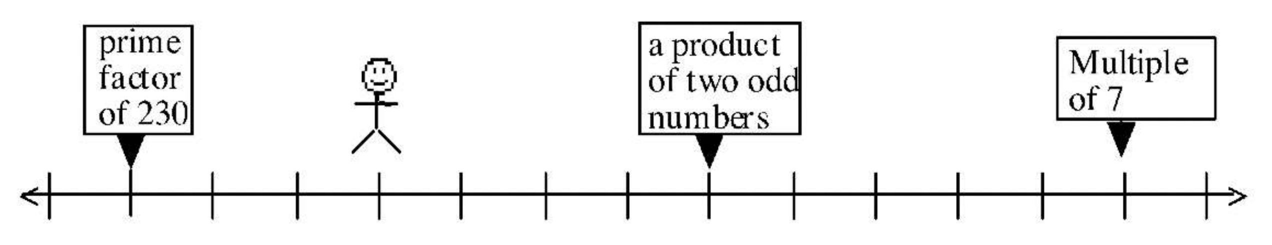

factor of 40
Sum of two odd numbers
Multiple of 5

Exercise 52

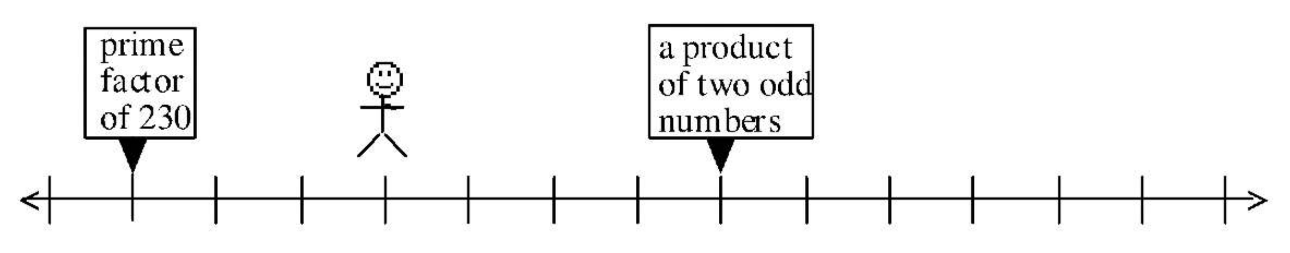
(There are actually nine possibilities between 1 and 1000 for this problem.)


Multiple of 11
Multiple of 10

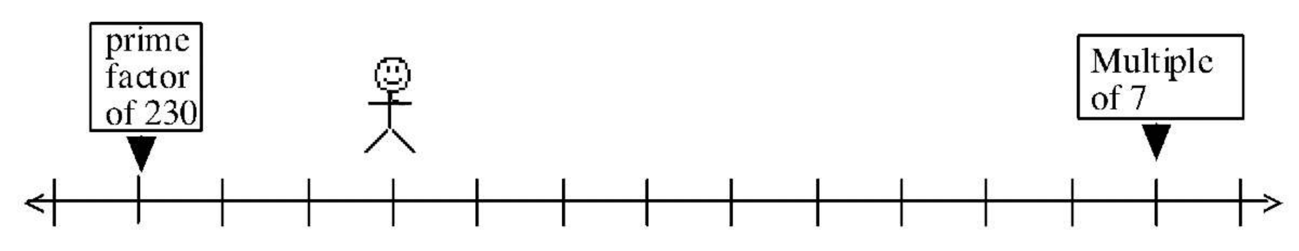
Exercise 53


prime factor of 230
a product of two odd numbers
Multiple of 7

Exercise 54


prime factor of 230
a product of two odd numbers

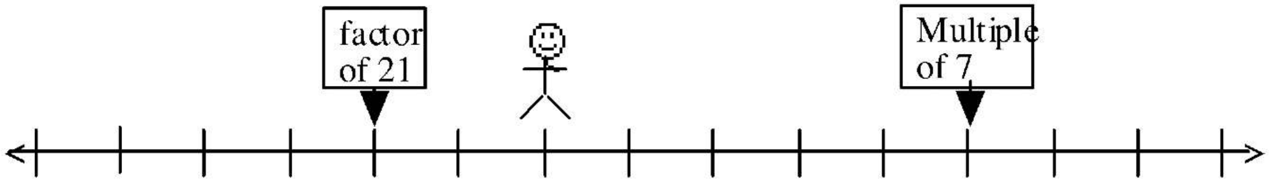
Exercise 55


prime factor of 230
Multiple of 7

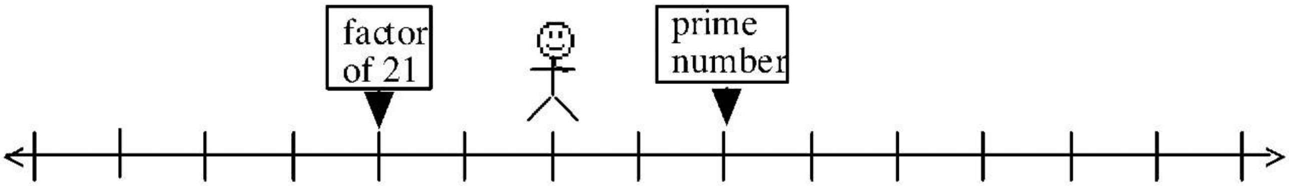
Exercise 56

Which of the clues in #53 was was not needed if the man is only standing on one single number? #54 and #55 should help you answer this. For instance, if you got more than one possibility for #54 or #55, then that problem didn't provide enough clues. For the one that gave exactly one possibility, that was enough clues, so the clue from #53 that was missing wasn't really needed.

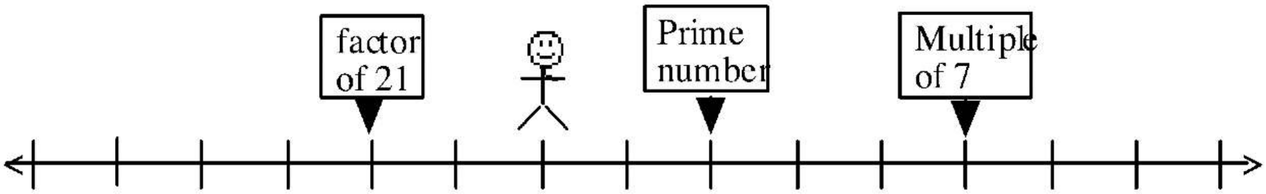
Exercise 57



Exercise 58



Exercise 59



Exercise 60

If the man in exercise #59 is standing on only one number in this problem, are all three clues given needed?

Yes or No:__________ Explain:

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8.4: Homework

- Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)
- Start a new module on the front side of a new page and write the module number on the top center of the page.
- Answers without supporting work will receive no credit.
- Some solutions are given in the solutions manual.
- You may work with classmates but do your own work.

? HW #1

Find GCF(252, 350) using:

a. prime factorization;
b. Old Chinese Method
c. Euclidean Algorithm
d. Compute LCM(252,350)

? HW #2

Find GCF(140, 315) using:

a. prime factorization;
b. Old Chinese Method
c. Euclidean Algorithm
d. Compute LCM(252,350)

? HW #3

Use the Euclidean Algorithm to compute the greatest common factor of the numbers given. Use correct notation, and show each step. Then, show how you check your answer. Also, compute the LCM of the two numbers.

a. GCF(3525, 658) LCM(3525, 658)
b. GCF(1075, 1548) LCM(1075, 1548)

? HW #4

a. If 6|482__354 What digits could go on the blank? Explain.
b. If 6|482354__ What digits could go on the blank? Explain.

? HW #5

State whether each of the following statements is true or false. If it is false, provide a counterexample. If it is true, provide an example.

a. If (a + b)|c, then a|c and b|c

b. If a|b and a|c, then a|(bc)

c. If a|b and a|(b + c), then a|c

d. If a|bc, then a|b and a|c

e. If a|b and a|c, then a|(b + c)

? HW #6

Write the prime factorization for the following numbers. If it is prime, write "prime" and explain how you know it is prime.

a. 371	b. 429	c. 197	d. 287

? HW #7

Assume m and n are composite whole numbers in each of the following. Find the following. Then provide an example using numbers for m (and n where used). Remember not to use prime numbers in your example.

a. GCF(m,m) =

b. LCM(m,m) =

c. GCF(m,0) =

d. GCF(m,1) =

e. If GCF(m,n) = 1, then LCM(m,n) =

f. If GCF(m,n) = m, then LCM(m,n) =

g. If LCM(m,n) = mn, then GCF(m,n) =

? HW #8

a. Formally prove that the sum of two odd numbers is even.
b. Formally prove that the product of two odd numbers is odd.

? HW #9

Find the following sums using methods from this module: Show all work

a. 1 + 2 + 3 + . . . + 313 + 314 + 315 =

b. 111 + 112 + 113 + . . . + 287 + 288 + 289 =

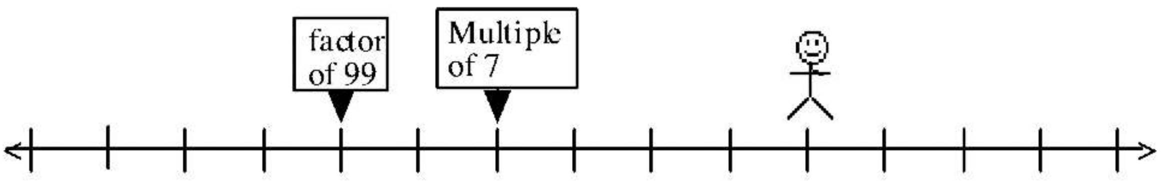
c. 15 + 30 + 45 + . . . + 900 + 915 + 930 =

d. 102 + 105 + 108 + . . . + 300 + 303 + 306 =

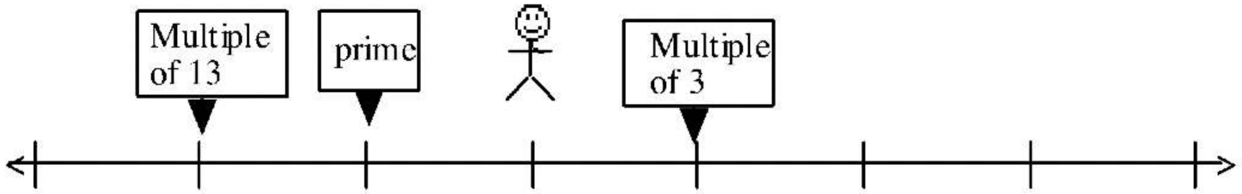
? HW #10

On each number line, state all whole number possibilities less than 100 that the man could be standing on.

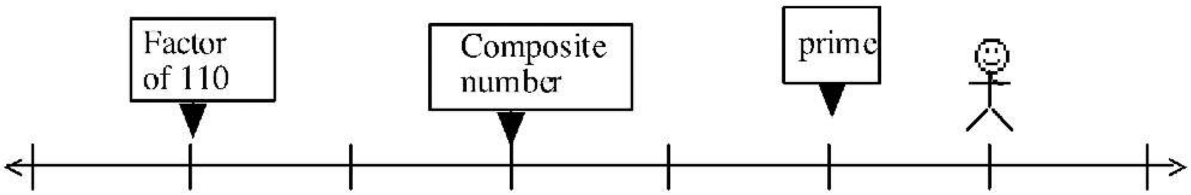
a.


factor of 99
Multiple of 7

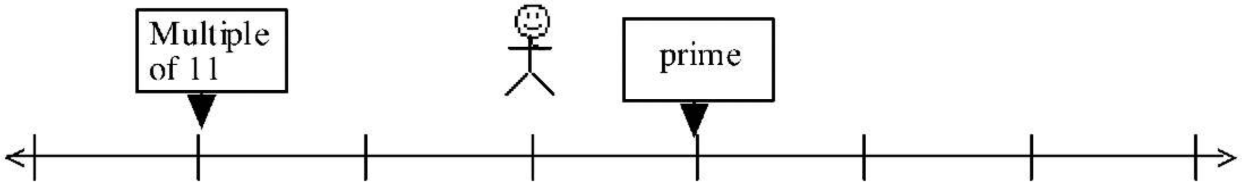
b.


Multiple of 13
prime
Multiple of 3

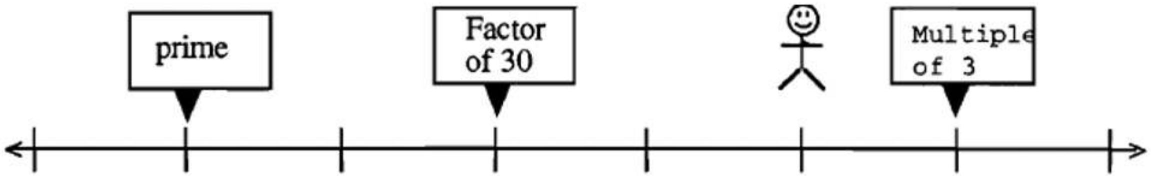
c.


Factor of 110
Composite number
prime

d.


Multiple of 11
prime

e.


prime
Factor of 30
Multiple of 3

? HW #11

The factors of a number that are less than the number itself are called *proper factors*. For instance, the proper factors of 10 are 1, 2 and 5. A number is classified as deficient if the sum of its proper factors is less than the number itself. 10 is a deficient number since 1 + 2 + 5 < 10. A number is classified as abundant if the sum of its proper factors is greater than the number itself. For instance, the proper factors of 18 are 1, 2, 3, 6, and 9. 18 is a deficient number since 1 + 2 + 3 + 6 + 9 > 18. A number is classified as perfect if the sum of its proper factors equals the number itself. For each number, list its proper factors. Then find the sum of its proper factors. Then, classify each number as deficient, abundant or perfect.

a. Proper factors of 6: ___________ ; Sum: _____ ; Classification:

b. Proper factors of 7: ___________ ; Sum: _____ ; Classification:

c. Proper factors of 8: ___________ ; Sum: _____ ; Classification:

d. Proper factors of 9: ___________ ; Sum: _____ ; Classification:

e. Proper factors of 11: ___________ ; Sum: _____ ; Classification:

f. Proper factors of 12: ___________ ; Sum: _____ ; Classification:

? HW #12

Are prime numbers deficient, perfect, or abundant? ________ Explain why.

? HW #13

Answer true or false for each of the following. If it is true, provide an example. If it is false, provide a counterexample.

a. Every prime number is odd.

b. If a number is divisible by 6, then it is divisible by 2 and 3.

c. If a number is divisible by 2 and 6, then it is divisible by 12.

d. If a number is divisible by 3 and 4, then it is divisible by 12.

e. If $a \neq b$, then GCF(a, b) < LCM(a, b).

f. If 6 is a factor of mn, then 6 is a factor of m or a factor of n.

g. If 5 is a factor of mn, then 5 is a factor of m or a factor of n.

? HW #14

Can the sum of two odd prime numbers be a prime number? Explain why or why not.

? HW #15

Find the least common multiple of the following sets of numbers:

a. LCM(2, 4, 5, 7, 8, 12, 14, 15)

b. LCM(3, 4, 6, 8, 9, 10, 12, 18)

? HW #16

If GCF(30, x) = 6 and LCM(30, x) = 180, then what is x? (Hint: see page 65)

? HW #17

The theory of biorhythm states that your physical cycle is 23 days long, your emotional cycle is 28 days long, your intellectual cycle is 33 days long. If your cycles all occur on the same day, how many days until your cycles again occur on the same day? About how many years is this?

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CHAPTER OVERVIEW

9: Rational Numbers

The teacher announced that the class would be learning more about rational numbers.

An English major raised his hand and said that numbers weren't rational After all; a number certainly couldn't think let alone be rational!

'The teacher said she didn't mean that kind of rational.

Another student said she'd heard numbers could be rational; as well as irrational; but she didn't think it had anything to do with thinking.

The teacher agreed. The root in the word 'rational' is ratio". A rational number is simply a number that can be written as the ratio of an integer and a nonzero integer.

Someone asked if that definition would be on the test, and if so, would it be on an English test or math test?

The teacher finally said that they were going to learn more about fractions today.

- 9.1: Understanding Fractions With The C-Strips
- 9.2: Rational Numbers
- 9.3: Facts About Comparing Fractions
- 9.4: Decimals
- 9.5: Homework

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9.1: Understanding Fractions With The C-Strips

When dealing with fractions, the denominator tells you how many equal parts it takes to make 1 unit. The numerator tells you how many of those equal parts are taken.

✓ Example 1

If H (hot pink) represents 1 unit, then which C-strip represents $\frac{3}{4}$?

Solution

The denominator is 4; so it takes 4 equal parts to make one unit, and each of those 4 equal parts = $\frac{1}{4}$. Find the C-strip such that a train of 4 of them is as long as one unit (H). Since a train of 4 light green C-strips (L) is the length of H (1 unit), then each light green C-strip makes up one part of a whole, and is therefore worth $\frac{1}{4}$. You need to find which C-strip represents $\frac{3}{4}$, so make a train of 3 light green C-strips and find the C-strip having this length. This would be the Blue (B) C-strip. Therefore, the answer is B.

For exercises 1 - 13, explain how to find the solution. Do each step using your C-strips.

a. State how many C-strips (each an equal part of the whole) make up one unit.

b. State which C-strip makes up one part of the whole.

c. State the fraction that the C-strip in part b represents.

d. State how many of the C-strips in part b you need to make into a train.

e. State which C-strip is the length of the train you made in part c, this is the answer!!!

✓ Exercise Example

If H represents 1 unit, then which C-strip represents $\frac{3}{4}$?

a. **4**	b. **L**	c. $\frac{1}{4}$	d. **3**	e. **B**

? Exercise 1

If H represents 1 unit, then which C-strip represents $\frac{1}{6}$?

a. ___	b. ___	c. ___	d. ___	e. ___

? Exercise 2

If H represents 1 unit, then which C-strip represents $\frac{5}{12}$?

a. ___	b. ___	c. ___	d. ___	e. ___

? Exercise 3

If O represents 1 unit, then which C-strip represents $\frac{2}{5}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 4

If B represents 1 unit, then which C-strip represents $\frac{2}{3}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 5

If D represents 1 unit, then which C-strip represents $\frac{5}{3}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Exercise 6

If N represents 1 unit, then which C-strip represents $\frac{3}{4}$?

a. ___	b. ___	c. ___	d. ___	e. ___

Example 2

If N (brown) represents $\frac{4}{5}$, then which C-strip represents 1 unit?

Solution

The denominator is 5, so it takes 5 equal parts to make up 1 whole unit, where each equal part is $\frac{1}{5}$. Since N is only $\frac{4}{5}$, then a train of only 4 of the 5 equal parts will be the length of N. A train of 4 red C-strips is the same length as N. So a red C-strip is one of the 5 equal parts that make up a whole. Since it takes 5 equal parts (5 reds) to make one unit, form a train of 5 reds and see which C-strip has this length. It is the orange C-strip (O). Therefore, the answer is O.

For exercises 7 - 11, explain how to find the solution. Do each step using your C-strips.

a. State how many C-strips will make up the named C-strip stated in the problem. Look at the numerator.

b. Which C-strip makes up one equal part?

c. State the fraction that the C-strip in part b represents. Look at the denominator.

d. State how many of the C-strips in part b will make up one unit.

e. Form the unit by making a train from the equal parts (C-strip in part b) and state which C-strip has the same length as that train.

Exercise Example

If N represents $\frac{4}{5}$, then which C-strip is 1 unit?

a. 4	b. R	c. 1/5	d. 5	e. O

Exercise 7

If D represents $\frac{1}{2}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

? Exercise 8

If L represents $\frac{1}{3}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

? Exercise 9

If P represents $\frac{2}{3}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

? Exercise 10

If N represents $\frac{4}{5}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

? Exercise 11

If R represents $\frac{2}{5}$, then which C-strip is 1 unit?

a. ___	b. ___	c. ___	d. ___	e. ___

The type of problems on this page are a little more challenging. They take more steps. From the first piece of information, figure out which C-strip is the whole unit – just like you did in problems 7 - 11. Then, start over using that unit C-rod, and figure out the second part of the question – just like you did in problems 1 - 16.

✓ Example 3

If N represents $\frac{2}{3}$, then which C-strip represents $\frac{3}{4}$?

Solution

Begin these the same way the previous problems were done by first figuring out what the unit C-strip is. After doing the same steps you did for exercises 14-18, you will conclude that **H is the unit C-strip**. Now, on to part 2: Start over with H as the unit C-strip, and find $\frac{3}{4}$ the same way you did it for the first 13 exercises. The key is to start over by looking only at the stated unit C- strip (H in this case), and not getting that confused with the first part of the problem. In other words, now that you have determined that the unit is H, determine what is 3/4 (in relation to the unit - H). You will find that **the answer is B.**

For exercises 12-14, discuss how to find the solution. Do each step using your C-strips.

a. State which C-strip is one unit.

b. State which C-strip is the answer.

✓ Example 3

If N represents $\frac{2}{3}$, then which C-strip represents $\frac{3}{4}$?

a. H	b. B

? Exercise 12

If P represents $\frac{2}{3}$, then which C-strip represents $\frac{3}{2}$?

a. _____	b. _____

? Exercise 13

If O represents $\frac{5}{6}$, then which C-strip represents $\frac{3}{4}$?

a. _____	b. _____

? Exercise 14

If D represents $\frac{2}{3}$, then which C-strip represents $\frac{1}{3}$?

a. _____	b. _____

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9.2: Rational Numbers

Materials: Fraction Circles, Fraction Arrays, Multiple Strips, C-strips

Use your marked fraction circles to do these first few activities.

Exercise 1

Take one wedge from each fraction circle (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9, 1/10, and 1/12). One fraction is bigger than another if it covers more space than the other fraction. Compare them and put them in order from smallest to largest, and write them in order below, using the less than symbol (<).

Exercise 2

From what you discovered in exercise 1, circle the larger fraction in each case.

a. 1/90 or 1/95	b. 1/32 or 1/33

If you take 5 of the fraction pieces that say 1/8, then together you have 5/8. Use this fact to do the next exercise.

Exercise 3

Use the fraction circles to compare 5/6, 5/8, 5/9, 5/10, and 5/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 4

From what you discovered in exercise 3, circle the larger fraction in each case.

a. 15/37 or 15/40	b. 89/100 or 89/200

Exercise 5

Use the fraction array to order these fractions: 4/5, 4/6, 4/7, 4/8, 4/9, 4/10, 4/11 and 4/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 6

Explain the pattern you learned from doing exercises 1-5.

Exercise 7

Use fraction circles to compare these fractions: 1/2, 2/3, 3/4, 4/5, 5/6, 7/8, 8/9, 9/10, and 11/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 8

Use the fraction array to order these fractions: 1/2, 2/3, 3/4, 4/5, 5/6, 7/8, 8/9, 9/10, 11/12. Put them in order from smallest to largest using the less than symbol (<).

Exercise 9

From what you discovered in exercises 7 and 8, circle the larger fraction in each case.

a. 89/90 or 94/95	b. 56/57 or 31/33

Exercise 10

Order these fractions: 45/46, 71/72, 34/35, 99/100, 25/26, 13/14, 51/52. Put them in order from smallest to largest using the less than symbol (<).

Exercise 11

Explain the pattern you learned from doing exercises 7 - 10.

Exercise 12

Draw a model of your own that would help to convince someone why 4/5 was bigger than 2/3.

A rational number is simply the formal name for a fraction. Below is a formal definition of a rational number.

A **rational number** is a number that **can** be written as the quotient (ratio) of an integer, m, and a nonzero integer, n, which is written like this: $\frac{m}{n}$.

When written in this form, m is called the numerator and n is called the denominator. If GCF(m, n) = 1 (i.e., m and n have no factors in common), then the rational number is said to be in simplest (or reduced) form.

The following is one way to represent a rational number, $\frac{m}{n}$.

Divide a (defined) unit into an equal number of n parts or subsets. Which means n parts make up a whole. Each one of these parts (or subsets) represents $\frac{1}{n}$. Then, m of those parts represents $\frac{m}{n}$.

Examples

For each rational number, $\frac{m}{n}$,

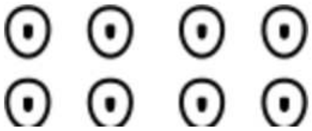
a. Define a unit in terms of dots. State the value of each dot.
  i. Divide the unit into an equal number (the denominator) of parts.
  ii. Show a representation of $\frac{1}{n}$.
  iii. Show a representation of $\frac{m}{n}$.
b. Define the unit differently, and do the three parts (i, ii and iii) again.

Example 1: 3/8

a. Let one unit be defined by 8 dots, as shown below. Each dot = 1/8

. . . .
. . . .

i. The unit is divided into 8 equal parts, as shown below.



ii. Below is a representation of 1/8, which is one of the equal parts shown in part b.

.

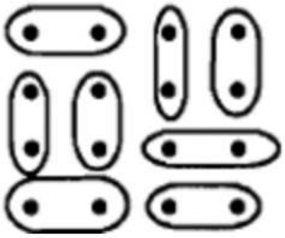
iii.Below is a representation of 3/8.

. . .

b. Let one unit be defined by 16 dots, as shown below. Each dot = 1/16

. . . .
. . . .
. . . .
. . . .

i. The unit is divided into 8 equal parts, as shown below.



ii. Below is a representation of 1/8, which is one of the equal parts shown in part b.

. .

iii.Below is a representation of 3/8.

. . . . . .

Notice there are 3 dots for the answer in part a, where each dot represents 1/8. So, the 3 dots represents the number 3/8. For part b, the answer has 6 dots. Since each dot represents 1/16, this also represents 6/16. Therefore, 3/8 and 6/16 must represent the same number. **Two fractions that represent the same number are called equivalent fractions.**

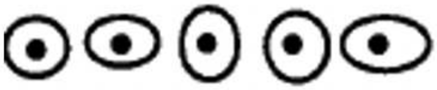
It is crucial to define your unit before beginning any exercise!!

✓ Example 2: 2/5

a. Let one unit be defined by 5 dots, as shown below. Each dot = 1/5

. . . . .

i. The unit is divided into 5 equal parts, as shown below.



ii. Below is a representation of 1/5, which is one of the equal parts shown in part b.

.

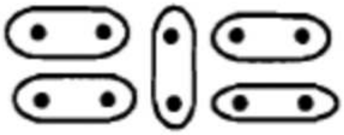
iii.Below is a representation of 2/5.

. .

b. Let one unit be defined by 10 dots, as shown below. Each dot = 1/10

. . . . .
. . . . .

i. The unit is divided into 5 equal parts, as shown below.



ii. Below is a representation of 1/5,which is one of the equal parts shown in part b.

. .

iii.Below is a representation of 2/5

. . . .

Notice there are 2 dots for the answer in part a, where each dot represents 1/5. So, the 2 dots represents the number 2/5. For part b, the answer has 4 dots. Since each dot represents 1/10, this also represents 4/10. Therefore, 2/5 and 4/10 must represent the same number. **Two fractions that represent the same number are called equivalent fractions.**

IT IS CRUCIAL TO DEFINE YOUR UNIT BEFORE BEGINNING ANY EXERCISE!!

? Exercise 13

Use the method in the last two examples to show two different representations for 3/4.

? Exercise 14

Use your fraction arrays to determine all fractions on the fraction array that are equivalent to 1/2. Do this by finding 1/2 on the array, and seeing what other numbers are the same length.

? Exercise 15

Use your fraction arrays to determine all fractions on the fraction array that are equivalent to 2/3. Do this by finding 2/3 on the array, and seeing what other numbers are the same length.

You'll now be using your multiple strips to identify equivalent fractions. To use your strips, you need to cut out the strips by rows. To find fractions equivalent to 3/7, align the 3 strip above the 7 strip as shown below:

3	6	9	12	15	18	21	24	27	30	33	36
7	14	21	28	35	42	49	56	63	70	77	84

Now, you can see eleven other equivalent fractions for 3/7: 6/14, 9/21, 12/28, 15/35, 18/42, 21/49, 24/56, 27/63, 30/70, 33/77, and 36/84.

? Exercise 16

Use your multiple strips to write 6 fractions equivalent to 2/9

? Exercise 17

Use your multiple strips to write 6 fractions equivalent to 4/5

? Exercise 18

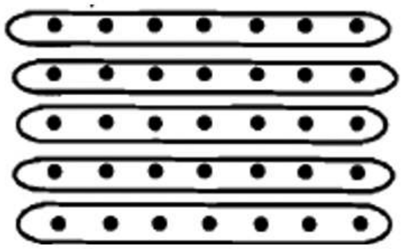
What is the rule for finding a fraction equivalent to a given fraction? Give an example how to find some fractions that are equivalent to 5/6.

We are now going to work again with models, using dots, to compare two fractions, add two fractions, subtract two fractions, or multiply or divide two fractions. For each problem, it is CRUCIAL that you begin each problem by explicitly stating the following:

1. Be specific about what you are using for the unit. It will be easiest if you use an array of dots, where the denominator of one fraction is the number of rows in the array, and the denominator of the other fraction is the number of columns in the array.

2. State the value of each dot, each column and each row.

Okay, let's go on to an example: **Use models to compare 2/5 and 3/7.**

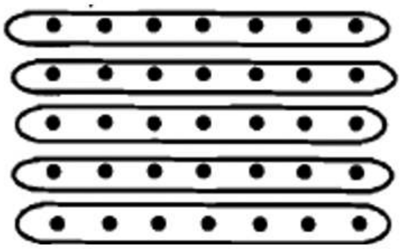
Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit.	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts by circling the rows, as shown below:  Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = ...	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = ...
Step 4: Now that we have properly defined the unit, we are ready to show what 2/5 looks like. Since 1/5 is 1 row of dots, then 2/5 must be 2 rows of dots. Therefore, 2/5 is shown below: . . . . Notice that 2/5 contains 14 dots, which is equivalent to 14/35.	Step 5: Similarly, we can show what 3/7 looks like. Since 1/7 is 1 column of dots, then 3/7 must be 3 columns of dots. Therefore, 3/7 is shown below: . . . . Notice that 3/7 contains 15 dots, which is equivalent to 15/35.	

Since 2/5 contains less dots than 3/7, 2/5 must be less than 3/7. Answer: **2/5 < 3/7**.

? Exercise 19

Compare 3/4 and 4/5 using models. Show **all** of the steps, and explain the procedure as shown in the previous example.

Okay, let's go on to an example of addition: **Use models to add 2/5 and 3/7.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit. ·	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts by circling the rows, as shown below:  Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = · · · · · · ·	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = · · · · ·
Step 4: Now that we have properly defined the unit, we are ready to show what 2/5 looks like. Since 1/5 is 1 row of dots, then 2/5 must be 2 rows of dots. Therefore, 2/5 is shown below: · · · · · · · · · · · · · · Notice that 2/5 contains 14 dots, which is equivalent to 14/35.	Step 5: Now, we add the two models together, as shown below. **The answer is 29/35.** · · · · · · · · · · · · · · · Notice that 3/7 contains 15 dots, which is equivalent to 15/35.	

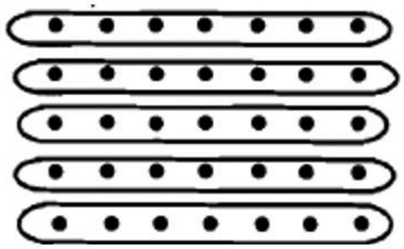
Step 5: Now, we add the two models together, as shown below. **The answer is 29/35.**

$$\begin{array}{ccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} + \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} = 29 \text{ dots } = 29/35$$

? Exercise 20

Add 1/4 and 2/5 using models. Show all of the steps, and explain the procedure as shown in the previous example.

Okay, let's go on to an example of subtraction: **Use models to do the following subtraction: 3/7 – 2/5.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit.	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts by circling the rows, as shown below:  Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = ...	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = ...
Step 4: Now that we have properly defined the unit, we are ready to show what 2/5 looks like. Since 1/5 is 1 row of dots, then 2/5 must be 2 rows of dots. Therefore, 2/5 is shown below: . . . . Notice that 2/5 contains 14 dots, which is equivalent to 14/35.	Step 5: Similarly, we can show what 3/7 looks like. Since 1/7 is 1 column of dots, then 3/7 must be 3 columns of dots. Therefore, 3/7 is shown below: . . . . Notice that 3/7 contains 15 dots, which is equivalent to 15/35.	

Now, we subtract, as shown below. **The answer is 1/35.**

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} - \begin{array}{ccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} = 1 \text{ dot } = 1/35$$

? Exercise 21

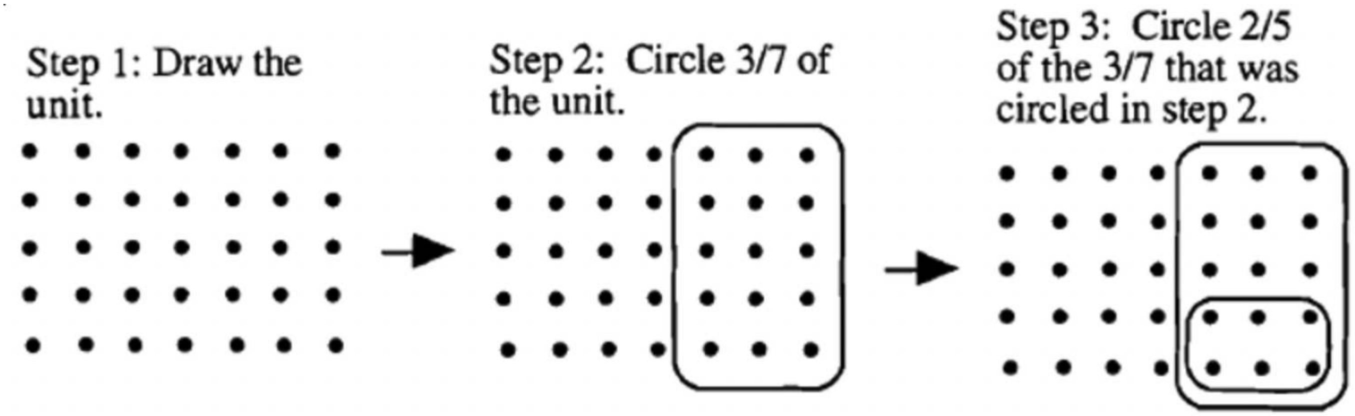
Do the following subtraction using models: 7/8 – 2/3. Show all of the steps, and explain the procedure as shown in the previous example.

Multiplying fractions is a little bit trickier. The repeated addition model doesn't make sense when you are multiplying two fractions. When you see two fractions multiplied together, like 2/5 · 3/7, think of this as 2/5 OF 3/7. In other words, you have to take 3/7 of a unit, and then you have to take 2/5 of that. Note that you have to represent the second fraction BEFORE you can do the multiplication.

Okay, let's go on to an example of multiplication: **Use models to do the following multiplication: 2/5 · 3/7.**

Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. Since there are 35 dots to a unit, each dot = 1/35 of a unit:	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts. Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = ...	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = ...
Step 4: Now that we have properly defined the unit, we first have to show what 3/7 (the second number in the multiplication) looks like. Since 1/7 is 1 column of dots, then 3/7 must be 3 columns of dots. Therefore, 3/7 is shown below:	Step 5: This is where there is a difference in how you proceed. You now need to find 2/5 of the 3/7 that is shown in step 4. Since there are 5 rows of dots in step 4, each row (of 3 dots) represents 1/5. So, you only want 2 rows of the dots from step 4, as shown below. . . . . . . This is the answer. There are 6 dots, so this represents 6/35.	

Multiplication can be shown all on one unit by first showing the unit, second circling the part that represents the second fraction in the multiplication, and then circling the part that represents the first fraction in the multiplication. Below is the sequence of steps for this problem.

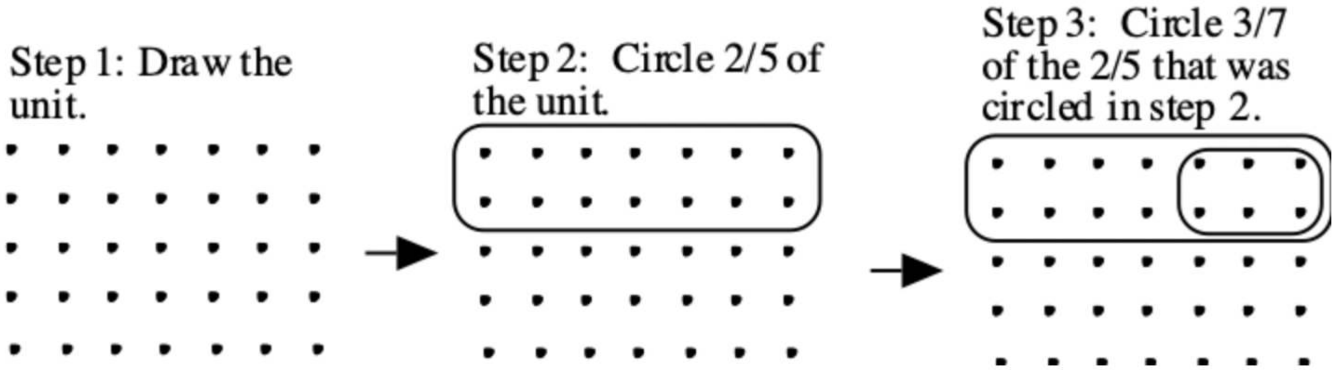

Step 1: Draw the unit.
Step 2: Circle 3/7 of the unit.
Step 3: Circle 2/5 of the 3/7 that was circled in step 2.

Okay, let's see what it looks like if we switch the order of the fractions: **Use models to do the following multiplication:** $\mathbf{3/7 \cdot 2/5}$**.**

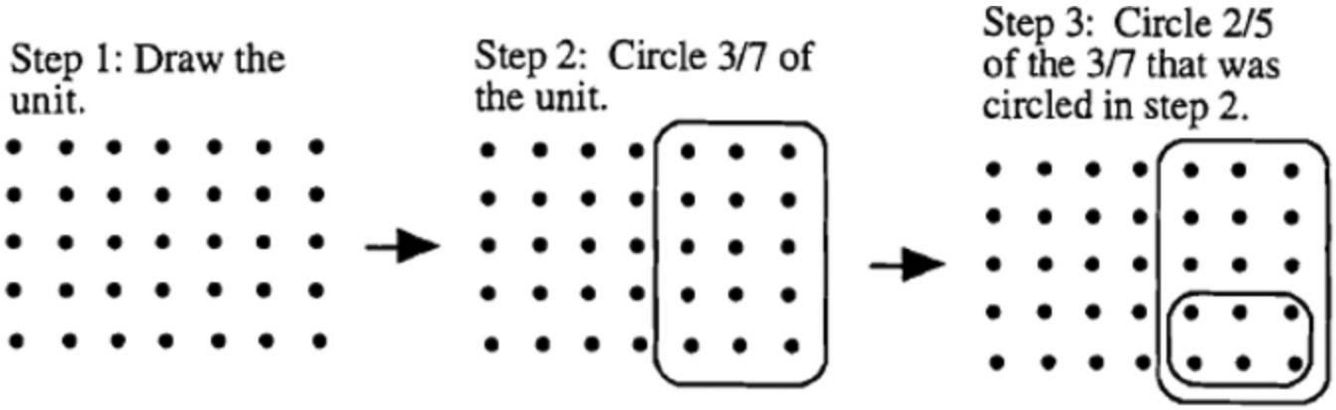
Step 1: Let 1 unit = 5 rows of dots by 7 columns of dots, for a total of 35 dots, as shown below. There are 35 dots to a unit, so each dot = 1/35 of a unit.	Step 2: Since there are 5 rows, the unit can be broken up into 5 equal parts. Therefore, each row is 1/5 of a unit. Notice there are 7 dots in 1/5 of a unit: 1/5 = ...	Step 3: Similarly, since there are 7 equal columns, each column is 1/7 of a unit: 1/7 = ...

Step 4: Now that we have properly defined the unit, we first have to show what 2/5 (the second number in the multiplication) looks like. Since 1/5 is 1 column of dots, then 2/5 must be 2 columns of dots. Therefore, 2/5 is shown below:	Step 5: You now need to find 3/7 of the 2/5 that is shown in step 4. Since there are 7 columns of dots in step 4, each column (of 2 dots) represents 1/7. So, you only want 3 rows of the dots from step 4, as shown below. . . . . . . This is the answer. There are 6 dots, so this represents 6/35.	

Again, this multiplication can be shown all on one unit by first showing the unit, second circling the part that represents the second fraction in the multiplication, and then circling the part that represents the first fraction in the multiplication. Below is the sequence of steps for this problem.


Step 1: Draw the unit.
Step 2: Circle 2/5 of the unit.
Step 3: Circle 3/7 of the 2/5 that was circled in step 2.

Although the answer is the same for 3/7 · 2/5, and 2/5 · 3/7, the sequence of steps is not. Note the difference between step 3 above (3/7 · 2/5) and step 3 below (2/5 · 3/7).


Step 1: Draw the unit.
Step 2: Circle 3/7 of the unit.
Step 3: Circle 2/5 of the 3/7 that was circled in step 2.

? Exercise 22

Do the following multiplications using models. Show all of the steps, and explain the procedure as shown in the previous examples.

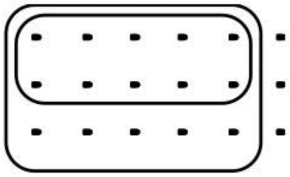
a. 4/5 · 2/3

b. 3/4 · 5/6

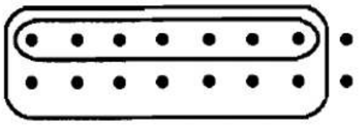
Exercise 23

By looking at the final drawing someone made to model a multiplication of two fractions, determine which multiplication was performed, and then state the answer. Circle which multiplication represents the correct choice.

a. 5/6 · 2/3 OR 2/3 · 5/6

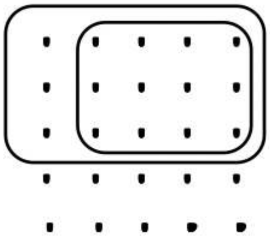


b. 1/2 · 7/8 OR 7/8 · 1/2

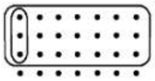


Exercise 24

If all of the dots shown for each problem represent 1 unit, determine the multiplication problem that someone did to get the answer, and state the answer.



a.



b.

Another model you can use to do multiplication is the C-strips, although it is somewhat limiting.

Example 1

Multiply $3/4 \cdot 1/2$.

Solution

This means 3/4 of 1/2 of a unit. So, we'll have to define an appropriate unit, first take 1/2 of the unit, and then take 3/4 of the answer obtained after taking 1/2. The choice for 1 unit using C-strips is to multiply the denominators together to get the length of the unit rod you need. In this case, 4 · 2 = 8, so choose the Brown C-strip as the unit.

Let 1 unit = N. First, take 1/2 of Brown (N), which is purple (P). Then, take 3/4 of purple, which is light green (L). The answer, L, represents 3/8, since the unit C-strip was N. Therefore, 3/4 · 1/2 = 3/8.

Example 2

Multiply $1/2 \cdot 3/4$.

Solution

This means 1/2 of 3/4 of a unit. So, we'll have to define an appropriate unit, first take 3/4 of the unit, and then take 1/2 of the answer obtained after taking 3/4. The choice for 1 unit using C-strips is to multiply the denominators together to get the length of the unit rod you need. In this case, 4 · 2 = 8, so choose the Brown C-strip as the unit.

Let 1 unit = N. First, take 3/4 of Brown (N), which is dark green (D). Then, take 1/42 of dark green (D), which is light green (L). The answer, L, represents 3/8, since the unit C-strip was N. Therefore, 1/2 · 3/4 = 3/8.

Example 3

Multiply $2/3 \cdot 4/5$.

Solution

This means 2/3 of 4/5 of a unit. So, we'll have to define an appropriate unit, first take 4/5 of the unit, and then take 2/3 of the answer obtained after taking 4/5. The choice for 1 unit using C-strips is to multiply the denominators together to get the length of the unit rod you need. In this case, 3 · 5 = 15. There isn't one C-strip that long, so use and orange + yellow as the unit.

Let 1 unit = O + Y. First, take 4/5 of O + Y, which is hot pink (H). Then, take 2/3 of hot pink (H), which is brown (N). The answer, N, represents 8/15, since the unit C-strip was O + Y. Therefore, 2/3 · 4/5 = 8/15.

Exercise 25

Use the C-strips to multiply 1/3 · 3/4. Explain the steps.

The process of doing multiplication using C-strips can be shown in a chart. I've shown the steps for the above three examples on the next page.

Exercise 26

Fill in the chart showing how to do the following multiplications using C-strips. The multiplication is in the first column. State an appropriate choice for the unit (name a C-strip, or sum of two C-strips) in the second column. Write the C-strip obtained after the first part of the multiplication (which is the second fraction as a part of the unit) in the third column. Then, do the final multiplication, and write the C-strip obtained in the fourth column. In the fifth column, write a fraction using C-strips putting the final unit obtained in the fourth column as the numerator, and the unit in the denominator. Then, in the last column, write the answer as a fraction. Do not simplify.

Example	$3/4 \cdot 1/2$	N	P	L	$\frac{L}{N}$	$\frac{3}{8}$
Example	$1/2 \cdot 3/4$	N	D	L	$\frac{L}{N}$	$\frac{3}{8}$
Example	$2/3 \cdot 4/5$	O+Y	H	N	$\frac{N}{O+R}$	$\frac{8}{15}$
a.	$1/3 \cdot 3/4$					
b.	$1/2 \cdot 1/4$					
c.	$3/2 \cdot 1/4$					
d.	$2/3 \cdot 1/2$					

On to Division with Rational Numbers...

Remember that the answer to the division $a \div b$ can be obtained by answering this question: "**How many sets of b are contained in a**?"

Let's use this idea to find the answer to the following division problem:

Example 1

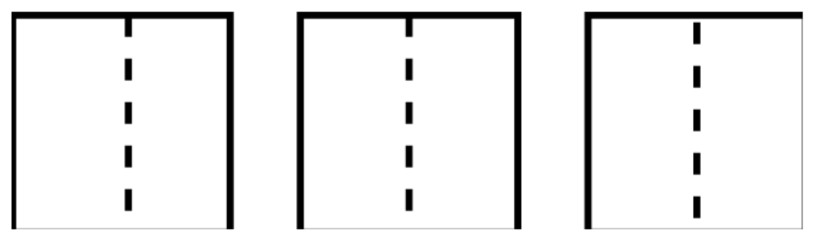
Use a model to compute: $\mathbf{3 \div 1/2}$

Solution

The answer to $3 \div 1/2$ is the answer to this question: How many 1/2's are contained in 3? Below are two possible methods you can use to find the solution.

Method 1: Let 1 unit = 1 square. Then count how many 1/2's are contained in 3 squares.

Draw three squares to represent 3. Then, divide each square into 1/2's. Count how many 1/2's are in the 3 squares. From the model, it is clear there are 6 1/2's in 3 squares. So the answer to 3 ÷ 1/2 is **6**.



Method 2: Let the unit be defined similarly to how we did it for the previous problems.Multiply the denominators together to determine how many dots are in a unit. In this case, 3 is the same as 3/1. Let the unit = 1 row of dots by 2 columns of dots for a total of 2 dots.

Unit = · · Then represent the number 3 based on this unit, and the number 1/2 based on this unit.

3 = 3 sets of 2 dots = · · · · · · and 1/2 = 1/2 of the 2 dots = ·

In the set representing 3, circle as many sets as possible that represent 1/2: ⊡ ⊡ ⊡ ⊡ ⊡ ⊡

Count how many 1/2's are in 3. There are **6** 1/2's in 3, so the answer is **6**.

Method 3: Let 1 unit = the dark green C-strip. Represent 3 and 1/2 as a C-strip based on this unit. Since 3 = 3 dark green C-strips, then 3 = the orange + brown C-strip (or 3 dark greens, or hot pink + dark green). Since 1/2 of the dark green C-strip is light green, then 1/2 = the light green C-strip.

We have to answer the question: How many 1/2's are in 3? Since 1/2 = L, and 3 = O + N, count how many light green C-strips make up the length of the O + N C-strip. There are **6**. Once again, the answer is **6**.

Unit:	D

3:	H	D

1/2	L

Here is how to get the answer. Show how many light greens are contained hot pink + dark green.

L	L	L	L	L	L

Notice that you begin by defining a unit C-strip, and then use that unit to define the two numbers in the division problem. But you do NOT refer back to the unit to compute the division problem. You simply count how many of the divisor (second C-strip) is in the dividend (first C-strip).

✓ Example 2

Use a model to compute: $1/2 \div 1/4$.

Solution

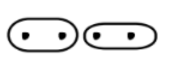
The answer to $1/2 \div 1/4$ is the answer to this question: How many 1/4's are contained in 1/2?

Let 1 unit = 2 rows of dots by 4 columns of dots:

· · · ·
· · · ·

Then represent the number 1/2 based on this unit, and the number 1/4 based on this unit.

1/2 = · · · · and 1/4 = :

In the set representing 1/2, circle as many sets as possible that represent 1/4: 

Count how many 1/4's are in 1/2. There are **2** 1/4's in 1/2, so the answer is **2**.

This can also be done using C-strips. Let the unit = N. Then 1/2 = P and 1/4 = R. Since there are **2** reds in the purple C-strip, 1/2 ÷ 1/4 = **2**.

? Exercise 27

Use a model to do the following division: 1/5 ÷ 1/10. Use boxes, dots, or C-strips. First define the unit. Then explain and show all of the steps.

? Exercise 28

Perform the following division using each of the methods (boxes, dots, C-strips):

1/3 ÷ 1/9. First define the unit. Then explain and show all of the steps.

? Exercise 29

Perform the following division using each of the methods (boxes, dots, C-strips):

2/3 ÷ 1/6. First define the unit. Then explain and show all of the steps.

? Exercise 30

Perform the following division using each of the methods (boxes, dots, C-strips):

3 ÷ 1/4. First define the unit. Then explain and show all of the steps.

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9.3: Facts About Comparing Fractions

Assume a and c are whole numbers, and b and d are counting numbers.

Two fractions, a/b and c/d, are *equivalent* if and only if ad = bc. In other words,

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$$

Note: ⇔ means "if and only if", so if ad = bc, then the two fractions a/b and c/d are equivalent, and vice versa. Using this method to determine if two fractions are equivalent is called **comparing cross products**.

Examples: Determine if the following statements are true or false by comparing cross products. State whether the pair of fractions are equivalent or not.

Example 1

6/8 = 9/12

Solution

Since $6 \cdot 12 = 8 \cdot 9$, the statement is true. Therefore, 3/4 and 9/12 are equivalent fractions.

Example 2

15/24 = 10/18

Solution

Since $15 \cdot 18 \neq 24 \cdot 10$, the statement is false. Therefore, 15/24 and 10/18 are not equivalent fractions.

Exercise 31

Determine if the following statements are true or false by comparing cross products. State whether the pair of fractions are equivalent or not. Show your work and reasoning.

a. 14/25 = 28/50

b. 25/35 = 10/14

c. 12/16 = 25/40

The Fundamental Law of Fractions:

For any rational number, a/b and any integer c, a/b = ac/bc. The fractions a/b and ac/bc are called equivalent fractions.

Exercise 32

Write five equivalent fractions for each fraction given:

a. 2/3: ____

b. 5/7: ____

c. 3/8: ____

Two fractions, a/b and c/d, are *equivalent* if and only if the numerators are equal after writing each fraction with a common denominator.

Determine if the following statements are true or false. State whether the pair of fractions are equivalent or not.

✓ Example 1

6/8 = 9/12

Solution

Write each fraction with a common denominator of 24 by applying the fundamental law of fractions: 6/8 = 18/24, and 9/12 = 18/24. Since the numerators are equal when each fraction is written with a common denominator, 6/8 and 9/12 are equivalent.

✓ Example 2

15/24 = 10/18

Solution

Write each fraction with a common denominator of 72 by applying the fundamental law of fractions: 15/24 = 45/72, and 10/18 = 40/72. Since the numerators are not equal when each fraction is written with a common denominator, 15/24 and 10/18 are not equivalent.

? Exercise 33

Determine if the following statements are true or false by writing each fraction with a common denominator. State whether the pair of fractions are equivalent or not. Show your work and reasoning.

a. 14/25 = 28/50

b. 25/35 = 10/14

c. 12/16 = 25/40

A fraction, a/b, is in *simplest* (reduced) *form* if GCF(a,b) = 1.

One way to simplify fractions is to prime factor the numerator and denominator, and then "cancel out" any common factors. Another way is to

(1) find the GCF of the numerator and denominator,

(2) rewrite both the numerator and denominator as the product of the GCF(a,b) and another factor, and then

(3) "cancel out" any common factors. When you cancel out everything from either the numerator and/or denominator, there is always a factor of 1 that is still there.

✓ Example

Write 315/350 in simplest form by using each of the two methods just described.

Method 1: Prime factorization:

$$\frac{315}{350}=\frac{3\times 3\times 5\times 7}{2\times 5\times 5\times 7}=\frac{9}{10}$$

Method 2: GCF(315, 350) = 35:

$$\frac{315}{350}=\frac{9\times 35}{10\times 35}=\frac{9}{10}$$

? Exercise 34

Write each fraction in simplest form **using each of the two methods**:

(1) prime factorization and

(2) finding GCF as shown in the previous example.

Show the actual factorization for each method, and then reduce to lowest terms.

a. Method 1: $\frac{378}{675}$

Method 2: $\frac{378}{675}$

b. Method 1: $\frac{247}{323}$

Method 2: $\frac{247}{323}$

Two fractions, a/b and c/d, are *equivalent* if the two fractions are equal when they are both written in simplest form.

Examples: Determine if the following statements are true or false. State whether the pair of fractions are equivalent or not.

✓ Example 1

6/8 = 9/12

Solution

In simplest form, 6/8 = 3/4, and 9/12 = 3/4. Therefore, 6/8 and 9/12 are equivalent.

✓ Example 2

15/24 = 10/18

Solution

In simplest form, 15/24 = 5/8, and 10/18 = 5/9. Therefore, 15/24 and 10/18 are not equivalent.

? Exercise 35

35. Determine if the following statements are true or false by writing each fraction in simplest form. State whether the pair of fractions are equivalent or not. Show your work and reasoning.

a. 14/25 = 28/50

b. 25/35 = 10/14

c. 12/16 = 25/40

Below is a way to compare two fractions that are unequal by using cross products.

$$\frac{a}{b} < \frac{c}{d} \Leftrightarrow ad < bc \qquad \text{and} \qquad \frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc$$

As you write the fractions, a/b < c/d, a is the first number written and d is the last number written. a and d are called the extremes (outside numbers when written as a/b < c/d), and their product is written on the left of the inequality sign when you take the cross products. The other two numbers, b and c, are called the means (when written as a/b < c/d, these are the inside numbers), and their product is written on the right of the inequality sign when you take the cross products. There are only three cases possible when comparing two fractions, a/b and c/d. Either the first fraction (a/b) is equal to the second (c/d), in which case ad = bc; the first (a/b) is less than the second (c/d), in which case ad < bc; or the first (a/b) is more than the second (c/d), in which case ad > bc.

✓ Example 1

Compare 2/5 and 3/7 using cross products.

Solution

Multiply the extremes (2 and 7) and put on the left. Multiply the means (5 and 3) and put on the right. Compare the products. Since $2 \cdot 7 < 5 \cdot 3$, then 2/5 < 3/7.

✓ Example 2

Compare 6/7 and 5/6 using cross products.

Solution

Multiply the outside numbers (6 and 6) and put on the left. Multiply the inside numbers (7 and 5) and put on the right. Compare the products.
Since 6·6 > 7·5, then 6/7 > 5/6.

? Exercise 36

Use cross products to compare each of the following fractions. Use < or >.

a. 4/5 and 5/8	b. 12/35 and 11/18	c. 13/15 and 14/17

At the beginning of this exercise set, you compared several fractions using the fraction circles and fraction array. One problem was to compare 1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/9 , 1/10, and 1/12, and put them in order from smallest to largest using the less than symbol.

The answer was: 1/12 < 1/10 < 1/9 < 1/8 < 1/6 < 1/5 < 1/4 < 1/3 < 1/2

This can easily be checked using cross products. You can check one at a time: 1/12 < 1/10 since 10 < 12, and 1/10 < 1/9 because 9 < 10, etc.

? Exercise 37

Check the validity of the answer you got for exercise 9a. Write the answer. Then check using cross products.

? Exercise 38

Check the validity of the answer you got for exercise 9b. Write the answer. Then check using cross products.

Adding and Subtracting Fractions

In order to add or subtract fractions, the fractions must have a common denominator. Below is the rule for adding or subtracting fractions that have a common denominator.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad \text{and} \qquad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

In order to add fractions that do not have a common denominator, you must first rewrite each fraction as an equivalent fraction so that both fractions have a common denominator, OR you may use the following rule for addition of fractions.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Note: If you are going to add fractions by rewriting each fraction with a common denominator, it is usually preferable to find the least common denominator, which is the least common multiple of the two denominators. Whether or not you do this, you should always write the answer in simplest form after adding or subtracting.

Multiplying two fractions

The rule for multiplying two fractions is to multiply the numerators together to obtain the new numerator, and multiply the denominators together to obtain the new denominator.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

It is easier to first "cancel out" any common factors before multiplying. This can be done by first prime factoring the numerators and denominators, and writing all the factors of the numerator in the numerator of the fraction, and all the factors of the denominator in the denominator of the fraction. Then, cancel any common factors before multiplying. If you do this, the fraction will already be in simplest form.

Dividing two fractions

To understand the rule for dividing two fractions, see if you can follow the reasoning below.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{1} = \frac{a}{b} \cdot \frac{d}{c}$$

The rule for dividing two fractions is this: Dividing by a fraction is the same as multiplying by the reciprocal of that fraction. Then, use the rule for multiplying as just explained.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

? Exercise 39

Compute each of the following using the order of operations. Write each answer in simplest terms. Show all work.

a.

$$\frac{3}{4} \times \frac{4}{5} - \frac{5}{6} \div \frac{7}{3}$$

b.

$$\frac{5}{6} \div \frac{4}{5} + \frac{5}{6}\left(\frac{1}{3} - \frac{2}{5}\right)$$

Unlike integers, there are infinitely many rational numbers between any two rational numbers. It doesn't make sense to talk about two consecutive rational numbers. For instance, between 4/9 and 5/9, there are infinitely many rational numbers. One easy way to list a few is to write equivalent fractions for 7/9 and 8/9. A simple common denominator would be 90, so look at 70/90 and 80/90. It's easy to see these rational numbers between 70/90 and 80/90: 71/90, 72/90, 73/90, 74/90, 75/90, 76/90, 77/90, 78/90, 79/90. Of course, had I chosen equivalent fractions with a larger denominator (900 or 900,000, etc.), you would easily be able to list many more rational numbers between 7/9 and 8/9.

If you wanted to list just one rational number between 7/9 and 8/9, you could choose one of the ones from the above list, or you could simply find the midpoint (or average) of the two. To find the average of any two numbers, add and divide by 2. Since dividing by 2 is the same as multiplying by one-half, add and multiply by one-half. Note the average of 7/9 and 8/9 is 1/2(7/9 + 8/9) = 1/2(15/9) = 15/18 or 5/6.

? Exercise 40

Find 5 rational numbers, written with a common denominator, between 2/5 and 3/5.

? Exercise 41

Find the average of 5/11 and 6/11.

? Exercise 42

Find the average of 3/7 and 4/7

If you are asked to find several rational numbers between two rational numbers that do not have a common denominator, first you should rewrite each fraction as equivalent fractions having a common denominator. For instance, to find 4 rational numbers between 4/7 and 5/6, first rewrite 4/7 and 5/6 with a common denominator of 42: 24/42 and 35/42. In this case, it is easy to find 4 rational numbers between 4/7 and 5/6: 25/42, 26/42, 27/42, etc.

Here's another one: Find 5 rational numbers between 3/4 and 4/5. First, rewrite 3/4 and 4/5 with a common denominator of 20: 15/20 and 16/20. Then, find equivalent fractions with a larger denominator than 20: 150/200 and 160/200 are easy choices, although you could have picked 90/120 and 96/120. In any case, there are infinitely many choices. I probably would choose 151/200, 152/200, 153/200, 154/200 and 155/200.

? Exercise 43

Find 3 rational numbers, written with a common denominator, between 1/3 and 2/5.

? Exercise 44

Find 3 rational numbers, written with a common denominator, between 1/2 and 1/3. (Be careful: Which number is smaller and which is larger?)

? Exercise 45

Find 5 rational numbers, written with a common denominator, between 5/6 and 4/5.

(Be careful: Which number is smaller and which is larger?)

? Exercise 46

Find the average of 5/8 and 6/7.

? Exercise 47

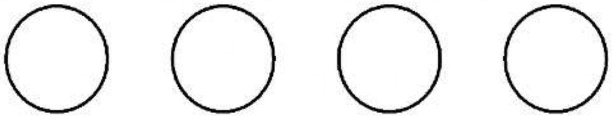
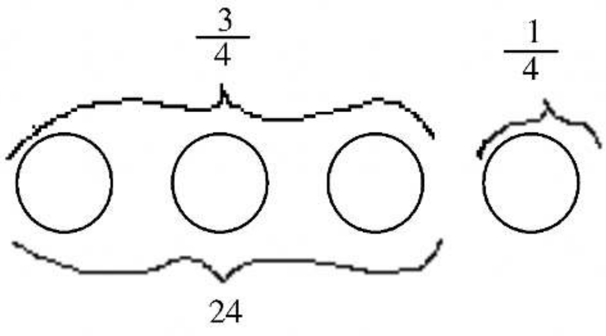
Find the average of 5/7 and 5/8.

Now, we'll work on some word problems where we can use the meaning of fractions to easily solve the problems. We'll use the fraction circles as manipulatives.

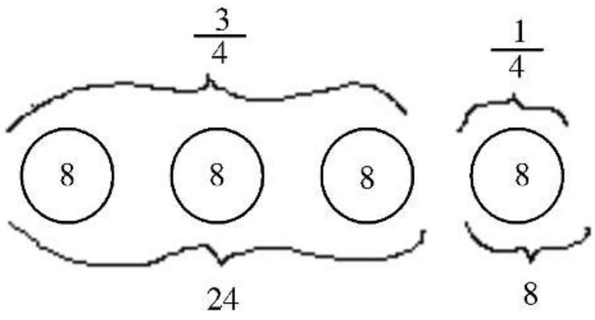
Problem: One day, 24 of my students showed up for class. This only represented 3/4 of my students. How many students were enrolled in the class? How many were absent?

Solution: We know that 24 is 3/4 of the total students in the class. That means 4/4 makes up all the students. Get out the fraction circle where 4 equal parts make up a whole. You know three of those parts represent 24 students. If 3 of those parts represent 24 students, then each of the equal parts represents 8 students. Since 4 equal parts represents the whole class, then there must be 8 · 4, or 32 students enrolled. One of the equal parts represents the absent students, so 8 students are absent.

You could also show this pictorially – you see circles in place of the fraction circles.

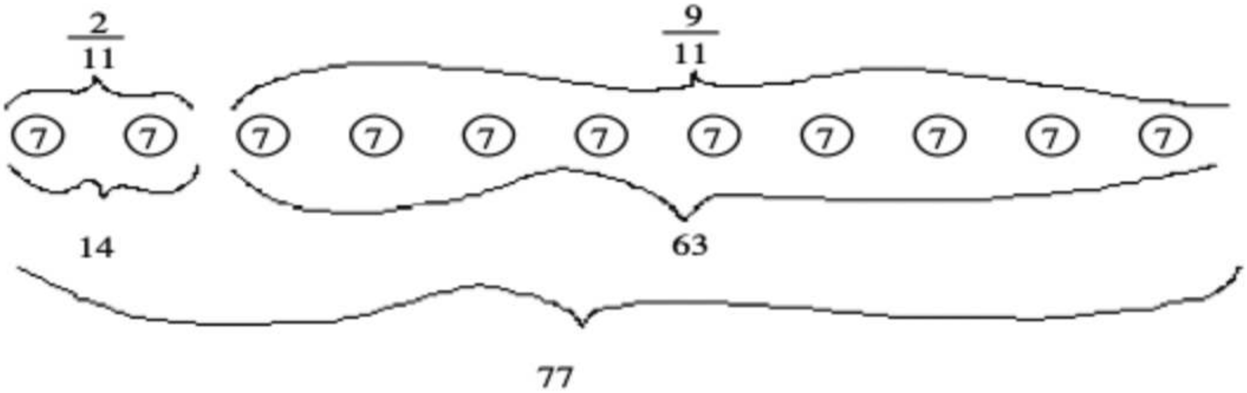
Step 1: 4 equal parts make up a whole.	
Step 2: 3 equal parts, or 3/4, represents 24 students. The part left over represents 1/4 of the students.	

Step 3: If those three circles represent 24, then 8 must be in each circle. Now it is clear there are 32 students in the class, and 8 are absent.



The picture in Step 3 is what the final picture looks like.

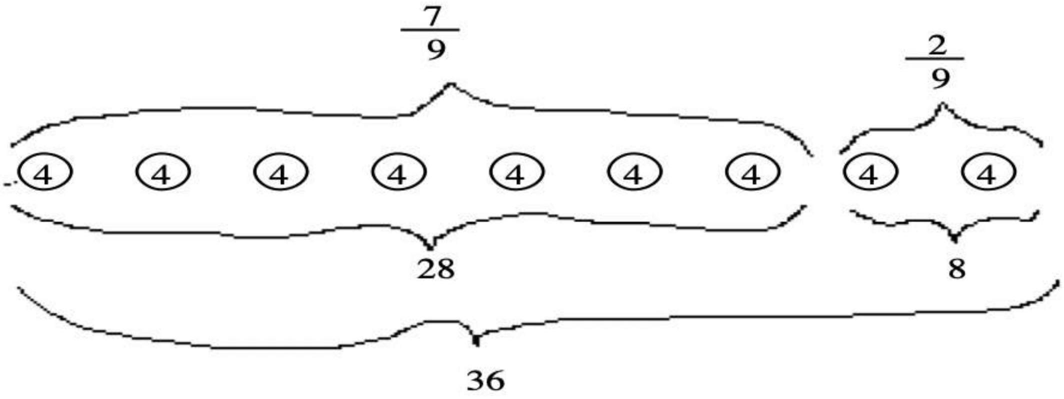
Here is another example. 14 teachers were absent one day. This represents 2/11 of the teachers working at the school. How many teachers work at the school? Since 11 equal parts make up a whole, and 2 of those parts represent 14, then each of the 11 equal parts represents 7 teachers. So, 11 of the equal parts represents 77 teachers.



Here is one more example:

One day, 7/9 of the people at a local business came to work. 36 people worked there. How many people came to work?

First, since 9 equal parts make up a whole, draw 9 circles. You know 7 of those represent 7/9 and 2 of those represent 2/9. You also know, the total equals 36, which means 4 goes into each equal part. Therefore, 28 came to work, and 8 did not. Here is the picture:



For each of the following word problems, draw model similar to the previous two examples to solve the problem. You may want to use fraction circles as well.

Exercise 48

66 students in my class passed the first test. That represents 11/12 of my students. How many students did not pass?

Exercise 49

36 students in first grade brought a lunchbox to school one day. This was 3/7 of the first graders. How many did not buy lunch?

Exercise 50

I have 120 students this semester. 5/8 of my students are female. How many are female and how many are male?

Exercise 51

Make up a word problem to solve using fractions. Make sure you ask a question. Then solve the problem using models. Explain how the model works.

Write the word problem here:

Solve the problem using models here:

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9.4: Decimals

The meaning of decimals is best understood once one has a real understanding of place value and fractions. The decimal notation we commonly use is an extension of place value in base ten. The decimal point indicates that succeeding digits represent tenths, hundredths, thousandths, etc. So a person must understand what these fractions mean in order to understand and make the connection to decimals.

The key to understanding the relationship between decimals and fractions (or mixed numbers) begins with READING the decimal correctly. Most people read 5.3 as "five point three" which doesn't help one understand its meaning. It should be read "five and three tenths." In doing so, the connection between the decimal 5.3 and the mixed number $5\frac{3}{10}$ is clear. Similarly, 18.035 should be read "Eighteen and thirty-five thousandths" and corresponds to the mixed number $18\frac{35}{1000}$. Here are a few more:

a. 0.309 and $\frac{309}{1000}$ are both correctly read "three hundred nine thousandths"
b. 10.04 and $10\frac{4}{100}$ are both correctly read "ten and four hundredths"

Note that the decimal point is read as "and". The decimal point (and the word "and") separates the whole part from the fractional part of a mixed number. This is the only correct use of the word "and" when reading numbers. 760 is read "seven hundred sixty." A common mistake is to read 760 as "seven hundred and sixty." If there is no decimal point, don't say the word "and".

People just learning about decimals (like elementary school children) should NOT read the number 5.4 as "five point four". This is a shortcut way of reading the number that is only appropriate to use once one really understands the connection between decimals and fractions. Remember that a number written in decimal form is really just a different way to write a mixed number where the denominator of the fractional part is a power of ten! The name of the fractional part (tenths, hundredths, thousandths, etc.) is the place value of the last digit of the number after the decimal point, which also happens to be the denominator of the number written in fractional form.

Look again at 0.309 and $\frac{309}{1000}$. In the decimal, there are three digits (or place values) after the decimal point. In the fraction, there are three zeroes after the 1 (which is the number 1000 in the denominator). The same applies to 18.035 and $18\frac{35}{1000}$. For 5.3 and $5\frac{3}{10}$, the decimal has one digit after the decimal point, and the fraction has one zero after the 1. It's best if students are allowed to discover this fact for themselves. It's always more meaningful to discover relationships (that often become rules) on your own. Students who are regularly asked to read decimals, fractions and mixed numbers the correct way are more likely to make this particular discovery by themselves.

Mixed numbers like 5.3 and 18.035 can also be written immediately as improper fractions. The denominator of the fraction will still be the same as if it were written as a mixed number. The numerator is the number without a decimal point at all. For instance, 5.3 can be written as $5\frac{3}{10}$ or $\frac{53}{10}$; and 18.035 can be written as $18\frac{35}{1000}$ or $\frac{18035}{1000}$.

Note that when you first write a decimal as a fraction, it isn't necessarily in simplest (or reduced) form.

? Exercise 1

On the line, write in words how to read each of the following decimals. Do not use the word "point". Then, underneath, write each decimal as a fraction. If the number is greater than or equal to 1, first write the fraction as a mixed number and then write it as an improper fraction. *Simplify any fraction that is not in simplest form.* **Show all steps** (including original fractions before simplifying).

a. 0.4 ____

0.4 =

b. 0.26 ____

0.26 =

c. 3.08 ____

3.08 =

d. 9.85 ____

9.85 =

e. 17.305 ____

17.305 =

? Exercise 2

Write each proper fraction as a decimal. Write each improper fraction first as a mixed number (don't reduce), and then also as a decimal. Do not use your calculator.

a. $\frac{14}{100}$ = ___________

b. $\frac{8}{1000}$ = ___________

c. $\frac{435}{100}$ = ________________ = ________________

d. $\frac{5638}{10}$ = ________________ = ________________

e. $\frac{305}{100}$ = ________________ = ________________

The rule you may remember for multiplying fractions is to multiply the numbers together as if there were no decimal point, and then move the decimal point in from the right the total number of places it is in for both numbers. For instance, (8)(0.4) is done by multiplying 8 times 4 and then moving the decimal point in one place to get 3.2. Similarly, (0.06)(0.7) is done by multiplying 6 times 7 and moving the decimal point in three places (two for 0.06 and one more for 0.7 for a total of three) to get 0.042. Many people make this harder than it really is and don't realize they can easily multiply 0.3 and 0.4 in their head. It's as simple as $3 \cdot 4$ and moving the decimal in two places to get 0.12. Now, how about 1.1 times 1.2? It's simply 11 times 12 with the decimal moved in two places: 1.32.

Let's observe why this rule for multiplying decimals works by rewriting the numbers as fractions first. The key here is to write numbers greater than or equal to one as improper fractions. Then multiply (without canceling or reducing) the fractions. Lastly, rewrite the fraction as a decimal.

✓ Exercise Example 1

$(8)(0.4) = \frac{8}{1} \cdot \frac{4}{10} = \frac{32}{10} = 3.2$ (notice the decimal point is one place in for .4)

✓ Exercise Example 2

$(1.2)(1.01) = \frac{12}{10} \cdot \frac{101}{100} = \frac{1212}{1000} = 1.212$

(notice the decimal point is three places in, one for 1.2 plus two more for 1.01)

? Exercise 3

Multiply the following decimals mentally, and write the answer on the blank. Then do it again by showing the same steps as shown in the previous two examples where each decimal is **first** written as a fraction, then multiply the numerators and denominators, and then convert that answer (don't simplify) to a decimal. Do not use your calculator.

a. (0.4)(.07) = ______________

(0.4)(.07) =

b. (1.6)(0.2) = ______________

(1.6)(0.2) =

c. (0.25)(0.3) = ______________

(0.25)(0.3) =

d. (2.2)(0.3) = ______________

(2.2)(0.3) =

Recognizing Equivalent Decimals and Comparing Decimals

Zeroes, which are behind a decimal's last non-zero digit, can be added or removed without changing the value of the decimal. If you look at some equivalent fraction, you'll see why this should be true. For example,

$$\frac{14}{10} = \frac{140}{1000} = \frac{1400}{10000} = \frac{14000}{100000}.$$

These are all equivalent to $\frac{14}{100}$ because the numerator and denominator was multiplied by some power of 10 (10, 100 or 1000) to get one of the other equivalent fractions. If we replace each of the four fractions above with their decimal equivalents, we get 0.14 = 0.140 = 0.1400 = 0.14000. Let's call any zeroes at the end of a decimal's last non-zero digit "**trailing zeroes**". Then, we could conclude that any number starting with .14 that has trailing zeroes is also equivalent, like 0.1400000.

Two decimals are equal only if one can be made to look identical to the other by adding or removing trailing zeroes. You can also determine if they are equivalent by removing any excess trailing zeroes from each to see if they are identical.

? Exercise 4

For the first decimal given, circle any of the next four decimals that are equal to it.

a.	1.900;	1.0900	1.9	1.90000	0.190
b.	4.034;	4.0340	4.0334	4.0034	4.3040

To compare two or more decimals that are not equal, but have the same number of digits after the decimal point, you can write each as a fraction with the same denominator and then compare the numerator. For instance 0.14 is less than 0.21 since fourteen hundredths is less than twenty-one hundredths. Basically, in this case, it's just like comparing whole numbers. You'll be able to determine which is larger by comparing each number as if there was no decimal point. But, keep in mind this only makes sense if the numbers you are looking at have the same number of digits after the decimal point.

? Exercise 5

Compare each of the following decimals using <, = or >.

a.	3.5	0.9
b.	35.06	35.0600
c.	0.089	0.098

To compare two or more decimals that are not equal that do not all have the same number of digits after the decimal point, first write each decimal with the same number of digits after the decimal point (by adding trailing zeroes to one or more if necessary). By doing that, you are comparing tenths with tenths, or hundredths with hundredth, etc., as you did in exercise 5.

? Exercise 6

Compare each of the following decimals using <, = or >.

a.	3.51	3.488
b.	35.061	35.35
c.	0.08933	0.0894

If a fraction is written with a power of ten in the denominator, it's basic to write the same number in decimal form. You did that in Exercise 2. Any fraction that is written with a power of 10 in the denominator can be written as a terminal decimal. This means that it's possible to write the number with trailing zeroes. But what if the fraction doesn't have a power of 10 (like 10, 100, 1000, etc.) in the denominator? Sometimes those can be tricky!

If a fraction WITHOUT a power of ten in the denominator CAN be written as an equivalent fraction WITH a power of ten in the denominator, then it can be written as a terminating decimal.

For instance, $\frac{1}{2}$ can be written as $\frac{5}{10}$ (by multiplying the numerator and denominator by 5)

Therefore, $\frac{1}{2} = \frac{5}{10} = 0.5$ (remember to read this as five-tenths)

Well, that one wasn't too hard, but what about $\frac{7}{80}$? The question is whether you can multiply the denominator, 80, by something to get 10, 100, 1000, 10000, etc. There is no whole number you can multiply 80 by to get 10, 100 or 1000. But, if you multiply 80 times 125, it equals 100000. So, by multiplying both 7 and 80 by 125, we get the equivalent fraction $\frac{875}{10000}$ for $\frac{7}{80}$ which can now be written as the decimal 0.0875.

How about writing $\frac{5}{6}$ as a terminating decimal? Well, there is nothing you can multiply 6 by to get 10 or 100 or 1000 or 10000. Is there maybe some number we could multiply 6 by to get some higher power of ten? Well, that's a good question! Actually, there isn't, but how could you be sure? You certainly can't try every power of ten since there are an infinite number of them to try.

It would be nice if there was an easy way to determine if any given fraction could be written as a terminating decimal. The key is to consider the factors of the denominator of a fraction that can be written as a terminating decimal. If a fraction can be written as a terminating decimal, then there is some equivalent fraction where the denominator must be a power of ten: 10 or 100 or 1000, etc.

Exercise 7

Write the prime factorization of each of the following:

a. 10 = _______________

b. 100 = _______________

c. 1000 = _______________

d. 10000 = _______________

e. 100000 = _______________

Exercise 8

What are the only prime factors of powers of 10? ______________

Exercise 9

If a power of ten has three factors of 5, how many factors of 2 does it have? ______

Exercise 10

If a power of ten has two factors of 2, how many factors of 5 does it have? _____

Powers of ten only have 2s and 5s as its prime factors and nothing else.

Let's go back to our three numbers , $\frac{1}{2}$, $\frac{7}{80}$, and $\frac{5}{6}$, that we were trying to write as terminating decimals and analyze the situation.

$\frac{1}{2}$ is simplified, and in its prime factored form, there is exactly one 2 in the denominator. To write as an equivalent fraction with a denominator that is a power of ten, it must have only 2s and 5s as prime factors in the denominator, and the same number of each! Therefore, multiplying by one more 5 in the numerator and denominator did the trick!

Prime factor the numerator and denominator of this reduced fraction: $\frac{7}{80} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$

We must determine if it is possible to multiply the denominator by something so that the resulting denominator will be made up of only 2s and 5s and the same number of each. Well, there are four factors of 2 and one factor of 5. Since we need the same number of each factor, making an equivalent fraction by multiplying the numerator and denominator by three more factors of 5 will do the trick.

$\frac{7}{80}=\frac{7}{2\cdot2\cdot2\cdot2\cdot5}\cdot\frac{5\cdot5\cdot5}{5\cdot5\cdot5}=\frac{875}{10000}=0.0875$

In both examples, note we either multiplied by extra factors of 2 or 5, but not both!

What about $\frac{5}{6}$? Well, it's reduced, and the prime factorization of the denominator is $2\cdot3$. No matter what the denominator is multiplied by, we'll be stuck with a factor of 3 in the denominator. Since the only prime factors of powers of 10 are 2 and 5, there can't be a prime factor of 3 in the denominator if we want to end up with only a power of 10 in the denominator. Therefore, since it is impossible to write $\frac{5}{6}$ as an equivalent fraction with a power of ten in the denominator, it cannot be written as a terminating decimal.

How about $\frac{63}{72}$? If we prime factor the denominator, we get $2\cdot2\cdot2\cdot3\cdot3$.

? Exercise 11

Well, what do you think? Can $\frac{63}{72}$ be written as a terminating decimal? ________

Explain why or why not.

One way to check that you are obtaining the correct results is to use a calculator. For $\frac{1}{2}$, we got 0.5, which you can check by doing the division $1\div2$ on your calculator.

? Exercise 12

Use your calculator to find the decimal equivalent for each of the following:

a. $\frac{7}{80}=$ ______________

b. $\frac{63}{72}=$ ______________

? Exercise 13

Hmmm, did you get 0.0875 for part a? _________

If you did part b on your calculator, did you get a terminating decimal of 0.875?

Is that what you expected? Why or why not?

The reason $\frac{63}{72}$ can be written as a terminating decimal is because in its ***simplified*** form, it only has 2's and/or 5's as its prime factors. Here is how to finish this problem by simplifying first, and then multiplying by any needed factors of 2 or 5 to get the same number of each:

$\frac{63}{72}=\frac{3\cdot3\cdot7}{2\cdot2\cdot2\cdot3\cdot3}=\frac{7}{2\cdot2\cdot2}\cdot\frac{5\cdot5\cdot5}{5\cdot5\cdot5}=\frac{875}{10000}=0.875$

? Exercise 14

For each fraction, determine if it can be written as an equivalent fraction with a power of ten in the denominator. If a fraction cannot be written as a terminal decimal, ***explain*** why not. Otherwise, show **ALL** of the steps (as shown in the previous examples) to write it as a terminal decimal. The steps are listed below.

a. Simplify if possible
b. Prime factor the denominator
c. Multiply the numerator and denominator by an appropriate number of factors of 2 or 5 so that the denominator will be a power of 10
d. Simplify the numerator and denominator
e. Write as a terminating decimal

Then, **check your answer with a calculator** by taking the original fraction and dividing the numerator by the denominator. You should get the same decimal obtained by doing the five steps outlined above.

a. $\frac{3}{4}$

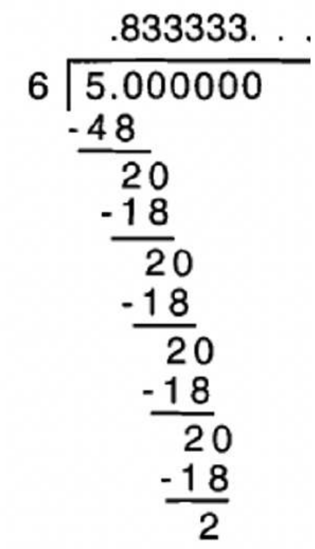
b. $\frac{9}{20}$

c. $\frac{9}{15}$

d. $\frac{18}{25}$

e. $\frac{5}{14}$

Okay, now we have to deal with those fractions that cannot be written as terminating decimals. Any simplified fraction that has at least one prime factor that isn't a 2 or 5 is in this category. Let's look at $\frac{5}{6}$ again. One way to write this as a decimal is to divide 5 by 6.



As you could see, I keep dividing 6 into 20, write 3, multiply to get 18, subtract from 20, get 2, bring down the 0, and start all over again. This can go on forever and ever. So, the 3's at the end will trail on forever. This is NOT a terminating decimal since the number cannot be written with trailing zeroes. In this case, there are trailing 3s. The **three dots at the end (called ellipses)** of the number are to show that the 3s repeat forever. The answer can be written as 0.8333333... or $0.8\bar{3}$. *Remember to put the ellipses (three dots) at the end of the number!* The bar over the 3 indicates that the 3 repeats forever and ever. Here are other ways to represent the same thing: $0.8\overline{33}$ or $0.83333\bar{3}$. In the first case, it says 33 repeats forever. In the second case, it says after the first three 3's, the 3 repeats forever. In both cases, when you write it out in long form, it looks like 0.8333333... so it's the same number. Usually, we'll write 0.8333333... or $0.8\bar{3}$.

Exercise 15

Write 0.8333333... or $0.8\bar{3}$ in three more ways, different than $0.8\overline{33}$ *or* $0.83333\bar{3}$.

____________________ , ____________________, and ____________________

One thing to notice about what happened when we divided 5 by 6 is that I kept getting 2 after each subtraction in the division, this is like the remainder. Also remember that when you divide, each remainder must be less than or equal than what you are dividing by.

Exercise 16

Write the possible remainders for each number.

a. 6: ______________________________

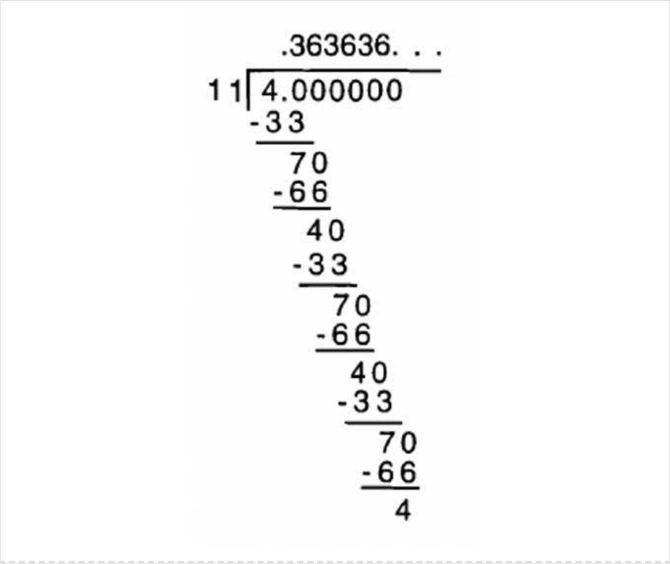
b. 7: ______________________________

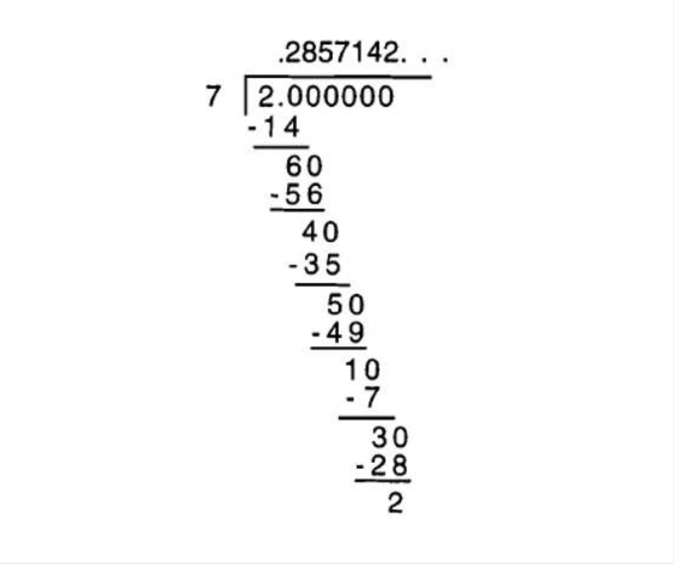
c. 9: ______________________________

d. 11: ______________________________

e. 3: ________________________________

Let's now look at what happens when you do long division compute $\frac{4}{11}$ and $\frac{2}{7}$.





In both of these cases, eventually, you get a remainder that you got previously, so the computation repeats itself. In the case of dividing 4 by 11, two remainders come up before there is some repetition 4 and 7. From 16d, you should have realized that the only possible remainders for 11 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Now if you got a remainder of 0 when dividing, you'd have a terminating decimal. So, when dividing by 11, the most remainders you might get in a row before one repeats is 10 in a row. But in the case of $\frac{4}{11}$, the remainders repeat after the 4 and 7. This means you keep dividing into 40 or 70 (since you bring the zero down) and that's why in the quotient the digits start repeating. ***Remember to put the ellipses (three dots) at the end of the number if there is a pattern of repeating digits.***

Now in the case of dividing 2 by 7, the remainders you get as you go along are 6, then 4, then 5, then 1, then 3, and then 2. When you bring down the 0, you are dividing 7 into 20 again, and hence the remainders repeat. Notice that the sequence of remainders (6, 4, 5, 1, 3, and 2) is different than the sequence of digits that repeat in the quotient. The digits that repeat in the quotient are 285714, so $\frac{2}{7}$ = 0.285714285714... or $0.\overline{285714}$.

Since there are only six possible remainders other than zero when dividing by 7, only a sequence of 6 digits could possibly repeat. In this case, all six possible remainders of 7 appeared in the long division.

There are many ways to express the infinite or repeating decimal 0.285714285714... or $0.\overline{285714}$. First of all, to establish what is repeating, you would want to see the sequence of digits repeating at least two times through. Therefore, 0.285714285714...is the shortest possible way of showing it when the ellipses (three dots) are used. If you simply wrote 0.285714..., it wouldn't be clear whether or not the 4 repeated or the 14 repeated, etc.

Here are a few other ways to write 0.285714285714... besides $0.\overline{285714}$.

0.285714285714285714285714...

0.2857142857142857142...

(in this case, you see it as the 857142 repeating from this point on)

0.2857142857142857...

(in this case, you see it as the 142857 repeating from this point on)

$0.28\overline{571428}$

(in this case, you see it as the 571428 repeating from this point on)

? Exercise 17

Write 0.285714285714... two more ways using ellipses (the three dots) and two more ways using a bar over a repeating sequence of digits.

? Exercise 18

Determine which of the following is equivalent to 0.383432432432...

One way to do this is write the number out in long form by carefully continuing the pattern and then check the digits one place value at a time.

a. 0.3834324324...
b. 0.383432432...
c. 0.3834323432...
d. $0.38343\overline{243}$
e. $0.38\overline{34324}$

Keep in mind that if you are finding a decimal equivalent for a fraction with 17 in the denominator that there may be up to sixteen digits in a row before you see any repetition. Now, obviously, there are only 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) in base ten. Note that the sequence of digits repeating in the quotient is totally different than the possible remainders you might get when you divide by 17 since a remainder may be more than a single-digit number.

? Exercise 19

a. How many possible remainders could there be if you divide a number by 33?

b. Do long division to write $\frac{31}{33}$ as a repeating decimal using ellipses and then using the bar over the repeating digits on this line: ______________________________

Show work here:

c. In the long division, how many digits repeat? ______

d. In the long division, what were the remainders you would keep getting if you continued the division forever and ever? _________________

? Exercise 20

Use your calculator to write each as a repeating decimal. Write the answer two ways, first with a bar over the repeating digits, and then in long form, **using ellipses**. Note that your calculator may or may not round the last digit shown. It can't show digits repeating forever, so you have to be savvy enough to know whether the decimal showing on the display is terminal or if it is an approximation.

a. $\frac{5}{9}$ = ________________ = _______________________________
b. $\frac{5}{7}$ = ________________ = _______________________________
c. $\frac{1}{6}$ = ________________ = _______________________________
d. $\frac{2}{3}$ = ________________ = _______________________________
e. $\frac{7}{11}$ = ________________ = _______________________________
f. $\frac{5}{12}$ = ________________ = _______________________________
g. $\frac{8}{15}$ = ________________ = _______________________________
h. $\frac{16}{45}$ = ________________ = _______________________________
i. $\frac{5}{66}$ = ________________ = _______________________________

Exercise 21

If you are changing a simplified fraction to a decimal where the denominator is x, how many digits at most can repeat in the quotient? _________

Well, you've learned how to write terminating decimals as fractions, and how to write fractions as decimals. You should be able to determine whether a fraction can be written as a terminating or repeating decimal before doing the actual division.

Rational Numbers are defined to be numbers which can be expressed as the ratio of two integers. Fractions as we usually refer to them (without decimal points or square roots, etc. in the numerator or denominator) are rational numbers. Since all fractions can be written as terminating or nonterminating (infinite, repeating) decimals, then all terminating and repeating decimals are also rational numbers. In Exercise 1, you wrote some terminating decimals as fractions. Since all nonterminating decimals came from a fraction, now we want to be able to go the other way around and write a repeating decimal as a fraction. We've got a nifty trick for doing just that!

Let's say we wanted to write $0.\overline{72}$ or 0.727272727272... as a fraction.

First, notice this is a very different number from 0.72, since 0.72 is seventy-two hundredths, or $\frac{72}{100}$. This simplifies to $\frac{18}{25}$. So,0 .72 and $\frac{18}{25}$ are the same number. You can check to make sure $\frac{18}{25}$ is the correct fraction by using a calculator and dividing 18 by 25. The calculator should read 0.72, which is where we started.

Okay, so how do we write $0.\overline{72}$ as a fraction? Since this is a repeating decimal that has infinitely many digits, there is no single power of ten we can put in the denominator. The trick is to use algebra to write a number in a way that eliminates the infinitely repeating part of the decimal. First, let's write out $0.\overline{72}$ the long way:

$0.\overline{72}$ = .727272727272... and call it the number x. So x = 0.727272727272...

Remember that if you multiply a decimal by 10, the decimal point moves one place to the right, and when you multiply a decimal by 100, the decimal point moves two places to the right, etc.

Exercise 22

If x = 0.727272727272..., then write out what 10x, 100x and 1000x equals. Write it out **without** using the bar over the repeating digits. Use the ellipses (three dots).

10x = ________________________________

100x = ________________________________

1000x = ________________________________

Exercise 23

Okay, let's work with x = 0.727272727272... and 100x = 72.727272727272...

a. In algebra, what is 100x - x? _________ (Hint: Subtract coefficients.)

b. Compute 100x - x another way: Line up the decimal points and subtract 0.727272727272... from 72.727272727272... Notice that if the decimals are lined up, the "tails" at the end of both repeating decimals are exactly the same so it should be easy to subtract. Show the work below.

c. Write an equation so that the answer to part a equals the answer to part b. Then use algebra to solve for x. Show work.

d. Simplify the fraction you got for x in part c. Show work.

e. Use a calculator to rewrite the fraction you got in part c as a decimal:

Is it equal to $0.\overline{72}$? __________

If so, you must have written $0.\overline{72}$ as the correct fraction!

Here's the trick for changing repeating decimals to simplified fractions:

Call the number you are trying to write as a decimal a variable, like n or x.

If there is one repeating digit, compute 10x; if there are two repeating digits, compute 100x; if there are three repeating digits, compute 1000x, etc. This aligns the repeating decimals up with each other so that the tail of x and the other number (10x, 100x, 1000x, etc.) is the same.

Then subtract x from the other number (10x, 100x, 1000x, etc.) The tails of both numbers will come off and you should have an algebraic problem to solve at this point. Make sure you write your answer as a reduced fraction with only integers in the numerator and denominator.

Here are two examples. *If you write the number in decimal form without the bar over the number, remember to put the ellipses (three dots) at the end!*

Write $0.\overline{2}$ as a simplified fraction. Let x = $0.\overline{2}$ = .222222... Since only one digit is repeating, multiply x by 10. Then 10x = 2.222222... $\begin{aligned} 10x &= 2.22222.... \\ \underline{-x} &\underline{= .2222.....} \\ 9x &= 2 \end{aligned}$ $x = \frac{2}{9}$ Use a calculator to check that $\frac{2}{9} = 0.\overline{2}$	Write $0.\overline{54}$ as a simplified fraction. Let x = $0.\overline{54}$ = .545454... Since two digits are repeating, multiply x by 100. Then 100x = 54.545454... $\begin{aligned} 100x &= 54.545454... \\ \underline{-x} &\underline{= .545454...} \\ 99x &= 54 \end{aligned}$ $x = \frac{54}{99} = \frac{6}{11}$ Use a calculator to check that $\frac{6}{11} = 0.\overline{54}$

? Exercise 24

Rewrite each of the following decimals as simplified fractions. For repeating decimals, use the technique shown in the previous examples. Then, check your answer using a calculator by dividing the numerator by the denominator to see if the result matches the original problem.

a. 0.4

b. $0.\overline{4}$ Begin by letting x = $0.\overline{4}$ or x = 0.44444...

c. 0.06 =

d. $0.\overline{06}$ Begin by letting x = $0.\overline{06}$ or x = 0.060606...

e. 0.9

f. $0.\overline{9}$ Begin by letting x = $0.\overline{9}$ or x = 0.9999... (this answer might surprise you)

g. 0.45

h. $0.\overline{45}$ Begin by letting x = $0.\overline{45}$ or x = 0.454545...

i. 0.084

j. $0.\overline{084}$ Begin by letting x = $0.\overline{084}$ or 0.084084084...

Sometimes, the arithmetic gets a little more challenging. Consider writing 0.14444... as a fraction. The 4 repeats starting two places after the decimal point. We plunge forward as before, but there's a little glitch at the end because one side of the equation will have a decimal point in it. If we were to divide by 9, the fraction will have a decimal point in the numerator so it isn't a reduced rational number; both the numerator and denominator have to be integers. Here are basic steps up until that point.

$10n = 1.444444...$ and $n = 0.144444$

Subtracting, we get $9n = 1.3$

One way to remedy this situation is to multiply both sides of the equation by 10 (or 100 or 1000 as needed) to eliminate the decimal. This is like clearing fractions by multiplying both sides of an equation by the least common denominator.

So multiply both sides of $9n = 1.3$ by 10 to get $90n = 13$. Then divide by 90 to get $\frac{13}{90}$.

Another way to remedy this situation is to write the right side of the equation as a fraction. Remember that 1.3 is $\frac{13}{10}$. So the equation is $9n = \frac{13}{10}$.

You can cross multiply to get $90n = 13$ and then divide by 90 to get $n = \frac{13}{90}$

In any case, remember that a reduced fraction is the ratio of two integers that are relatively prime.

? Exercise 25

Rewrite each repeating decimal, use the technique shown in the previous example, as a simplified fraction. Then, using your calculator, divide the numerator by the denominator and see if the result matches the original problem.

a. 0.02828...

b. 0.2888...

c. 0.00666...

d. 0.1011011...

e. 0.3999...

So, what did you think about the answer to 24f and 25e? Both of those are little mind-boggling. It's kind of hard to accept, but 0.999999... is really the same number as 1. It's not less than 1 it's exactly 1! For 24f, if you let $10n = 9.999...$ and $n = 0.999...$, then subtracting n from $10n$ yields $9n = 9$, so $n = 1$. That's hard to swallow, but it's the truth! Another way to see this is to realize that 1/3 + 2/3 = 1. But 1/3 = 0.333333.... and 2/3 = 0.666666... Then, 1/3 + 2/3 = 0.33333... + 0.66666... = 0.99999.... We know 1/4 + 2/3 = 1, so 0.99999... must also equal 1, as well. Oh, this stuff is just too cool!

For 0.39999..., the result of repeating 9s after a decimal makes 0.39999...= 0.4 = $\frac{4}{10} = \frac{2}{5}$.

So, basically, any number with a bunch of trailing 9s ends up being a terminating decimal.

All of the numbers we've been dealing with so far fractions, terminating decimals, and repeating decimals make up the rational numbers. Every rational number can be written as the ratio of two relatively prime integers, and can also be written as a terminating or repeating decimal. Conversely, every terminating and repeating decimal is a rational number.

Oh, but there's more...much more!

The rationals make up a very small part of the real numbers. To complete the real number system, we have to talk about the irrational numbers. Every real number is either rational or irrational. Those numbers which cannot be written as the ratio of two relatively prime integers is irrational. Those decimal numbers which neither terminate nor repeat are irrational. So, what do they look like?

One of the most commonly known irrational numbers is π. It is the number that is the ratio of the circumference of a circle to its diameter. Sounds like it's a ratio of two integers, but it's not! π can only be approximated. The most common approximation is 3.14 or 22/7. Neither of these is equal to π because both of these are rational numbers, and π is not!

Other irrational numbers are square roots of numbers that are not perfect squares, or cube roots of numbers that are not perfect cubes, etc.

For instance, these numbers are irrational: $\sqrt{5}$, $\sqrt{3}$, $\sqrt{12}$, $\sqrt{20}$, $\sqrt[3]{35}$, $\sqrt[4]{72}$

? Exercise 26

Write five irrational numbers not already listed above __________________________

? Exercise 27

Is $\sqrt{9}$ irrational? _____ Why or why not? ___________________________________

Another way to express an irrational number in decimal form is to make up a decimal that perhaps has some pattern to it, but never terminates or repeats. Two examples of this type are 2.12112111211112... and 5.01020304050607080910011012...

? Exercise 28

Write four irrational numbers in decimal form that shows a clear pattern.

? Exercise 29

Write both a rational number and an irrational number in decimal form that is between 0.53 and 0.54.

? Exercise 30

Write both a rational number and an irrational number in decimal form that is between 0.53333... and 0.54444...

Rational ________________________

Irrational ________________________

? Exercise 31

Classify each of the following numbers as rational or irrational.

a. 0.428222... _____________________

b. 0.283848... _____________________

c. $\frac{5}{13}$ _____________________

d. $\sqrt{80}$ _____________________

e. $\sqrt{100}$ _____________________

f. π _____________________

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9.5: Homework

- **Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)**
- **Start a new module on the front side of a new page and write the module number on the top center of the page.**
- **Answers without supporting work will receive no credit.**
- **Some solutions are given in the solutions manual.**
- **You may work with classmates but do your own work.**

? HW #1

Do each of the following steps using your C-strips.

i. State how many C-strips (each an equal part of the whole) make up one unit.
ii. State which C-strip makes up one part of the whole.
iii. State the fraction that the C-strip in part b represents.
iv. State how many of the C-strips in part b you need to make into a train.
v. State which C-strip is the length of the train you made in part c

a. If S represents 1 unit, then which C-strip represents $\frac{7}{11}$?

b. If H represents 1 unit, then which C-strip represents $\frac{2}{3}$?

c. If P represents 1 unit, then which C-strip represents $\frac{3}{2}$?

d. If L represents 1 unit, then which C-strip represents 3 ?

e. If Y represents 1 unit, then which C-strip represents $\frac{6}{5}$?

f. If O represents 1 unit, then which C-strip represents $\frac{1}{2}$?

g. If B represents 1 unit, then which C-strip represents $\frac{4}{3}$?

? HW #2

Do each step using your C-strips.

i. State how many C-strips will make up the named C-strip stated in the problem.
ii. Which C-strip makes up one equal part?
iii. State the fraction that the C-strip in part b represents.
iv. State how many of the C-strips in part b will make up one unit.
v. Form the unit by making a train from the equal parts (C-strip in part b) and state which C-strip has the same length as that train.

a. If O represents $\frac{5}{6}$, then which C-strip is 1 unit?

b. If W represents $\frac{1}{7}$, then which C-strip is 1 unit?

c. If D represents $\frac{3}{2}$, then which C-strip is 1 unit?

d. If N represents $\frac{4}{3}$, then which C-strip is 1 unit?

e. If D represents 3, then which C-strip is 1 unit?

f. If K represents $\frac{7}{9}$, then which C-strip is 1 unit?

? HW #3

Do each step using your C-strips.

i. State which C-strip is one unit.
ii. State which C-strip is the answer.

a. If N represents $\frac{2}{3}$, then which C-strip represents $\frac{1}{4}$?

b. If D represents $\frac{3}{4}$, then which C-strip represents $\frac{3}{2}$?

c. If B represents $\frac{3}{2}$, this which C-strip represents $\frac{4}{3}$?

HW #4

Use your fraction arrays to determine all fractions on the fraction array that are equivalent to 3/4. Do this by finding 3/4 on the array, and seeing what other numbers are the same length. Include a diagram.

HW #5

Use your multiple strips to write 6 fractions equivalent to 5/6. Draw the strips.

HW #6

Use your multiple strips to write 6 fractions equivalent to 3/8 Draw the strips.

HW #7

Compare 3/8 and 1/3 using models. Show **all** of the steps, and explain the procedure as shown in this module.

HW #8

Add 3/8 and 1/3 using models. Show all of the steps, and explain the procedure as shown in this module.

HW #9

Do the following subtraction using models: 3/5 – 1/4. Show all of the steps, and explain the procedure as shown in this module.

HW #10

Do the following multiplications using models. Show all of the steps, and explain the procedure as shown in this module.

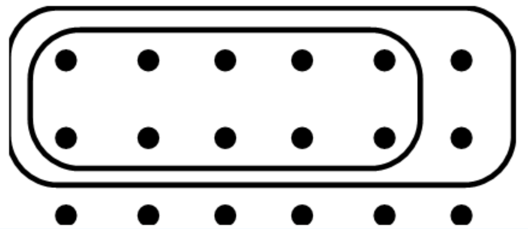
a. 3/8 · 2/5

b. 4/7 · 2/3

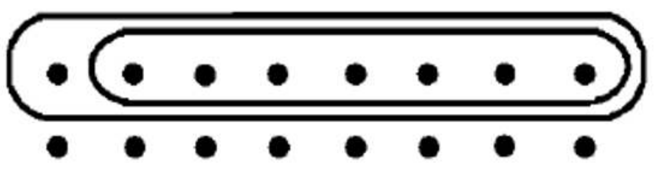
HW #11

By looking at the final drawing someone made to model a multiplication of two fractions, determine which multiplication was performed, and then state the answer.

a. 5/6 · 2/3 OR 2/3 · 5/6



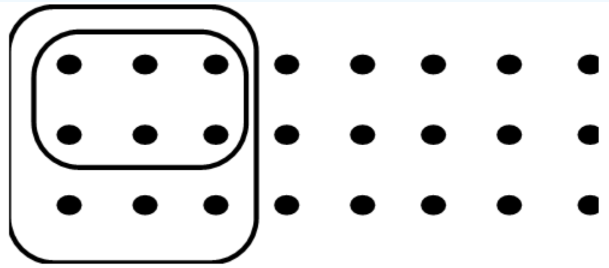
b. 1/2 · 7/8 OR 7/8 · 1/2



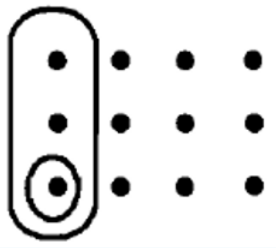
HW #12

If all of the dots shown for each problem represent 1 unit, determine the multiplication problem that someone did to get the answer, and state the answer.

a.



b.



HW #13

Fill in the chart showing how to do the following multiplications using C-strips. The multiplication is in the first column. State an appropriate choice for the unit (name a C-strip, or sum of two C-strips) in the second column. Write the C-strip obtained after the first part of the multiplication (which is the second fraction as a part of the unit) in the third column. Then, do the final multiplication, and write the C-strip obtained in the fourth column. In the fifth column, write a fraction using C-strips putting the final unit obtained in the fourth column as the numerator, and the unit in the denominator. Then, in the last column, write the answer as a fraction. Do not simplify.

a.	$\frac{1}{3}\cdot\frac{2}{3}$					
b.	$\frac{1}{2}\cdot\frac{5}{6}$					

HW #14

Perform the following division using the box and dot methods. First define the unit. Then explain and show all of the steps. Include diagrams.

a. 5 ÷ 1/3

b. 3/4 ÷ 1/3

HW #15

Determine if the following statements are true or false by comparing cross products.

a. 19/23 = 57/69

b. 24/37 = 68/91

HW #16

Write each fraction in simplest form **using each of the two methods:**

(1) prime factorization and

(2) finding GCF.

a. $\frac{216}{420}$

b. $\frac{195}{286}$

HW #17

Use cross products to compare each of the following fractions. Use < or >.

a. 18/23 and 5/8

b. 11/18 and 121/250

HW #18

Find 3 rational numbers, written with a common denominator, between 3/8 and 5/8.

HW #19

Find 3 rational numbers, written with a common denominator, between 1/2 and 4/7.

HW #20

a. 21 of John's students have cats at home. This represents 7/10 of John's students. How many students are in John's class? Solve the problem using models. Explain how the model works.

b. At an elementary school, 38 teachers drive alone to work. This represents 2/3 of the teachers. How many teachers work at the school? Solve the problem using models. Explain how the model works.

HW #21

Write in words how to read each of the following decimals.

a. 0.7

b. 0.67

c. 3.28

d. 19.835

HW #22

Multiply the following decimals mentally then do it again by showing the same steps as shown in this module..

a. (0.3)(0.8)

b. (1.2)(0.4)

c. (1.22)(2.3)

d. (3.2)(2.41)

? HW #23

For each fraction, determine if it can be written as an equivalent fraction with a power of ten in the denominator. If a fraction cannot be written as a terminal decimal, **explain** why not. Otherwise, show **ALL** of the steps to write it as a terminal decimal.

a. $\frac{11}{16}$

b. $\frac{3}{125}$

c. $\frac{1}{12}$

d. $\frac{9}{40}$

e. $\frac{21}{56}$

? HW #24

Rewrite each of the following decimals as simplified fractions. For repeating decimals, use the techniques shown in this module. Then, check your answer using a calculator by dividing the numerator by the denominator to see if the result matches the original problem.

a. $0.\bar{7}$

b. $0.\overline{72}$

c. $0.\overline{235}$

d. $0.2\bar{5}$

e. $0.3\overline{42}$

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CHAPTER OVERVIEW

10: Problem Solving Logic Packet

10.1: George Polya's Four Step Problem Solving Process
10.2: Reasoning and Logic
10.3: Basic Arguments- Using Logic
10.4: Review Exercises

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10.1: George Polya's Four Step Problem Solving Process

Step 1: Understand the Problem

- Do you understand all the words?
- Can you restate the problem in your own words?
- Do you know what is given?
- Do you know what the goal is?
- Is there enough information?
- Is there extraneous information?
- Is this problem similar to another problem you have solved?

Step 2: Devise a Plan: Below are some strategies one might use to solve a problem. Can one (or more) of the following strategies be used? (A strategy is defined as an artful means to an end.)

1. Guess and test.	11. Solve an equivalent problem.
2. Use a variable.	12. Work backwards
3. Draw a picture.	13. Use cases.
4. Look for a pattern.	14. Solve an equation.
5. Make a list.	15. Look for a formula.
6. Solve a simpler problem.	16. Do a simulation.
7. Draw a diagram.	17. Use a model
8. Use direct reasoning.	

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10.2: Reasoning and Logic

Logic is a tool to solve problems. In mathematical logic, a PROPOSITION (or STATEMENT) is a sentence that is either true or false. It cannot be both. One way to think about it is that in order for it to be true, it must be always be true. There can be no qualifiers.

The following are NOT statements in mathematical logic:

- He has black hair. (This depends on who “he” is, so we can’t tell if it is true or false.)
- x + 3y = 6 (This depends on the values of x and y.)
- 7 + 16 (This is not even a sentence.)
- Go to the store. (This has nothing to do with being true or false.)
- Are you a woman? (This is a question.)
- Napoleon was short. (This depends on how someone defines short.)

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10.3: Basic Arguments- Using Logic

An argument requires a number of **premises** (facts or assumptions) which are followed by a **conclusion** (point of the argument). The premises are used as justification for a conclusion. A conclusion which is correctly supported by the premises is known as a **valid argument**, while a **fallacy** is a deceptive argument that can sound good but is not well supported by the premises.

We will look at examples where the first two statements are the premises, and the third statement is the conclusion.

Example 1

Determine if the following argument is valid.

All men are mortal.

John Smith is a man.

John Smith is mortal.

There are two premises (the first 2 sentences) and one conclusion (the last sentence). If we think of the premises as a and b, and the conclusion as c, then the argument in symbolic form is: $a \wedge b) \to c$. In order for the argument to be valid, we need this conditional statement to always be true. If there is ever a time, *even just one time*, when this conditional statement is false, then it is an **invalid argument**. Another way to think of this is to say that the conclusion must follow from the premises. If the premises are true, then the conclusion *must* be true in order for the argument to be valid.

Look at the argument – if we *assume* that a and b are both true, then does the conclusion have to follow? YES! If all men are mortal, and if John Smith is a man, then John Smith must be mortal. This is valid.

Example 2

Determine if the following argument is valid.

All dogs are yellow.

Flippy is a dog.

Flippy is yellow.

All dogs are yellow is equivalent to “If it is a dog then it is yellow.” That is equivalent to “If it is not yellow, then it is not a dog” by the contrapositive. Assume the premises are true. Does the conclusion have to follow? YES! So this is valid!

In both of the examples above, the first statement of the premises could be written as an “if-then” statement. “All dogs are yellow” means the same thing as “If it is a dog, it is yellow."

The above examples are examples of **Modus Ponens**, which is always a valid argument.

Format of **Modus Ponens** (which is a valid logical argument)

$p \to q$

p

q

Basically Modus Ponens states that if p implies q, and p is true, then q must also be true!

One could create a truth table to show Modus Tollens is true in all cases : $[(p \to q) \wedge p] \to q$

Example

Determine if the following argument is valid. (Hint: rewrite the “all” as “if-then”, then also write the contrapositive)

All dogs are yellow.

Chipper is yellow.

Chipper is a dog.

“All dogs are yellow” is equivalent to “If it is a dog then it is yellow.” or “If it is not yellow, then it is not a dog” by the contrapositive. Assume the premises are true. Does the conclusion have to follow? It states all dogs are yellow, but doesn’t say anything about yellow things, or that everything yellow is a dog. It is possible to have something yellow (like a lemon) that is not a dog; that means the conclusion isn’t necessarily true. This argument is invalid. Let p stand for “It is a dog.” Let q stand for “It is yellow.” The format of the above argument, shown below, is not Modus Ponens.

It is an example of **Fallacy by Converse Error**.

$p \rightarrow q$
p
q

Remember the example where p is “You live in Vista” and q is “You live in California”? Consider

If you live in Vista, then you live in California.	p → q
Johns lives in Vista.	p
So, John lives in California.	q

This is a valid logical statement because it is of the form **Modus Ponens**.

If you live in Vista, then you live in California.	p → q
John lives in California.	q
So, Johns lives in Vista.	p

This is an invalid argument, and is an example of **Fallacy by Converse Error**.

If you live in Vista, then you live in California.	p → q
Johns does not live in Vista.	~ p
So, John does not live in California.	~q

This is also an invalid argument, and is an example of **Fallacy by Inverse Error**.

If you live in Vista, then you live in California.	p → q
John does not live in California.	~ q
So, John does not live in Vista.	~ p

This is a valid argument, and is an example of **Modus Tollens**.
Modus Tollens is based on the contrapositive. Remember that p → q *is logically equivalent to* (~ q) → (~ p)
So the above argument could be written in four steps:

If you live in Vista, then you live in California.	p → q
Rewrite as the contrapositive:	
If you do not live in California, you do not live in Vista.	(~ q) → (~ p)
John does not live in California.	~ *q*
So, John does not live in Vista.	~ *p*

The last three statements LOOKS like Modus Ponens. But the original argument only had three lines. It wasn't written as the contrapositive. So it's not called Modus Ponens. One could create a truth table to show Modus Tollens is true in all cases: [(p → q) $\wedge\ q] \rightarrow\ p$

Another reasoning argument is called the **Chain Rule (transitivity)**. Below is an example. The first two sentences are the premises, and the last is the conclusion. If the first two are true, the conclusion is true.
If I have a bus pass, I will go to school.
If I go to school, I will attend class.
If I have a bus pass, I will attend class.
So the idea is that if "if p, then q" and "if q, then r" are both true, then "if p, then r" is also true.
Symbolically, the chain rule is: [(p → q) $\wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
One could create a truth table to show the truth table is true in all cases, but it's more complicated because there are 3 statements, hence 8 rows in the truth table.
The format for the Chain Rule where the first two lines are the premises and the third is the conclusion is:
p → q
q → r
p → r

Exercise 17

17. If the two statements below are premises, use the Chain Rule to state the conclusion.

If Mia doesn't study, then Mia does not pass the final.

If Mia does not pass the final, then Mia does not pass the class.

What about a logic statement where all of the outcomes of a formula are true in every situation? When this happens, it is called a **tautology**. Modus Ponens, Modus Tollens, and the Chain Rule (transitivity) are tautologies. A truth table will show the statement true in each row of the column for that statement. **A fallacy is when all the outcomes of a logic statement are false.** An example of a fallacy in words is "I called Jim and I did not call Jim." If p is "I called Jim," the logic statement in symbols for this fallacy is $p \wedge\ p$). A tautology would be "I called Jim or I did not call Jim," which is written as $p \vee\ p$)

You will create your own truth tables for Modus Ponens and Modus Tollens in the next exercises. Create intermediate columns so it is clear how you get the final column, which will show each is a tautology.

Exercise 18

18. Make a Truth Table showing Modus Ponens is a valid argument. In other words, create and fill out a truth table where the last column is [(p → q) $\wedge p] \rightarrow q$, and show that in all four situations, it is true, which means it is a tautology

Exercise 19

19. Make a Truth Table showing Modus Tollens is a valid argument. In other words, create and fill out a truth table where the last column is [(p → q) $\wedge\ q] \rightarrow\ \ p$, and show that in all four situations, it is true.

SUMMARY of arguments, where the first two statements are premises, and the third is the conclusion.

Format of **Modus Ponens** (which is a valid logical argument)
p → *q*
p
q

Format of **Modus Tollens** (which is a valid logical argument)
p → *q*
~*q*
~*p*

Format of **Fallacy by the Converse Error** (an invalid argument)
p → *q*
q
p

Format of **Fallacy by the Inverse Error** (an invalid argument)
p → *q*
~*p*
~*q*

Format of **Chain Rule** (which is a valid logical argument)
p → *q*
q → *r*
p → *r*

Exercise 20

20. Determine if the following arguments are valid or not. If they are valid, write if it is by Modus Ponens, Modus Tollens, or the Chain Rule. If it is not valid, write if it is by Fallacy by Converse Error, or Fallacy by Inverse Error, or neither. If it looks like the chain rule, but has a false conclusion, write the correct conclusion.

If you are a gambler, then you are not financially stable.
Sean isn’t financially stable.
Sean is a gambler.

If you have a college degree, then you are not lazy.
Marsha has a college degree.
Marsha is not lazy.

If you have a college degree, then you are not lazy.
Shannon is lazy.
Shannon does not have a college degree.

If you have a college degree, then you are not lazy.
Beth is not lazy.
Beth has a college degree.

If you are a kitten, then you are a cat.
If you are a cat, then you can purr.
If you are a kitten, then you can purr.

If you are a comedian, then you are funny.
If you are funny, then you are smart.
If you are smart, then you are a comedian.

? Exercise 21

21. Write a conclusion that would make each argument valid, and state if you used Modus Ponens or Modus Tollens.

If you are a gambler, then you are not financially stable.
Hollis is financially stable.

If a defendant is innocent, then he does not go to jail.
Podric went to jail

? Exercise 22

22. Determine whether there is a problem with the person’s thinking. Explain your reasoning.

A) John’s mom told him “If you get home after 10pm, then you are grounded.” John got home at 9:30pm and was grounded. He was really ticked off because he said that she lied to him. Did she?

B) Marcia told her daughter: “If you get home before 10pm, then I will give back your cell phone.” Her daughter got home at 9:45pm, but her mom didn’t give back the cell phone. Did her mother lie?

? Exercise 23

23. Create a truth table for $p \vee (p \rightarrow q)$

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10.4: Review Exercises

Exercise 1

1. State George Polya's Four Step Problem Solving Process. Only list the 4 headings.

Exercise 2

2. State 4 strategies for Devising a Plan.

Exercise 3

3. Juan will give you $1 on the first day of April, $2 on the second day of April, $4 on the third day of April, and double the amount through April 15th, or he will give you $30,000 up front on the first day of April. Which is the better deal and why?

Exercise 4

4. Find the next three numbers in the sequence:

a. 0, 10, 10, 20, 20, 20, 30, 30, 30, 30, 40,

b. 5, 15, 45, 135,

Exercise 5

5. List the ways you can make change for $25 using $5, $10, and $20 bills.

Exercise 6

6. Explain the difference between inductive and deductive reasoning.

Exercise 7

7. State whether each of the following is a proposition are not. If not, state why not. If it is a proposition, state its truth value.

a. $3 + 4 = 7$

b. California is not in the U.S.

c. Mary is pretty.

Exercise 8

8. Determine whether each quantified statement is true or false.

a. All men have black hair.

b. Some women have blond hair.

c. No boys like pigs.

Exercise 9

9. State the negation of each statement.

a. Grass is green.

b. Tim married Tom.

c. It is not cold outside.

Exercise 10

10. State the negation of each quantified statement.

a. Everyone plays Pokemon Go.

b. Some kids wear hats.

c. No dogs jump on the couch.

Exercise 11

11. If p is true and q is false, state the truth value of each of the following.

a. $p \land q$
b. $p \lor (\ q)$
c. p → q
d. q → p
e. $\sim(p \lor q)$

Exercise 12

12. Negate each statement.

a. I am married if I wear a wedding ring.

b. It is hot outside and I am at the beach.

c. My name is Julie or my name is Beth.

Exercise 13

13. State the converse, inverse, and contrapositive of this conditional.

“If Omar doesn’t buy cat food, then Omar owns a fish.

Exercise 14

14. Determine if the following arguments are valid or not. If they are valid, write if it is by Modus Ponens, Modus Tollens, or the Chain Rule. If it is not valid, write if it is by Fallacy by Converse Error, or Fallacy by Inverse Error, or neither. If it looks like the chain rule, but has a false conclusion, write the correct conclusion.

If you are a child then you are not a legal adult. Sean isn’t a legal adult. Sean is a child.
If you drink wine, then you are at a bar. Marsha is at a bar. Marsha drinks wine.
If you eat hot dogs, then it is 4^{th} of July. It is 4^{th} of July. You are eating hot dogs.

If you like vegetables, then you are vegan.
Beth is not vegan.
Beth does not like vegetables.

If you are in college, then you have a car.
If you have a car, then you know how to drive.
If you are in college, then you know how to drive.

? Exercise 15

15. Write a conclusion to make each argument valid. State if you used Modus Ponens or Modus Tollens.

If you eat worms, then you are a fish.
Matt is not a fish.

If you have a certificate, then you have a job.
Isabel has a certificate.

? Exercise 16

16. What is a tautology? What is a fallacy?

? Exercise 17

17. Make a truth table for the following. State if it is a tautology, fallacy, or neither.

a. p → $(p \vee q)$

b. ~$(plandq) \to q$

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CHAPTER OVERVIEW

11: Material Cards

https://www.dropbox.com/home/LIbrete...s_reduced.docx

- 11.1: Coins
- 11.2: A-Blocks
- 11.3: Value Label Cards
- 11.4: Models for Base Two
- 11.5: Unit Blocks
- 11.6: Base Two Blocks
- 11.7: Base Three Blocks
- 11.8: Base Four Blocks
- 11.9: Base Five Blocks
- 11.10: Base Six Blocks
- 11.11: Base Seven Blocks
- 11.12: Base Eight Blocks
- 11.13: Base Nine Blocks
- 11.14: Base Ten Blocks
- 11.15: Base Eleven Blocks
- 11.16: Base Twelve Blocks
- 11.17: Supplementary Longs
- 11.18: Centimeter Strips
- 11.19: Counters
- 11.20: Number Squares
- 11.21: Fraction Circles
- 11.22: Strips and Arrays

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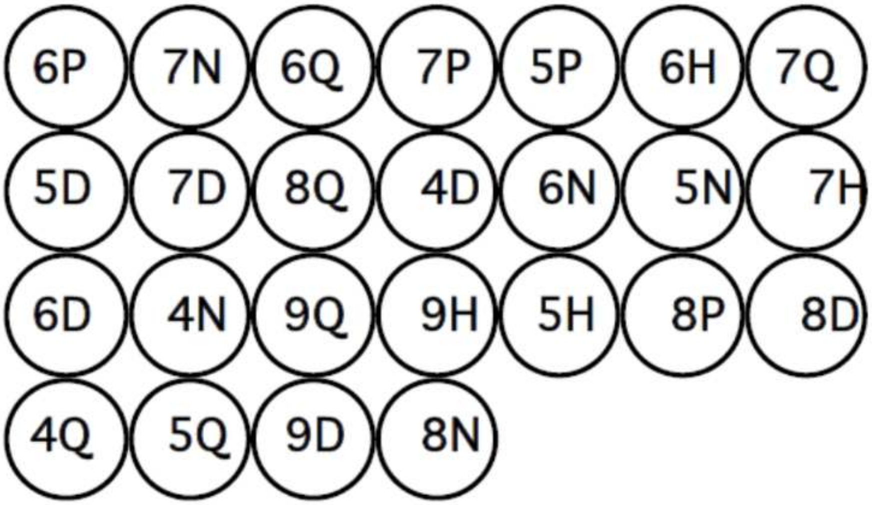
11.1: Coins

Below are two sets of coins. You can either make square coins, which are easier and faster to cut out or you can make round coins. **You only need one set.** If you lose any pieces, you can always use the second set. In any case, they are all the same size even though they represent pennies, nickels, dimes, quarters and half dollars. These represent coins from 1964-1969. The number represents the last digit of the year from 1964 through 1969 (**4, 5, 6, 7, 8** or **9**) and the letter represents the first letter of the denomination (**P,N,D,Q** or **H**). For instance, **6P** would stand for a 1966 penny.

You might want to keep these in an envelope and paperclip it to your workbook so they are readily available for use.

6P	7N	6Q	7P	5P	6H	7Q
5D	7D	8Q	4D	6N	5N	7H
6D	4N	9Q	9H	5H	8P	8D
4Q	5Q	9D	8N			



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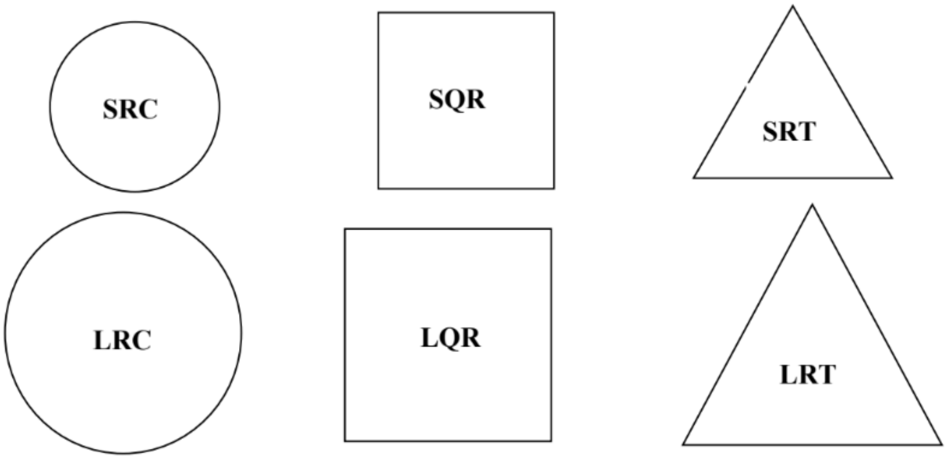
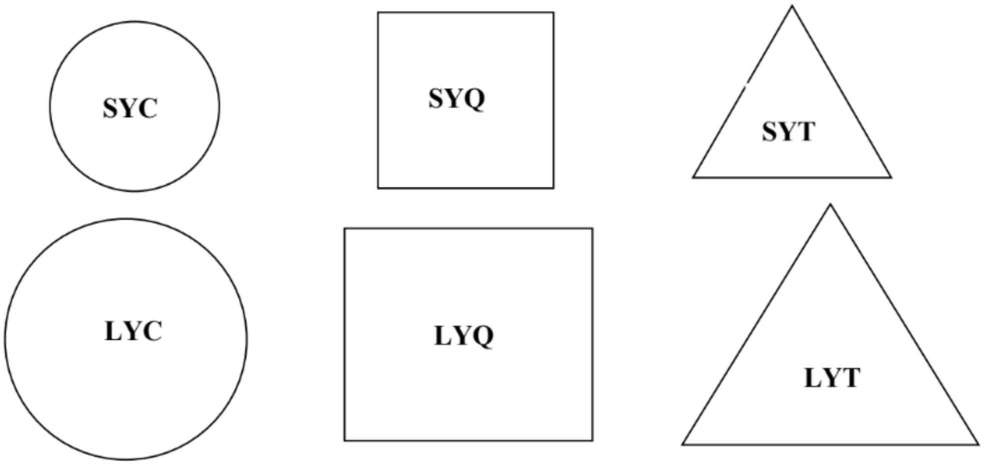
11.2: A-Blocks

The A-Blocks you'll be making will consist of 24 objects. There are two sizes (small and large), four colors (red, yellow, green and blue) and three shapes (circle, square and triangle).Each of the 24 elements is labeled with a three letter code. The first letter refers to its size (**S** or **L**), the second letter refers to its color (**R, Y, G** or **B**) and the last letter refers to its shape (**C, Q** or **T**). We are using **Q** to represent the square. For instance, **SGT** is the code for the small green triangle

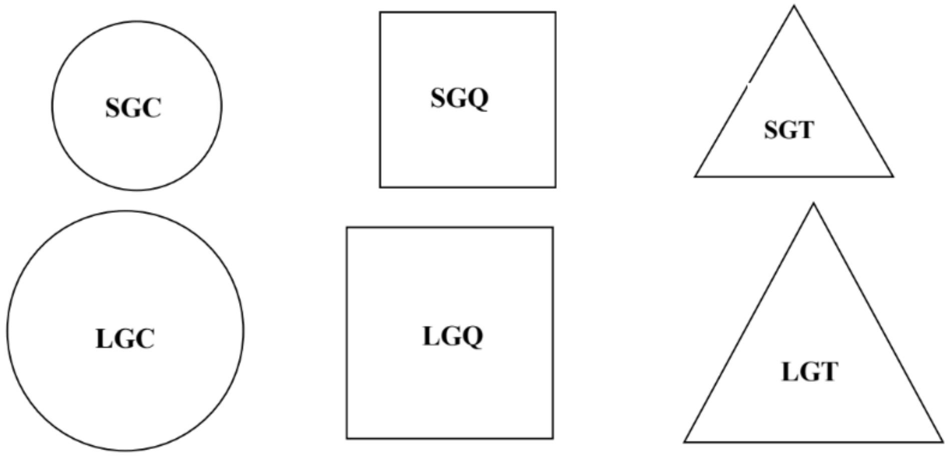
2A Red A-Blocks



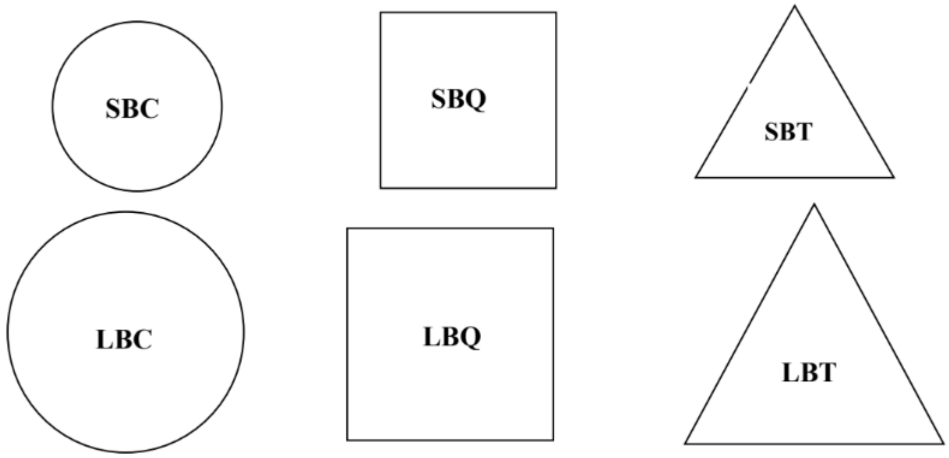
2B Yellow A-Blocks



2C Green A-Blocks


SGC
SGQ
SGT
LGC
LGQ
LGT

2D Blue A-Blocks


SBC
SBQ
SBT
LBC
LBQ
LBT

2E White

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11.3: Value Label Cards

Below are two sets of value label cards to be used with the set of A-blocks in various exercises. Each card has a value on it. The first set has the full value written out and the second set has the abbreviations. Use whichever set you prefer to work with. There is a negative value card for each positive value card. By negative, I am referring to the complement of a set. For instance, there is a subset of **BLUE** (**B**) A-blocks. There are six elements in this subset The negative value card would be **NOT BLUE** ($\mathbf{B}^c$) and this represents the set of all A-blocks that are not blue. There are 18 elements in that subset. **LARGE** and **SMALL** are complements of each other so there aren't separate cards for their negative values. There are some extra cards left blank so that you can fill in some creative labels like $\mathbf{B} \cap \mathbf{T}$, $\mathbf{B} \cup \mathbf{T}$ or some other combinations needed for certain exercises. These cards need to be cut out and saved along with your A-blocks. Preferably, this is on cardstock paper or laminated. If not and these labels are too flimsy, you can make some labels on index cards so they are sturdier.

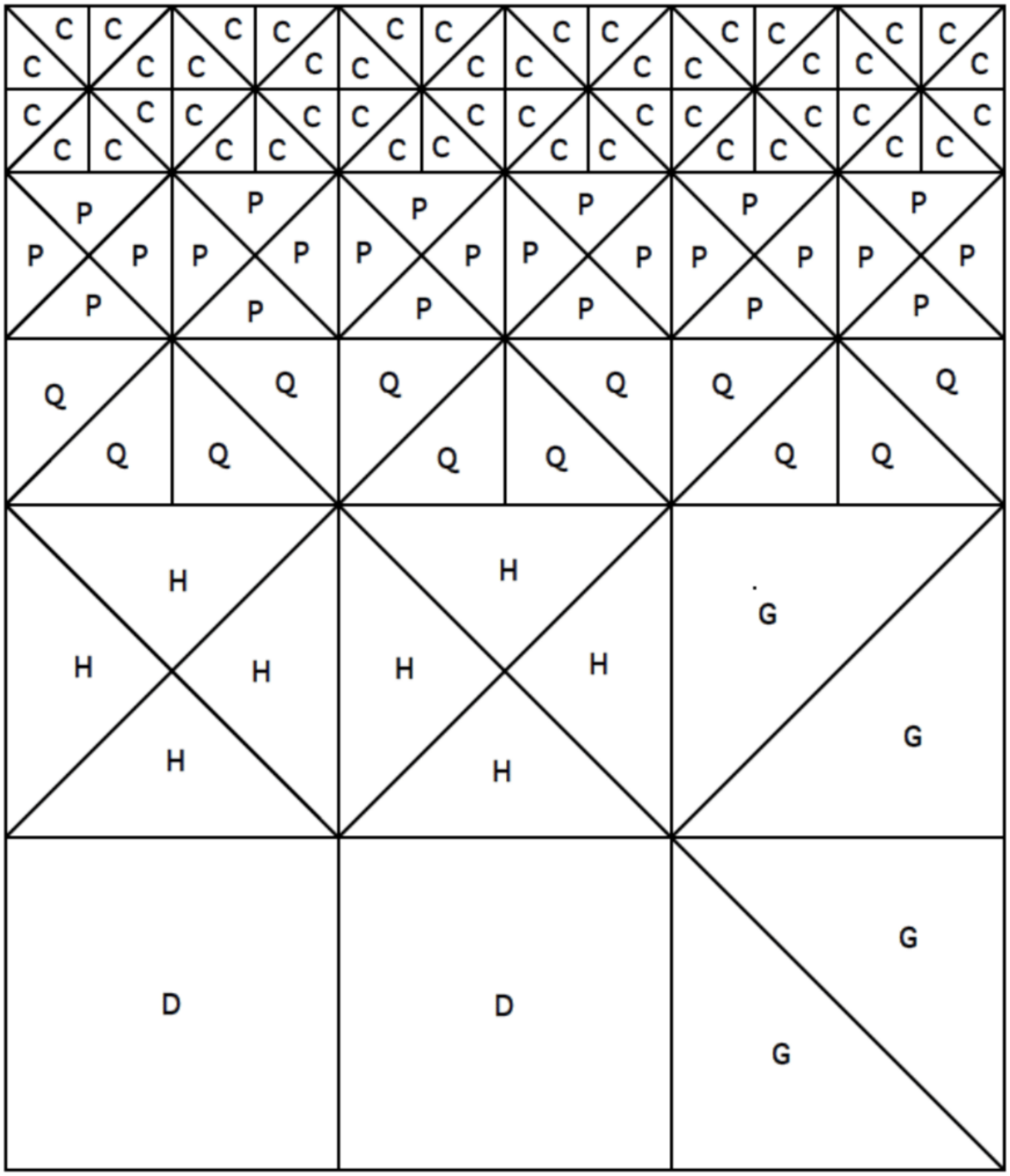
RED	**NOT RED**
BLUE	**NOT BLUE**
GREEN	**NOT GREEN**
YELLOW	**NOT YELLOW**
CIRCLE	**NOT CIRCLE**
SQUARE	**NOT SQUARE**
TRIANGLE	**NOT TRIANGLE**
LARGE	**SMALL**

R	$\mathbf{R}^c$
B	$\mathbf{B}^c$
G	$\mathbf{G}^c$
Y	$\mathbf{Y}^c$
C	$\mathbf{C}^c$
Q	$\mathbf{Q}^c$
T	$\mathbf{T}^c$
L	**S**

11.3: Value Label Cards is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.4: Models for Base Two

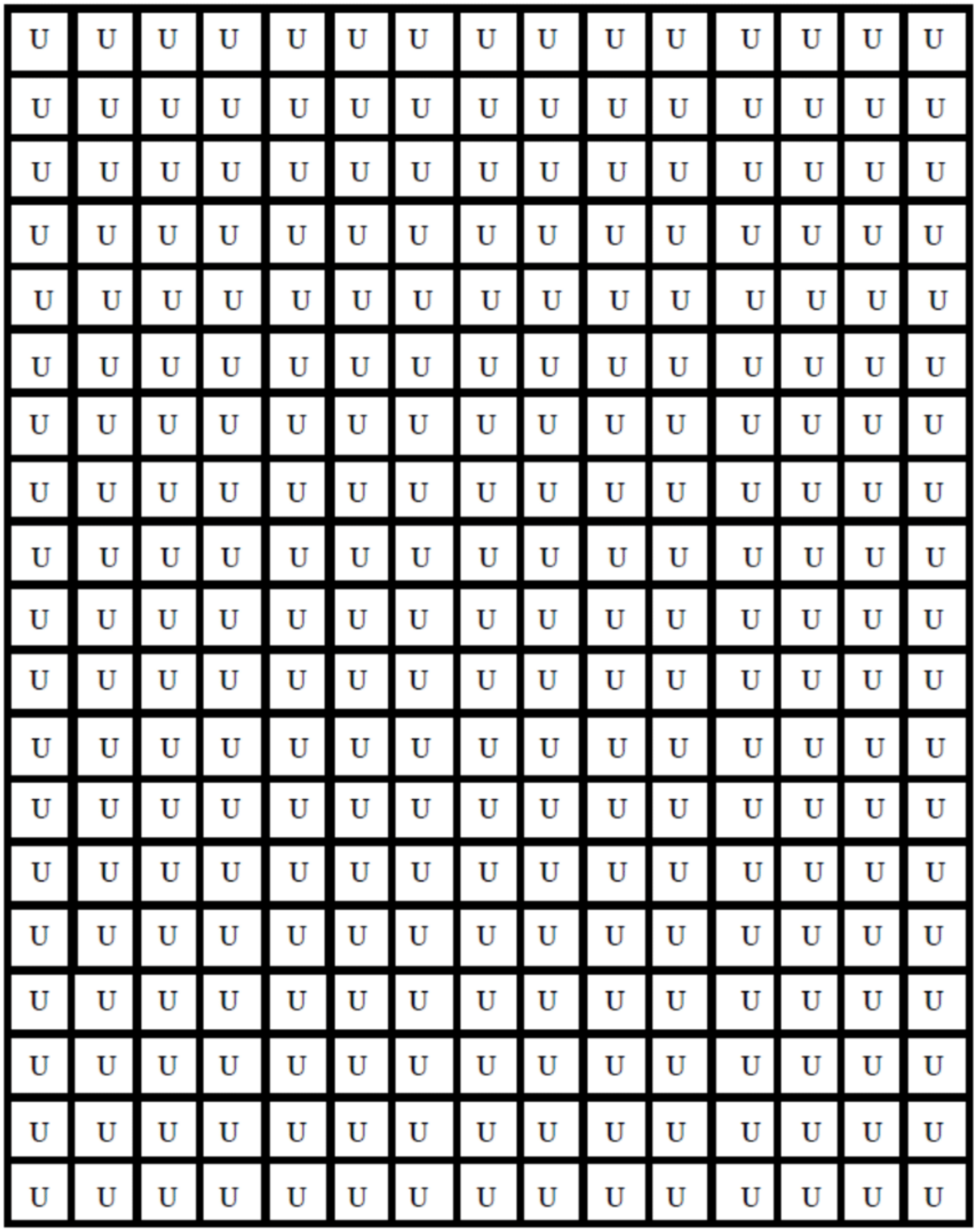
Cut along all of the line segments to make modelsof cups (**C**), pints (**P**) and quarts (**Q**), half-gallons (**H**), gallons (**G**) and double-gallons (**D**).



11.4: Models for Base Two is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.5: Unit Blocks

A set of Unit Blocks are on this material card. Cut along all solid line segments. Each piece is labeled "U". These can be used as the units with any set of Base Blocks. **There are more unit blocks here than you need to cut out. Cut out at least 60 units.** ***If you prefer, you can make extra copies of this sheet, and cut out more unit blocks so that each of the Base Blocks that follows has its own units in its baggie.***



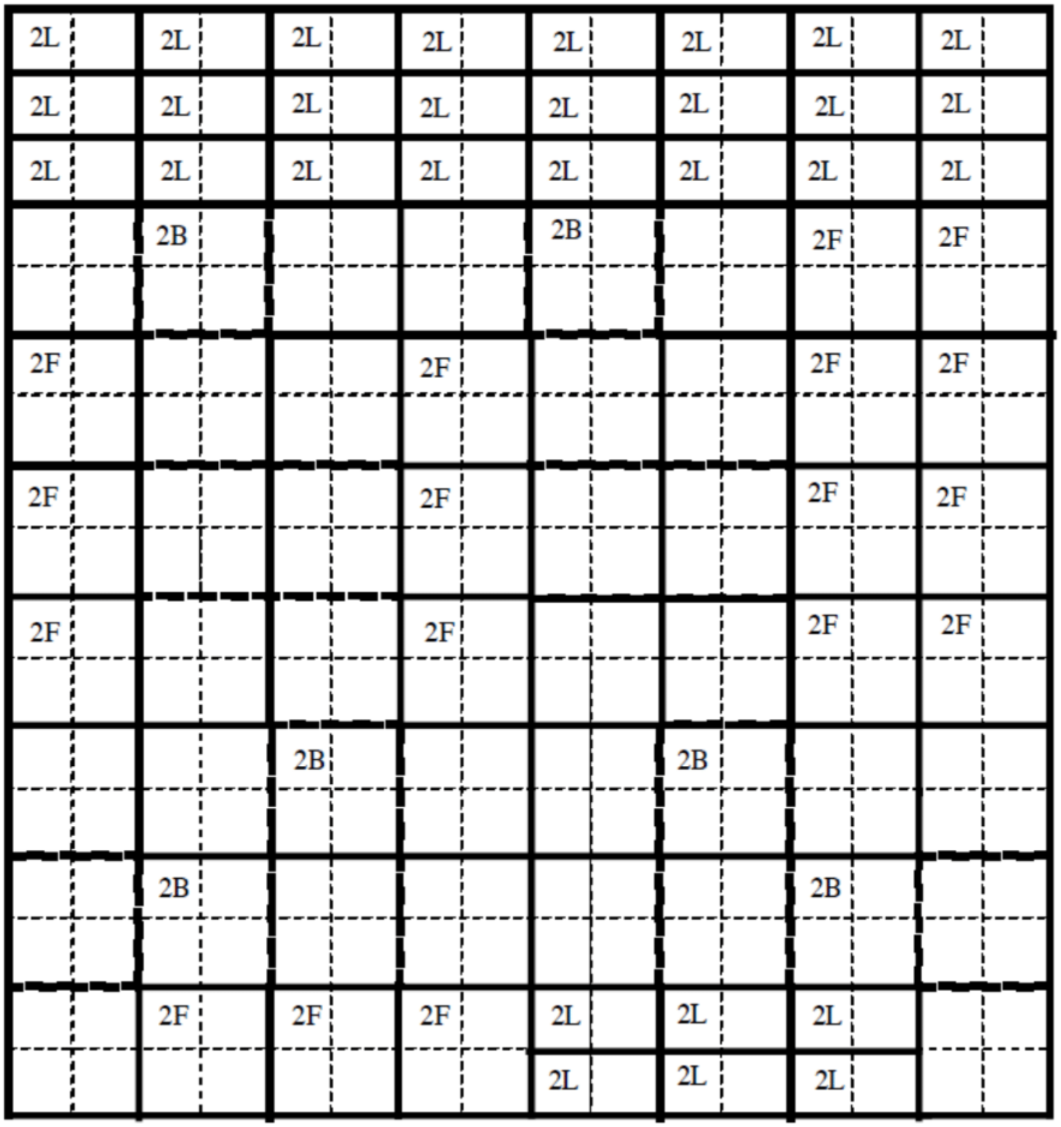
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U	U	U	U	U	U	U

11.5: Unit Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.6: Base Two Blocks

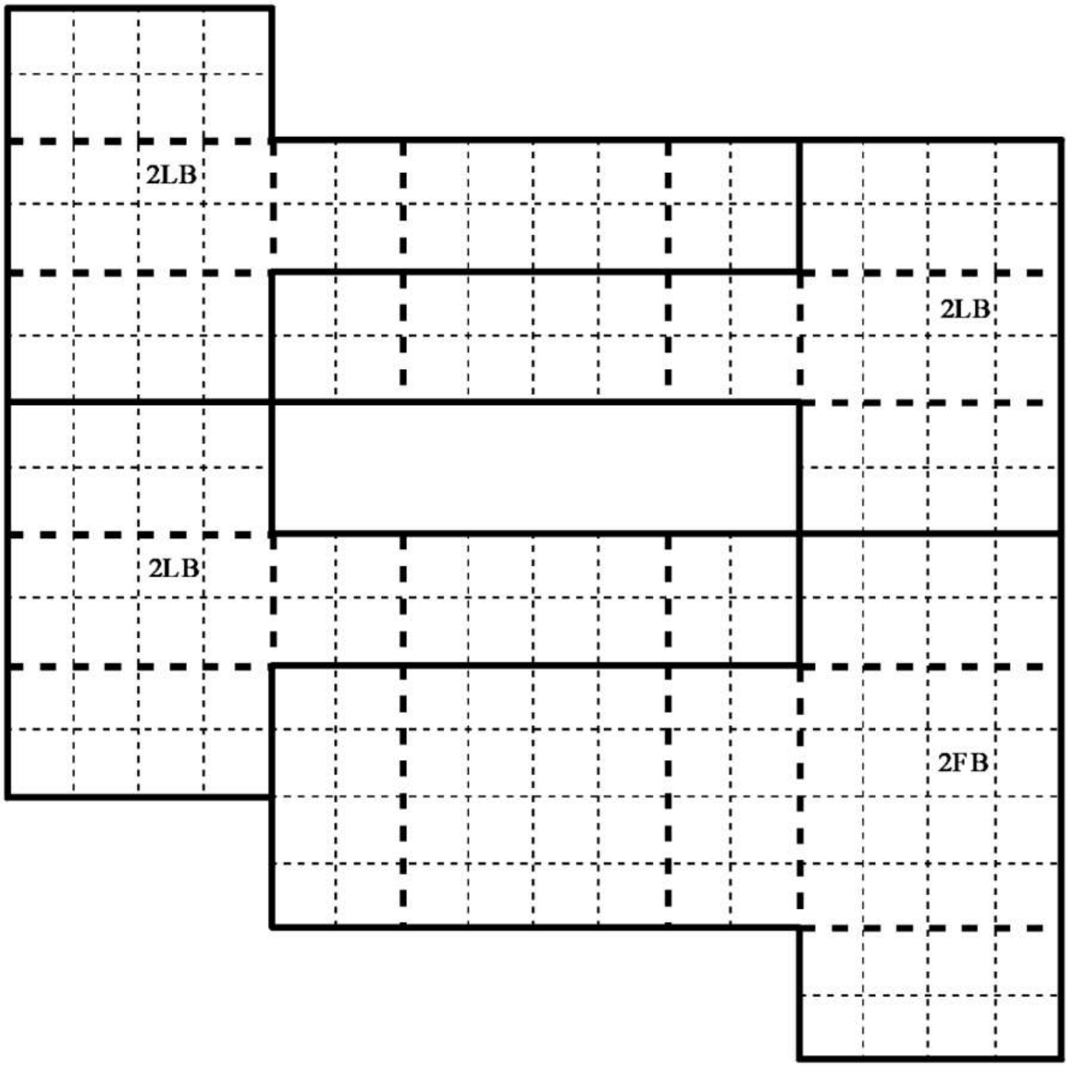
Longs (2L), Flats (2F), Blocks (2B)

A set of Base Two blocks are on this and the next material card (5B). Cut along all solid line segments. Each piece is labeled. A Base Two Long is 1 cm by 1 cm and is labeled "2L". A Base Two Flat is 2 cm by 2 cm and is labeled "2F". A Base Two Block is 2 cm by 2 cm by 2cm and is labeled "2B". Fold these 3-dimensional blocks along the dark dotted lines; then tape to make a cube. Directions for other blocks are on the next page. Use the unit blocks from Material Card 4 with this set of Base Blocks.


2L 2L 2L 2L 2L 2L 2L 2L
2L 2L 2L 2L 2L 2L 2L 2L
2L 2L 2L 2L 2L 2L 2L 2L
2B 2B 2F 2F
2F 2F 2F 2F
2F 2F 2F 2F
2F 2F 2F 2F
2B 2B
2B 2B
2F 2F 2F 2L 2L 2L
2L 2L 2L

Long Blocks (2LB) and Flat Blocks (2FB)

This page contains the remainder of the set of Base Two blocks. Cut along all solid line segments. Each piece is labeled. A Base Two Long Block is 4 cm by 2 cm by 2 cm and is labeled "2LB". The Base Two Flat Block is 4 cm by 4 cm by 2 cm and is labeled "2FB". Fold these 3-dimensional blocks along the dark dotted lines; then tape to make rectangular solids.

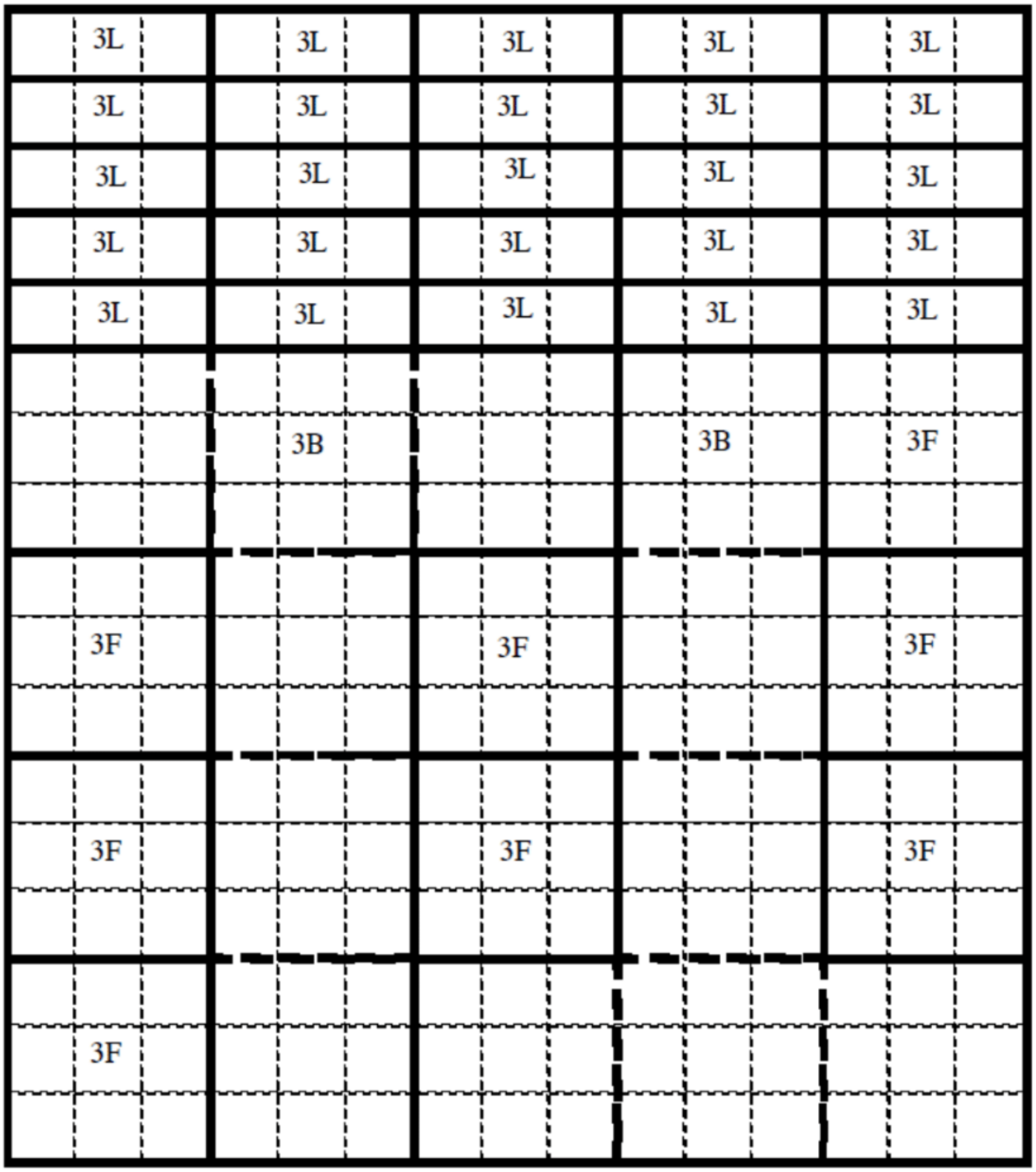


11.6: Base Two Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.7: Base Three Blocks

Longs (3L), Flats (3F) and Blocks (3B)

This page contains a set of Base Three blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks.. Cut along all solid line segments. Each piece is labeled. A Base Three Long is 1 cm by 3 cm and is labeled "3L". A Base Three Flat is 3 cm by 3 cm and is labeled "2F". A Base Three Block is 3 cm by 3cm by 3 cm and is labeled "3B". Fold these 3-dimensional blocks along the dark dotted lines; then tape to make a cube.

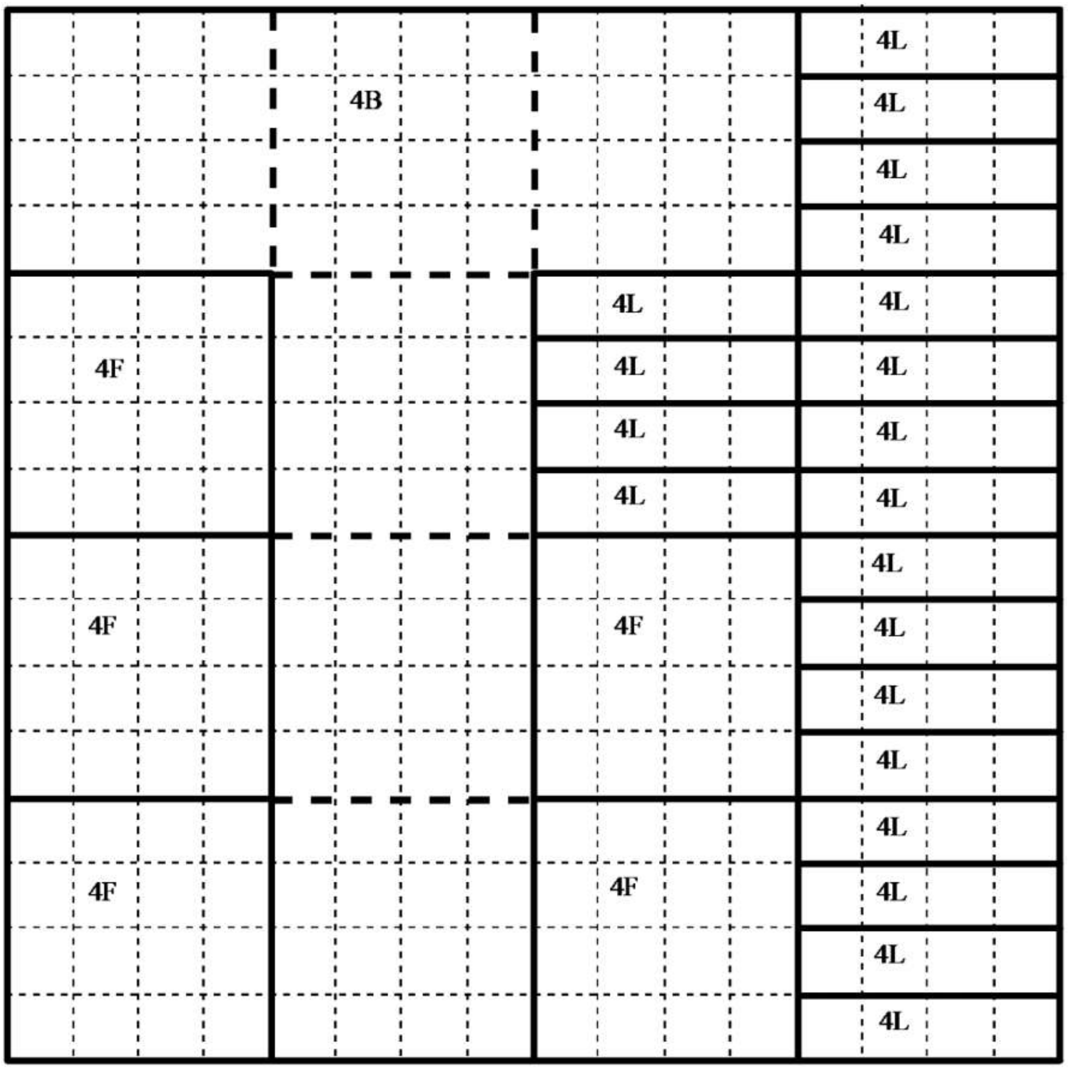

3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3L
3B
3B
3F
3F
3F
3F
3F
3F
3F
3F

11.7: Base Three Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.8: Base Four Blocks

Longs (4L), Flats (4F) and Blocks (4B)

This page contains a set of Base Four blocks.Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Four Long is 1 cm by 4 cm and is labeled "4L". A Base Four Flat is 4 cm by 4 cm and is labeled"4F". The Base Four Block is 4 cm by 4 cm by 4 cm and is labeled "4B". Fold this 3-dimensional block along the dark dotted lines; then tape to make a cube.

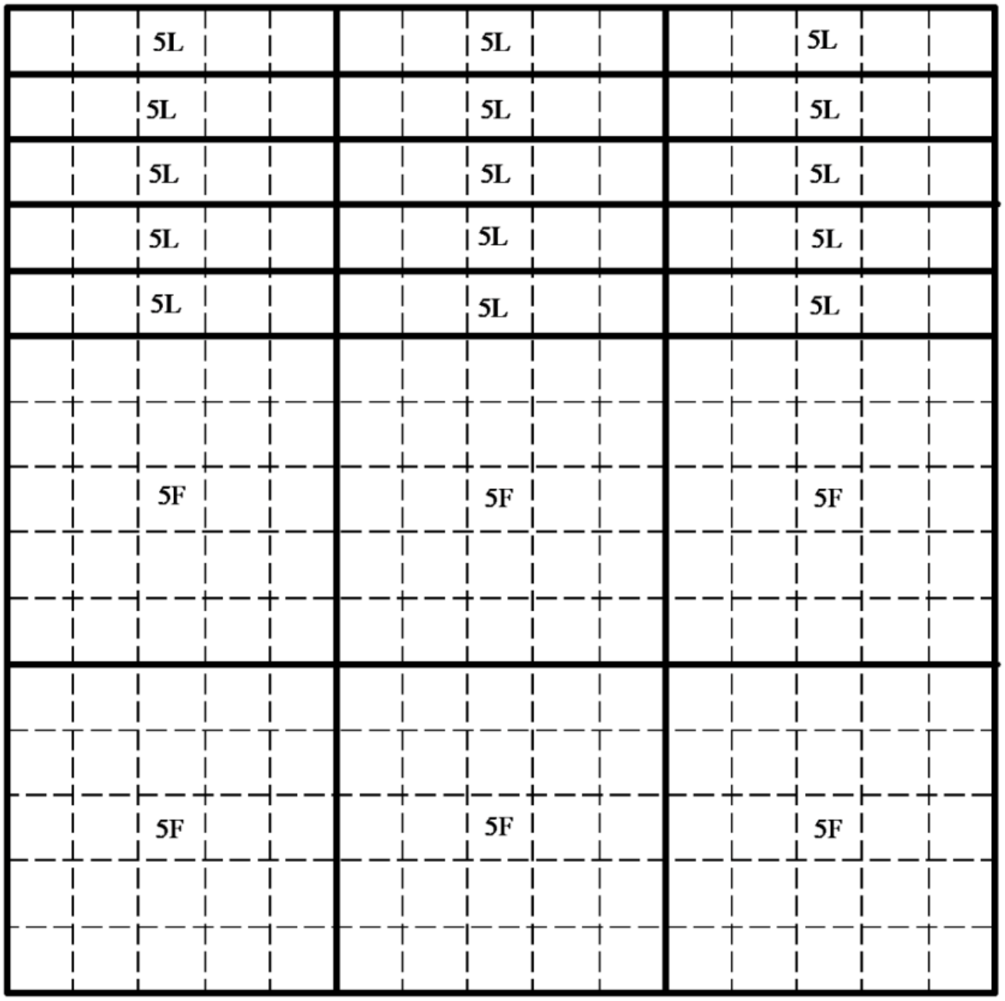


11.8: Base Four Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.9: Base Five Blocks

Longs (5L) and Flats (5F)

This page contains a set of Base Five blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Five Long is 1 cm by 5 cm and is labeled "5L". A Base Five Flat is 5 cm by 5 cm and is labeled "5F".

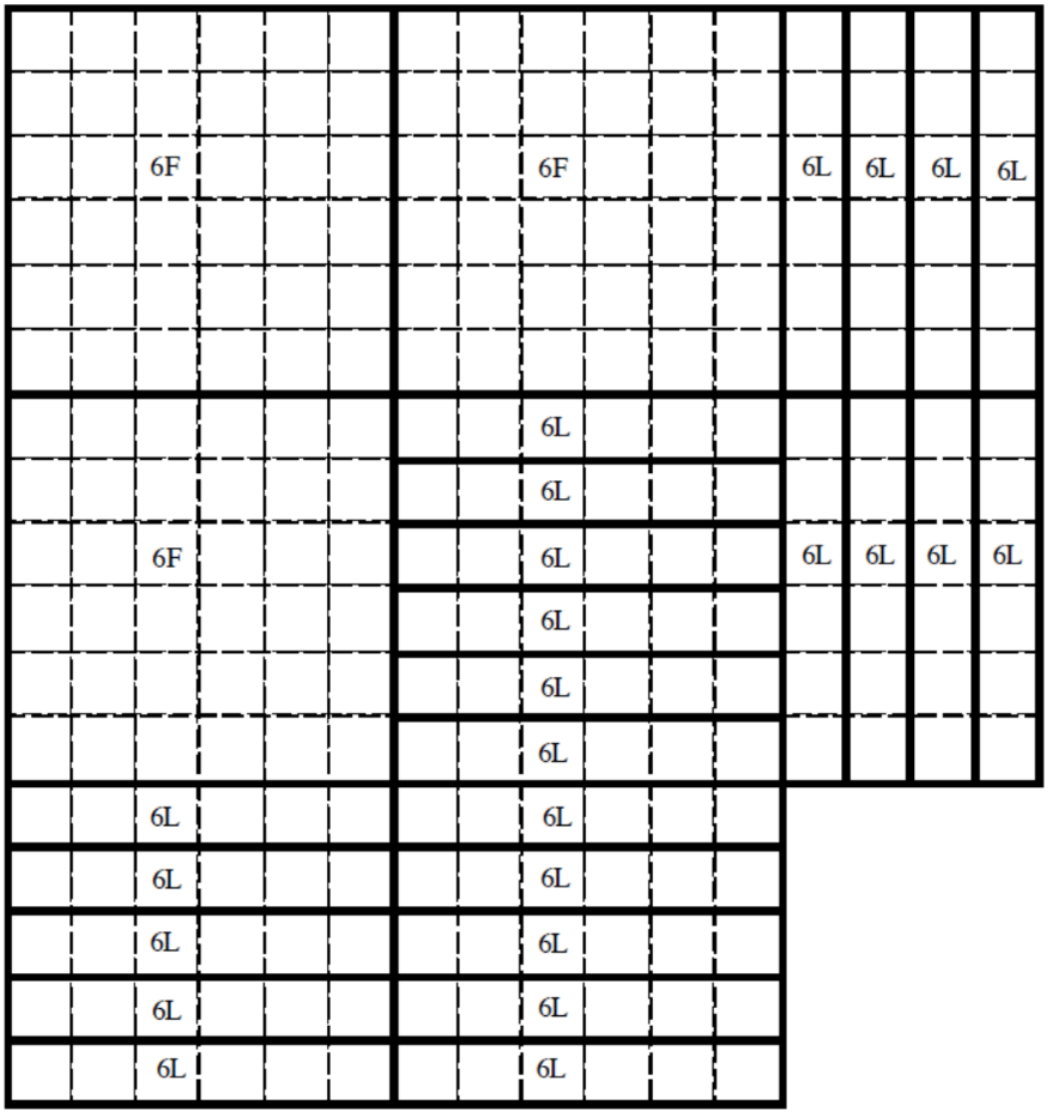

5L
5L
5L
5L
5L
5L
5L
5L
5L
5L
5L
5L
5L
5L
5L
5F
5F
5F
5F
5F
5F

11.9: Base Five Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.10: Base Six Blocks

Longs (6L) and Flats (6F)

This page contains a set of Base Six blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Six Long is 1 cm by 6 cm and is labeled "6L". A Base Six Flat is 6 cm by 6 cm and is labeled "6F".

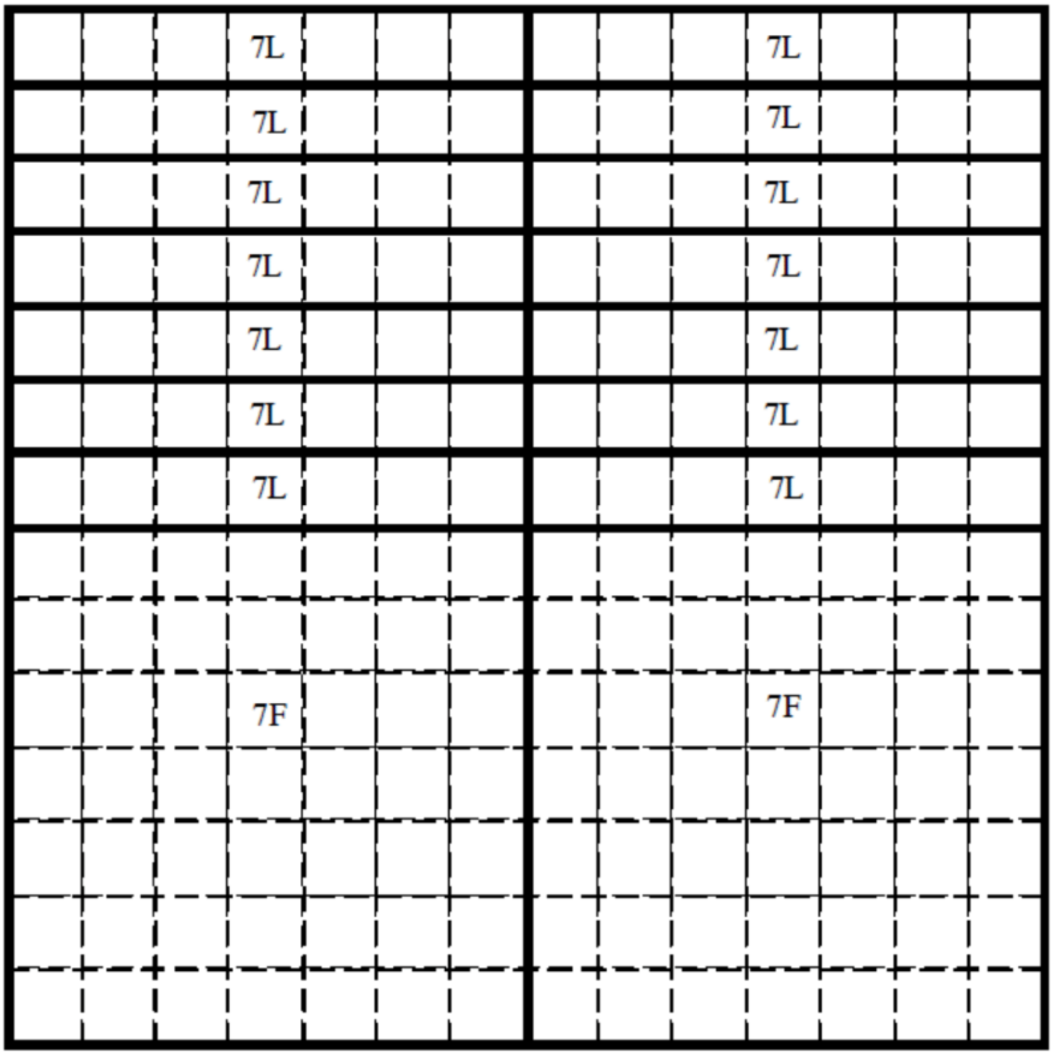

6F
6F
6L
6L
6L
6L
6L
6L
6F
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L
6L

11.10: Base Six Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.11: Base Seven Blocks

Longs (7L) and Flats (7F)

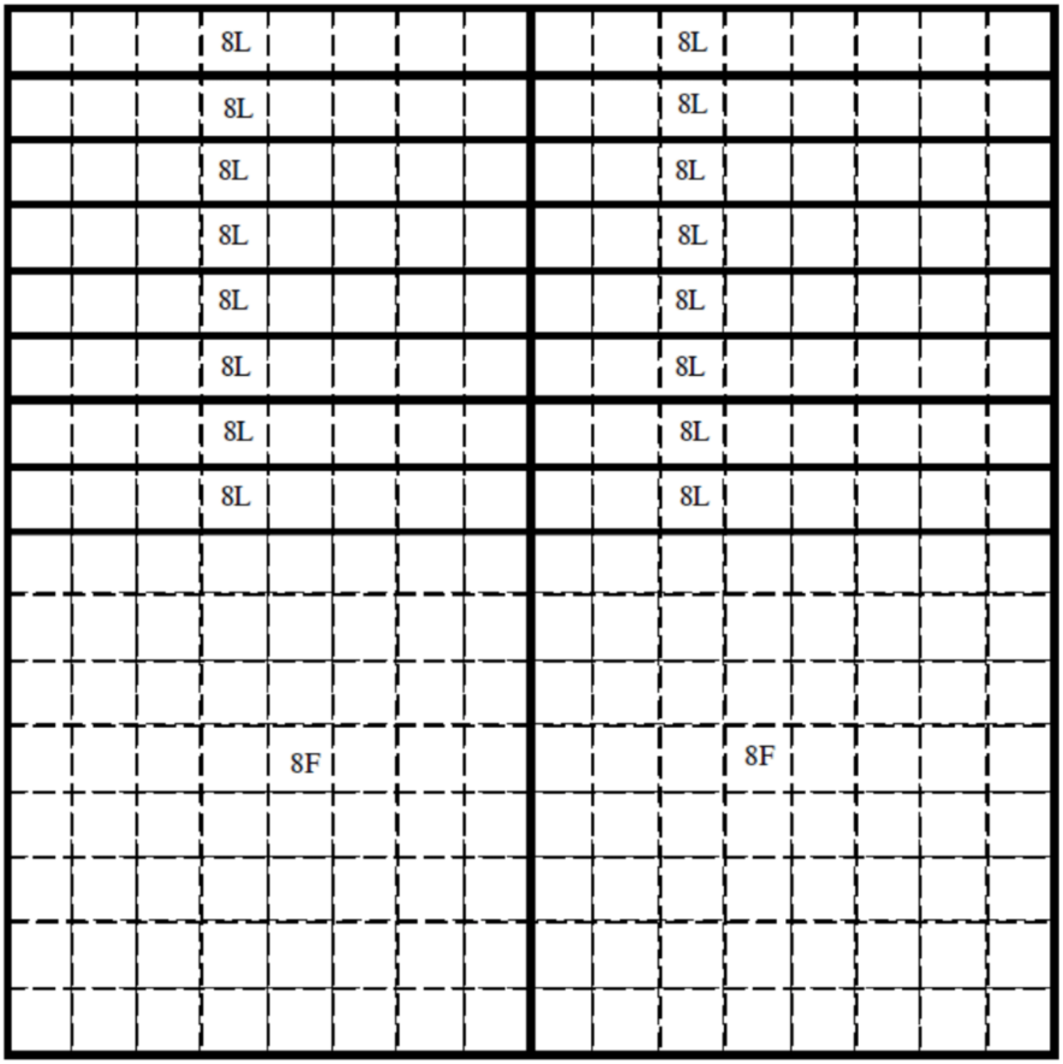
This page contains a set of Base Seven blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Seven Long is 1 cm by 7 cm and is labeled "7L". A Base Seven Flat is 7 cm by 7 cm and is labeled "7F".



11.11: Base Seven Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.12: Base Eight Blocks

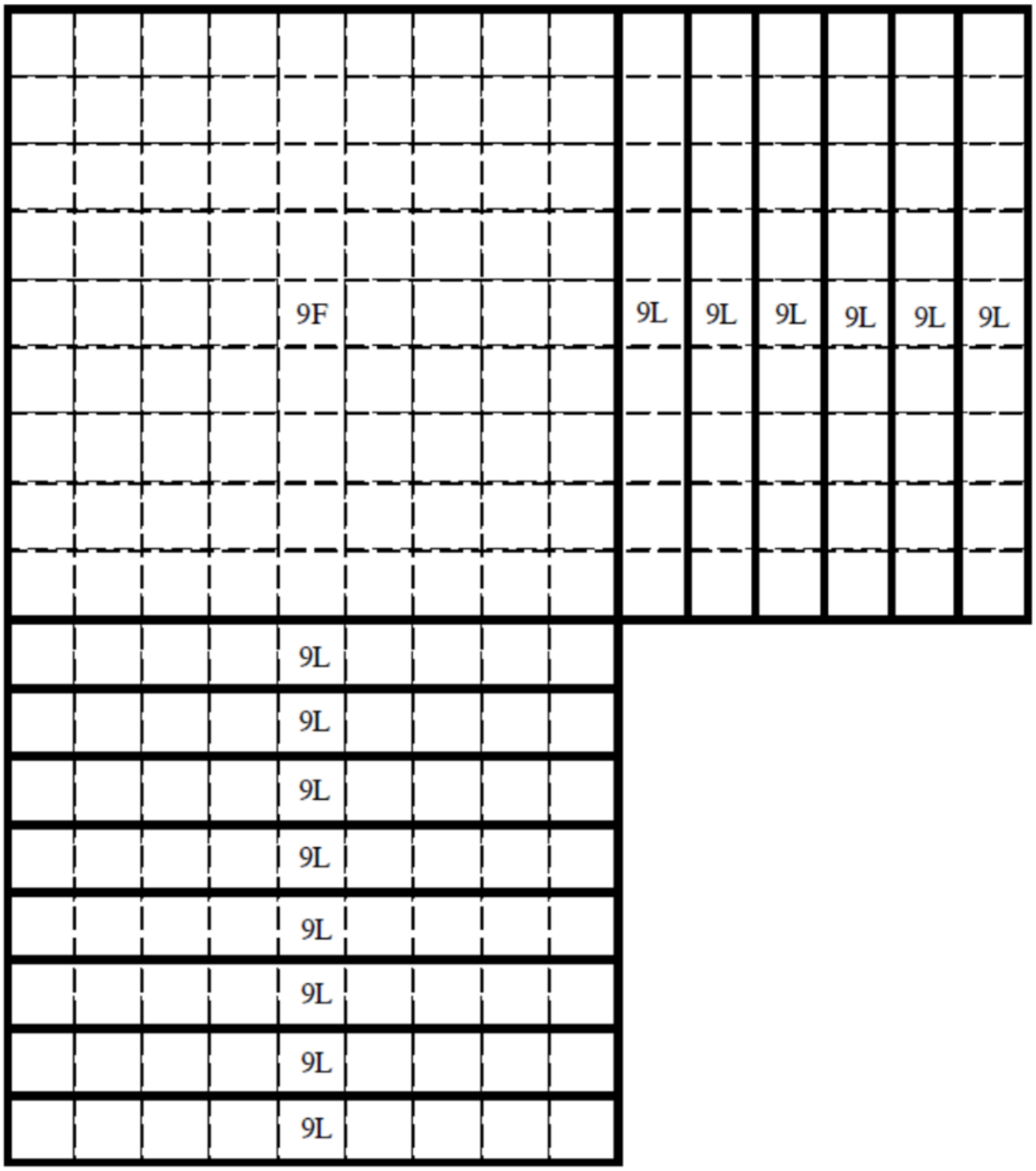
This page contains a set of Base Eight blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Eight Long is 1 cm by 8 cm and is labeled "8L". A Base Eight Flat is 8 cm by 8 cm and is labeled "8F".



11.12: Base Eight Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.13: Base Nine Blocks

This page contains a set of Base Nine blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Nine Long is 1 cm by 9 cm and is labeled "9L". The Base Nine Flat is 9 cm by 9 cm and is labeled "9F". Extra longs for Base Nine are on a separate Supplementary Material Card for Cards 12-15, found after Card 15.

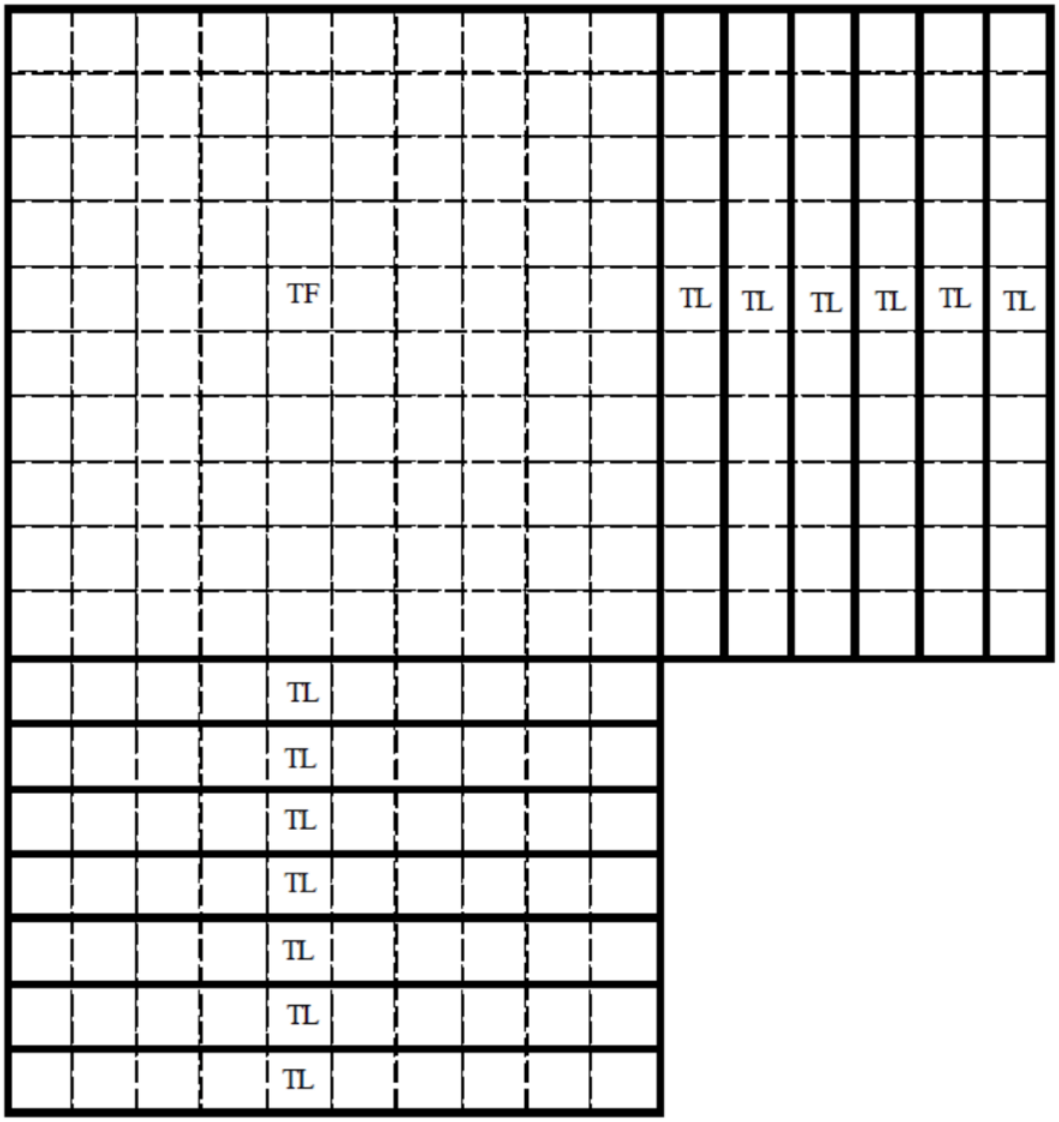


11.13: Base Nine Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.14: Base Ten Blocks

Longs (TL) and Flats (TF)

This page contains a set of Base Ten blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled. A Base Ten Long is 1 cm by 10 cm and is labeled "TL". The Base Ten Flat is 10 cm by 10 cm and is labeled "TF". Extra longs for Base Ten are on a separate Supplementary Material Card for Cards 12-15, found after Card 15.

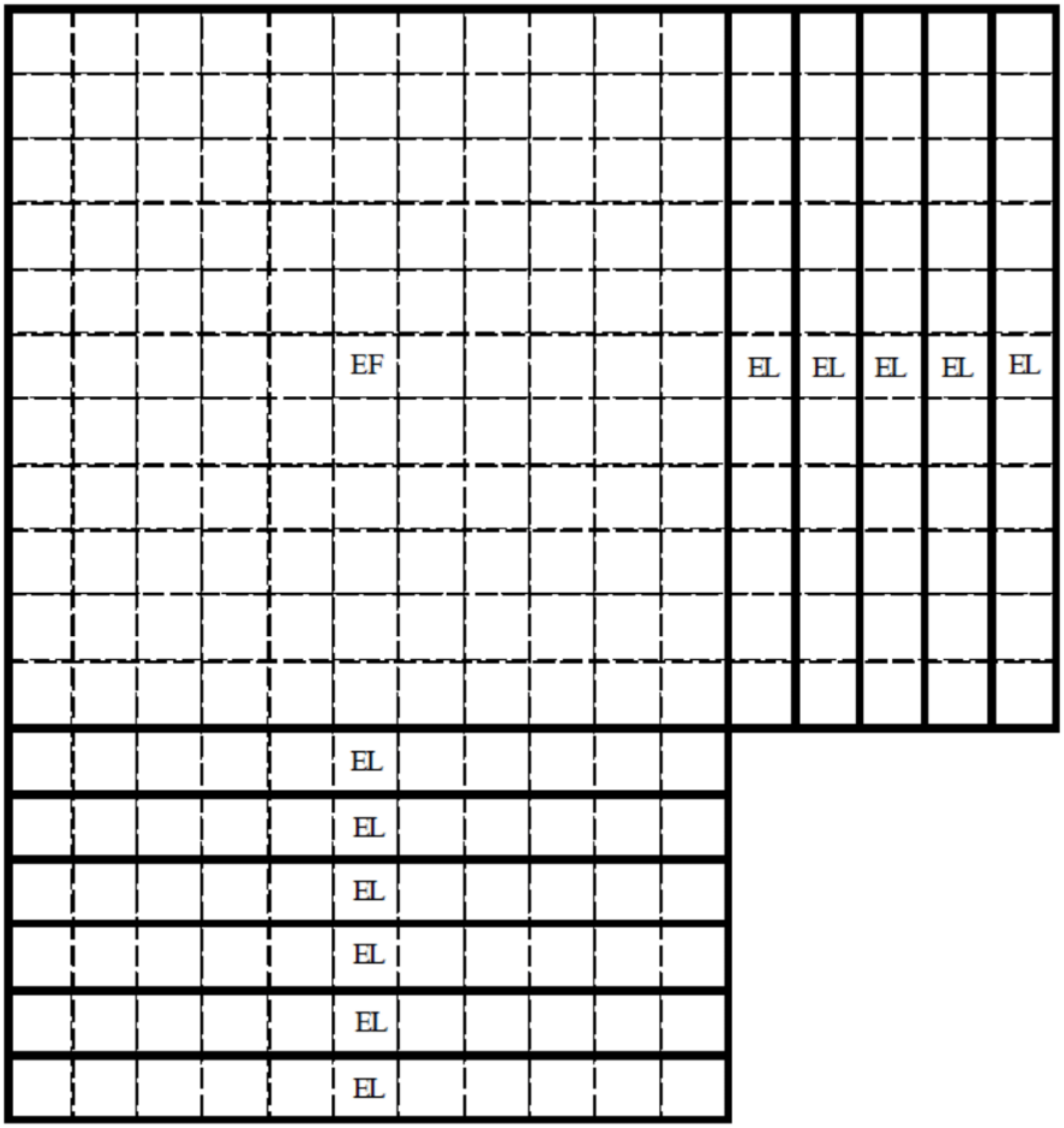


11.14: Base Ten Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.15: Base Eleven Blocks

Longs (EL) and Flats (EF)

This page contains a set of Base Eleven blocks. Use the unit blocks from Material Card 4 with this set of Base Blocks. Cut along all solid line segments.Each piece is labeled. A Base Eleven Long is 1 cm by 11 cm and is labeled "EL". The Base Eleven Flat is 11 cm by 11 cm and is labeled "EF". Extra longs for Base Eleven are on a separate Supplementary Material Card for Cards 12-15, found after Card 15.

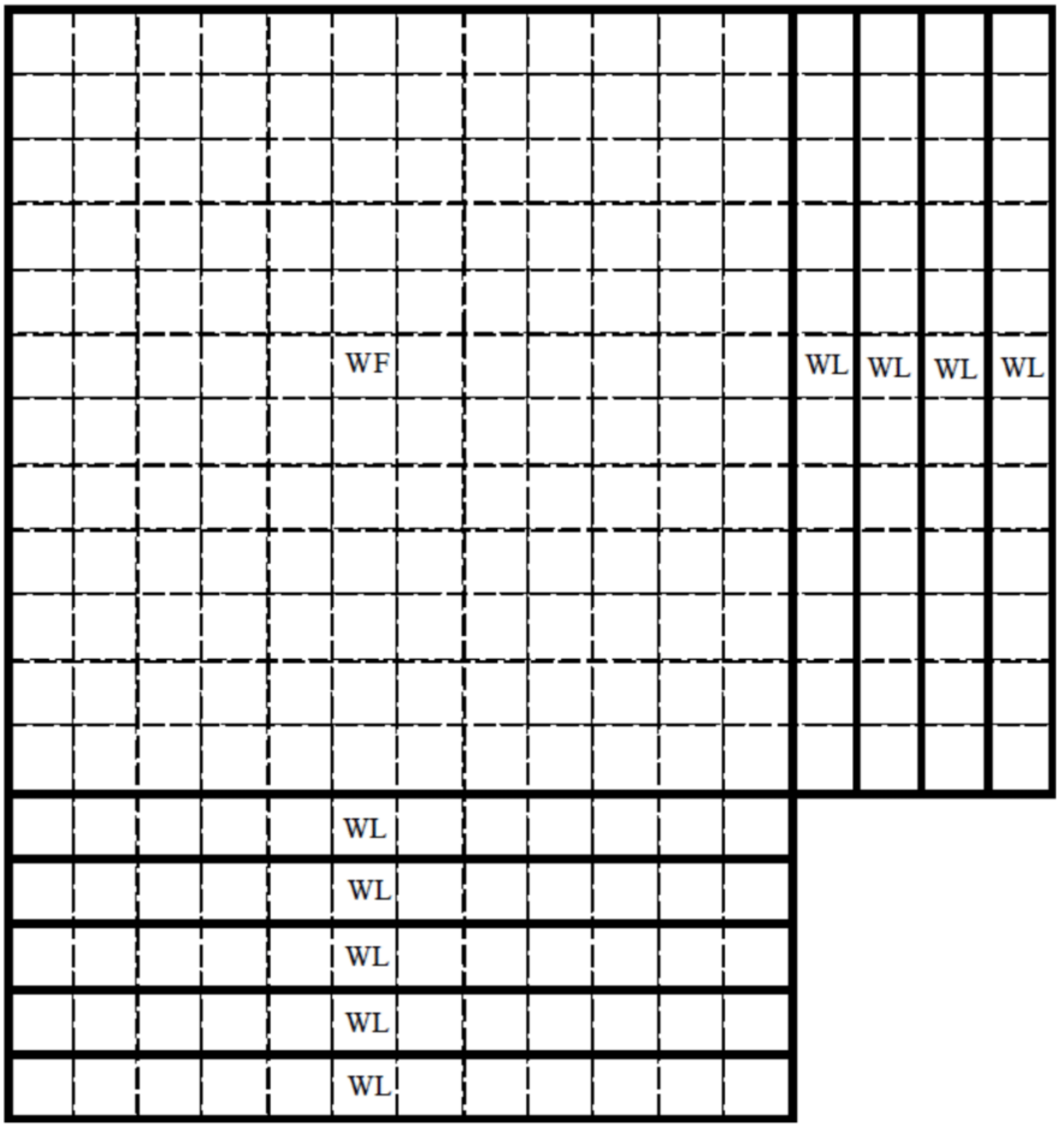


11.15: Base Eleven Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.16: Base Twelve Blocks

Longs (WL) and Flats (WF)

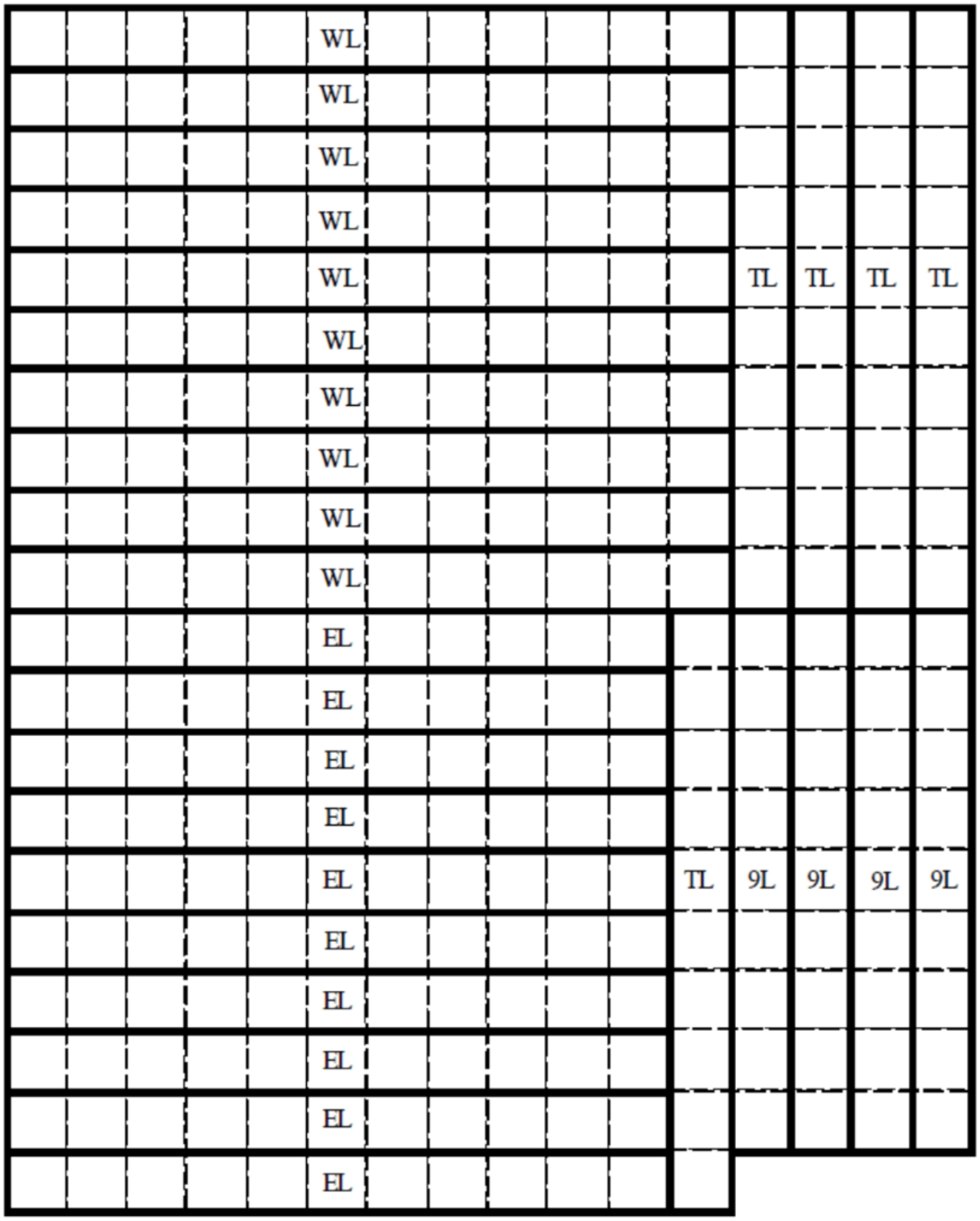
This page contains a set of Base Twelve blocks. Use the unit blocks from Material Card 9 with this set of Base Blocks. Cut along all solid line segments. Each piece is labeled.A Base Twelve Long is 1 cm by 12 cm and is labeled "WL". The Base Twelve Flat is 12 cm by 12 cm and is labeled "WF". Extra longs for Base Twelve are on a separate Supplementary Material Card for Cards 12-15, found on the next sheet.



11.16: Base Twelve Blocks is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.17: Supplementary Longs

For Base Nine (9L), Base Ten (TL), Base Eleven (EL) and Base Twelve (WL)


WL
WL
WL
WL
WL
TL
TL
TL
TL
WL
WL
WL
WL
WL
EL
EL
EL
EL
EL
TL
9L
9L
9L
9L
EL
EL
EL
EL
EL

11.17: Supplementary Longs is shared under a CC BY-NC license and was authored, remixed, and/or curated by LibreTexts.

11.18: Centimeter Strips

White Centimeter Strips

The dimensions of each strip are one centimeter by one centimeter. The abbreviation is **W**.

W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W
W	W	W	W	W	W	W	W	W	W	W	W	W	W	W

Red Centimeter Strips

The dimensions of each strip are two centimeters by one centimeter. The abbreviation is **R**.

R	R	R	R	R	R	R	R
R	R	R	R	R	R	R	R
R	R	R	R	R	R	R	R
R	R	R	R	R	R	R	R

Light Green Centimeter Strips

The dimensions of each strip are three centimeters by one centimeter. The abbreviation is **L**.

L	L	L	L	L
L	L	L	L	L
L	L	L	L	L
L	L	L	L	L
L	L	L	L	L

Purple Centimeter Strips

The dimensions of each strip are four centimeters by one centimeter. The abbreviation is **P**.

P	**P**	**P**	**P**
P	**P**	**P**	**P**
P	**P**	**P**	**P**
P	**P**	**P**	**P**
P	**P**	**P**	**P**

Yellow Centimeter Strips

The dimensions of each strip are five centimeters by one centimeter. The abbreviation is **Y**.

Y	**Y**	**Y**
Y	**Y**	**Y**
Y	**Y**	**Y**
Y	**Y**	**Y**
Y	**Y**	**Y**

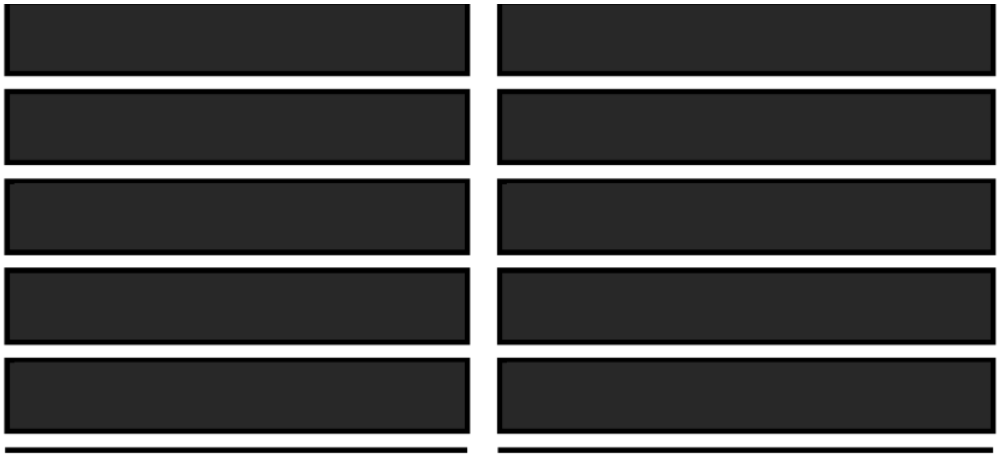
Dark Green Centimeter Strips

The dimensions of each strip are six centimeters by one centimeter. The abbreviation is **D**.

D	**D**	**D**
D	**D**	**D**
D	**D**	**D**
D	**D**	**D**
D	**D**	**D**

Black Centimeter Strips

The dimensions of each strip are seven centimeters by one centimeter. The abbreviation is **K**



Tan or Light Brown Centimeter Strips

The dimensions of each strip are eight centimeters by one centimeter. The abbreviation is **N**.

N	**N**
N	**N**
N	**N**
N	**N**
N	**N**

Blue Centimeter Strips

The dimensions of each strip are nine centimeters by one centimeter. The abbreviation is **B**.

B	**B**
B	**B**
B	**B**
B	**B**
B	**B**

Orange Centimeter Strips

The dimensions of each strip are ten centimeters by one centimeter. The abbreviation is **O**.

O
O
O
O
O

Silver or Gray Centimeter Strips

The dimensions of each strip are eleven centimeters by one centimeter. The abbreviation is **S.**

S
S
S
S
S

Hot Pink Centimeter Strips

The dimensions of each strip are twelve centimeters by one centimeter. The abbreviation is **H**.

H
H
H
H
H

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11.19: Counters

Postive Green Counters

Each counter represents a positive one (+1). THese are used to perform the four basic operations (addition, subtraction, multiplication and division) on the integers (positive and negative whole numbers and zero).

+	+	+	+	+	+	+
+	+	+	+	+	+	+
+	+	+	+	+	+	+
+	+	+	+	+	+	+

Negative Red Counters

Each counter represents a negative one (-1). These are used to perform the four basic operations (addition, subtraction, multiplication and division) on tthe integers (positive and negative whole numbers and zero).

−	−	−	−	−	−	−
−	−	−	−	−	−	−
−	−	−	−	−	−	−
−	−	−	−	−	−	−

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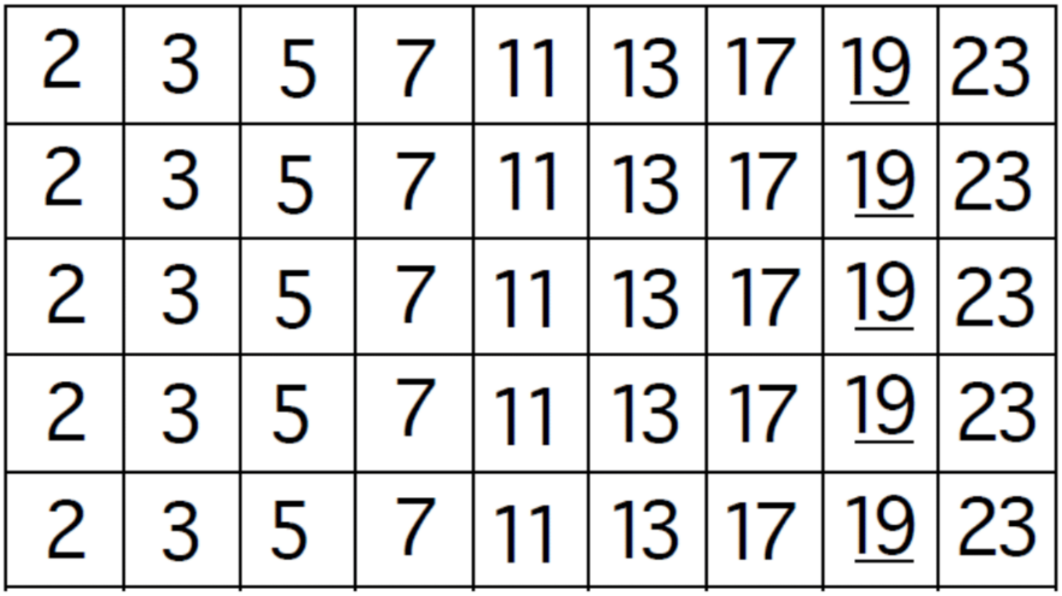
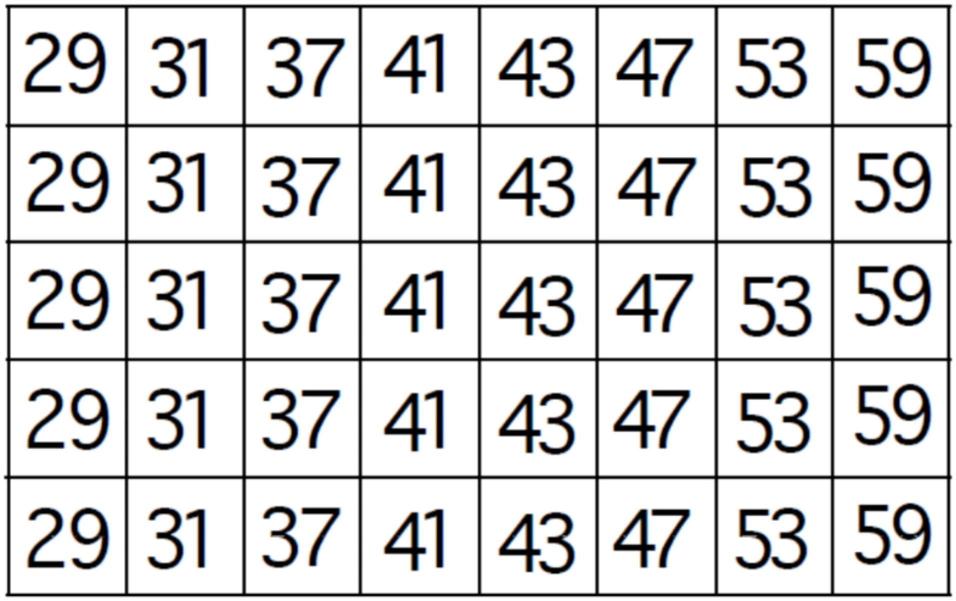
11.20: Number Squares

Prime Number Squares (2 - 23)



2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23
2	3	5	7	11	13	17	19	23

Prime Number Squares (29 - 59)

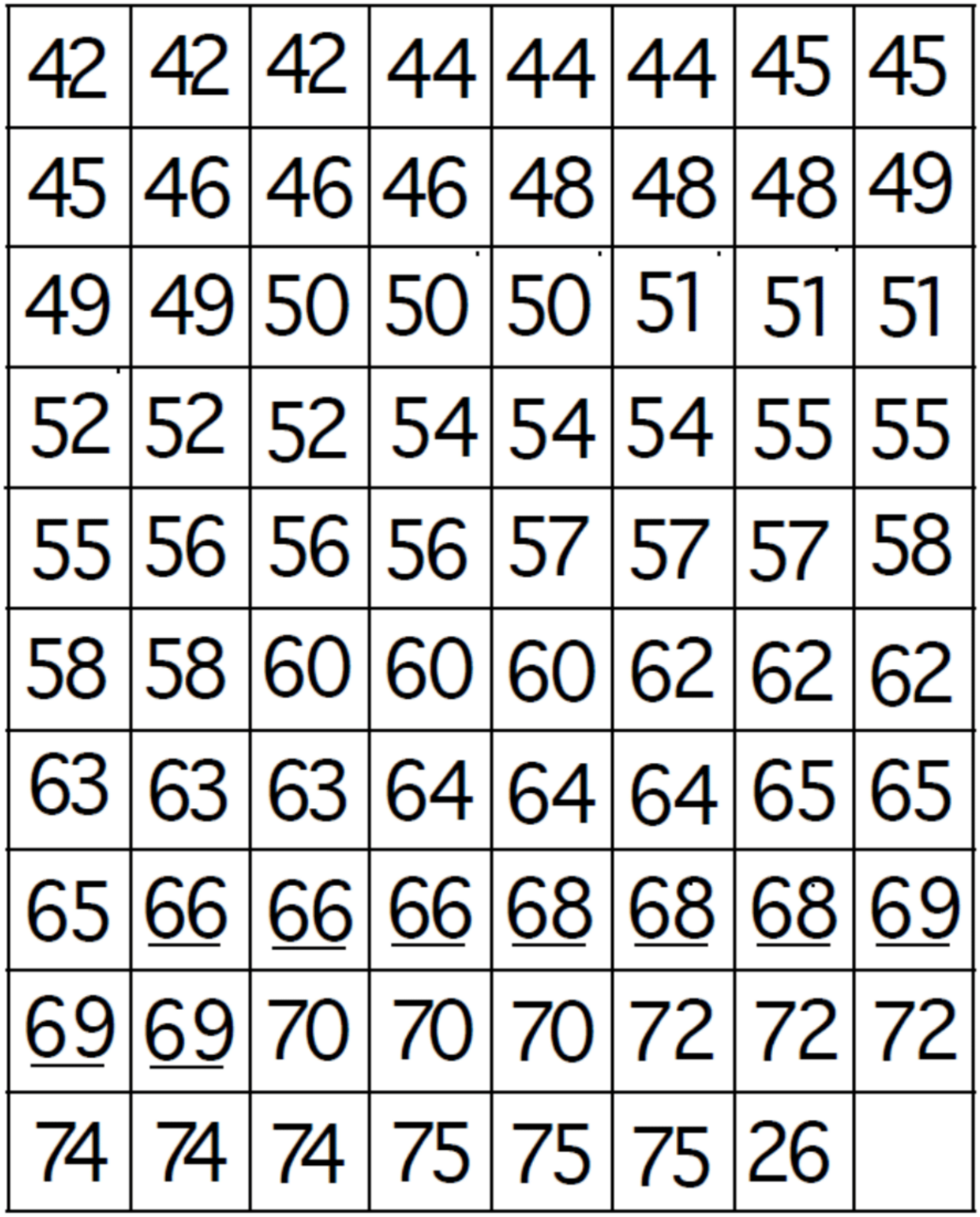


29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59
29	31	37	41	43	47	53	59

Composite Squares (4 - 40)

4	4	4	6	6	6	8	8
8	9	9	9	10	10	10	12
12	12	14	14	14	15	15	15
16	16	16	18	18	18	20	20
20	21	21	21	22	22	22	24
24	24	25	25	25	26	26	27
27	27	28	28	28	30	30	30
32	32	32	33	33	33	34	34
34	35	35	35	36	36	36	38
38	38	39	39	39	40	40	40

Composite Squares (42 - 75)



42	42	42	44	44	44	45	45
45	46	46	46	48	48	48	49
49	49	50	50	50	51	51	51
52	52	52	54	54	54	55	55
55	56	56	56	57	57	57	58
58	58	60	60	60	62	62	62
63	63	63	64	64	64	65	65
65	66	66	66	68	68	68	69
69	69	70	70	70	72	72	72
74	74	74	75	75	75	26	

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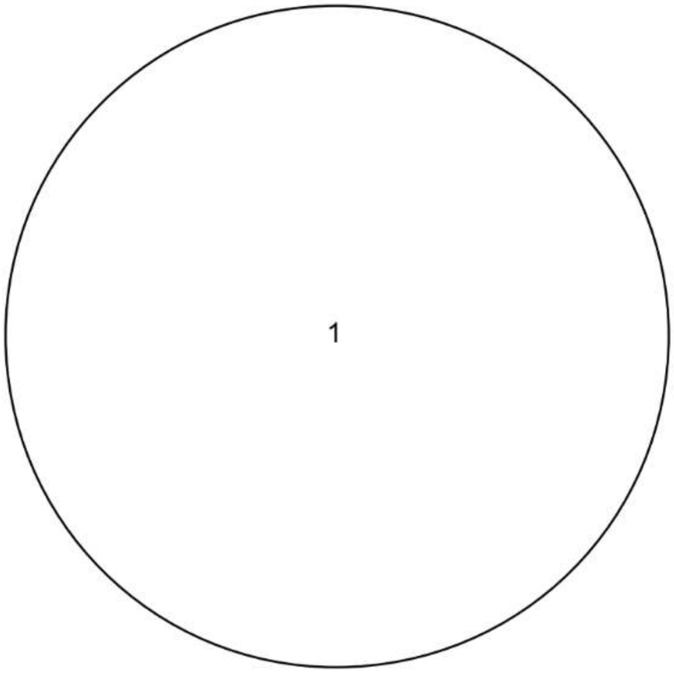
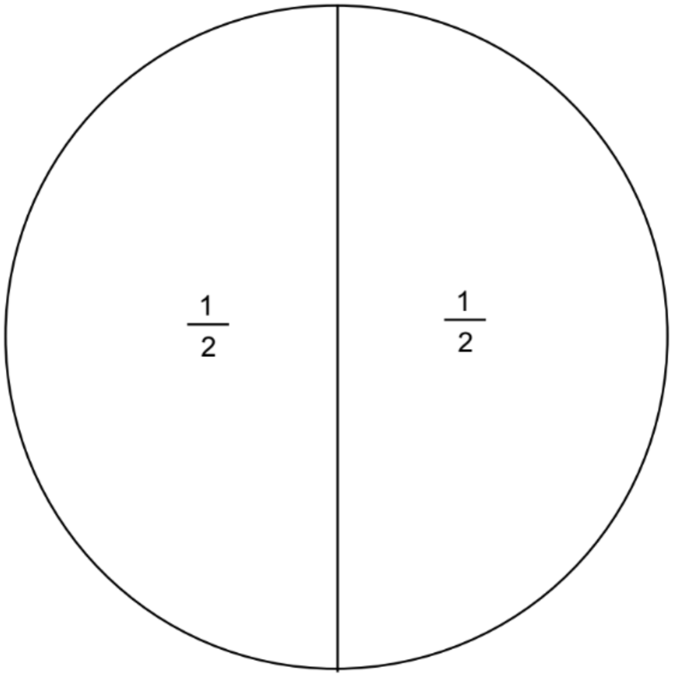
11.21: Fraction Circles

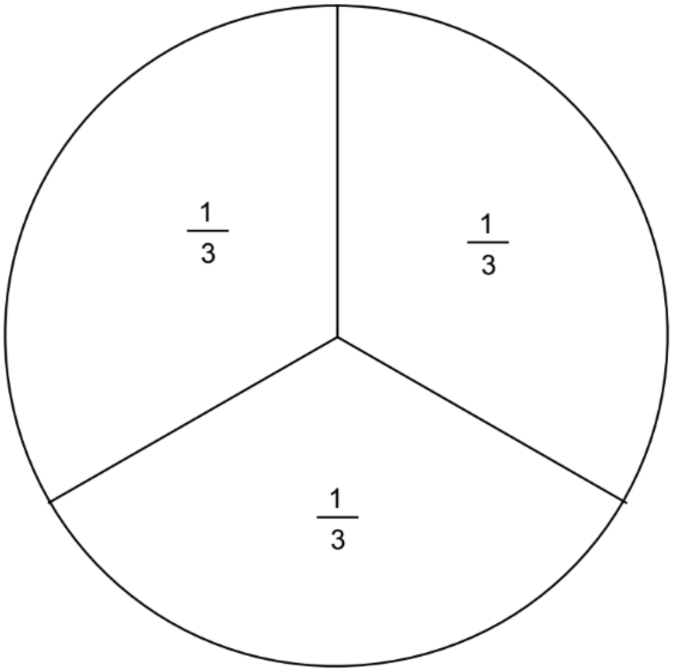
White Fraction Circle



Red Fraction Circle (halves)

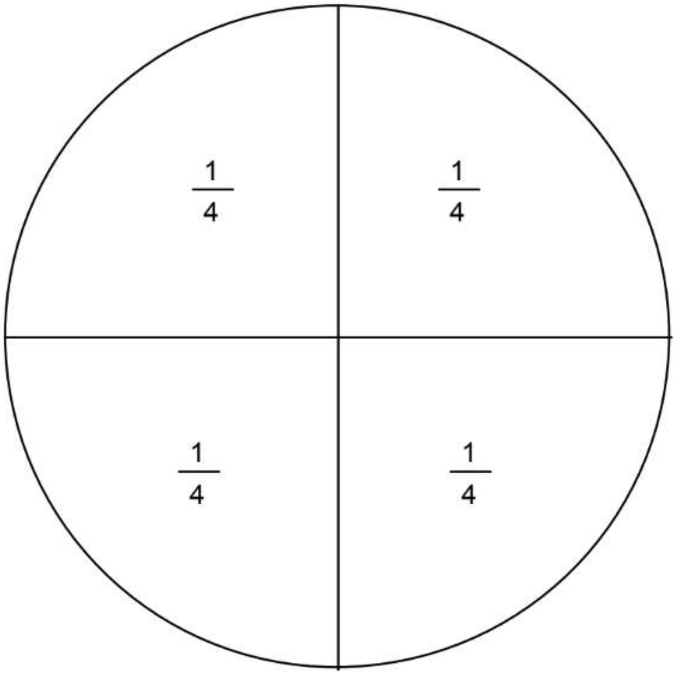


Light Green Circle (thirds)



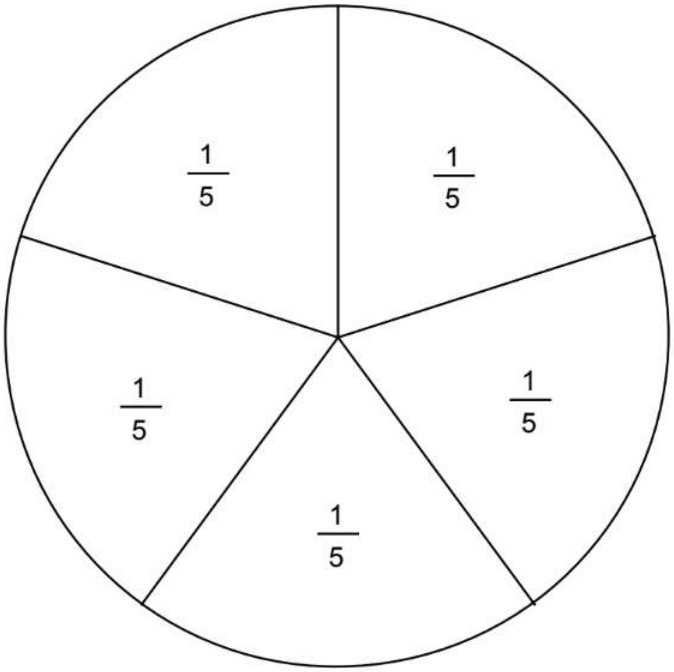
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$

Purple Circle (fourths)



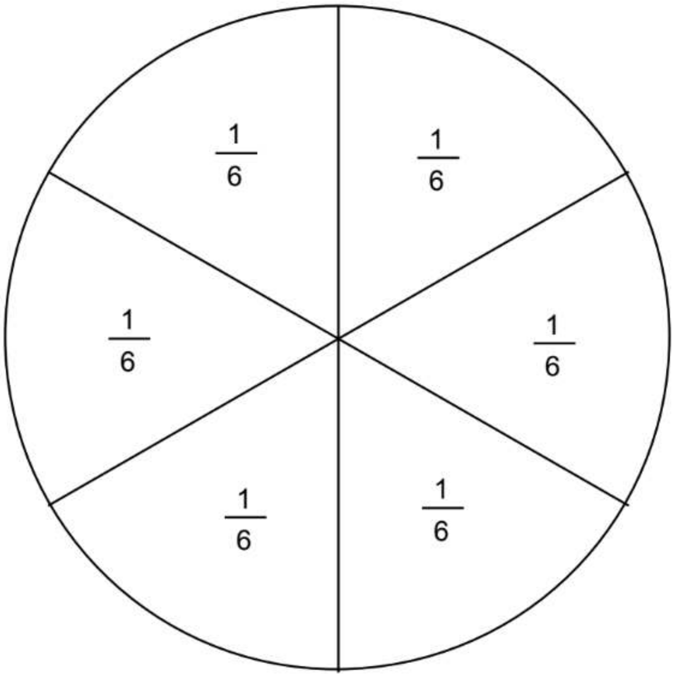
$\frac{1}{4}$
$\frac{1}{4}$
$\frac{1}{4}$
$\frac{1}{4}$

Yellow (fifths)



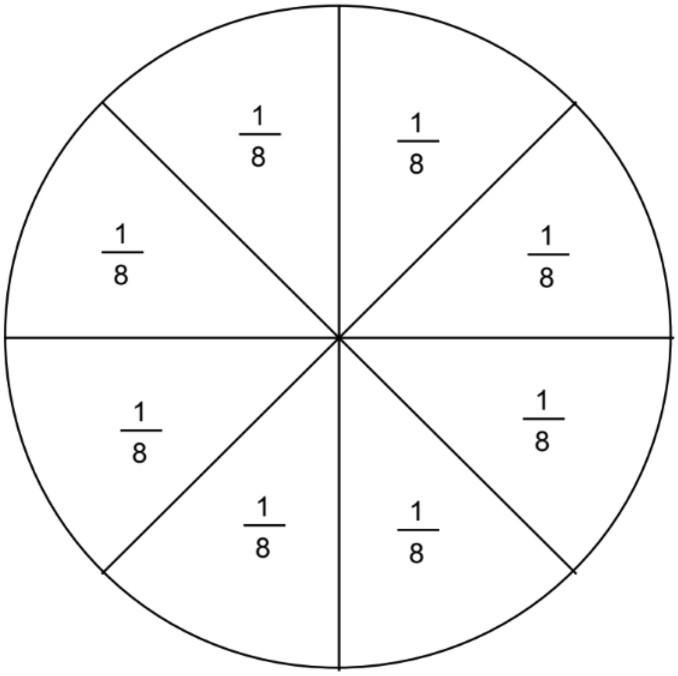
$\frac{1}{5}$
$\frac{1}{5}$
$\frac{1}{5}$
$\frac{1}{5}$
$\frac{1}{5}$

Dark Green (sixths)

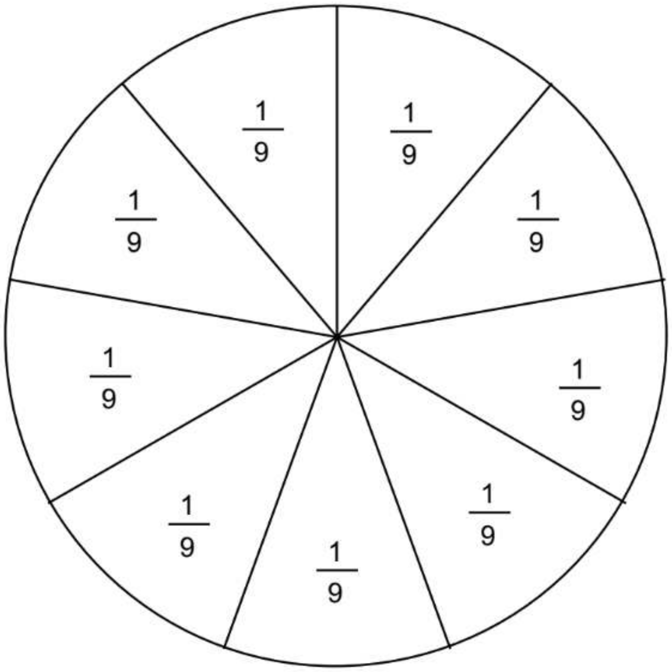


$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$
$\frac{1}{6}$

Brown (eights)


1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/8

Blue (ninths)


1/9
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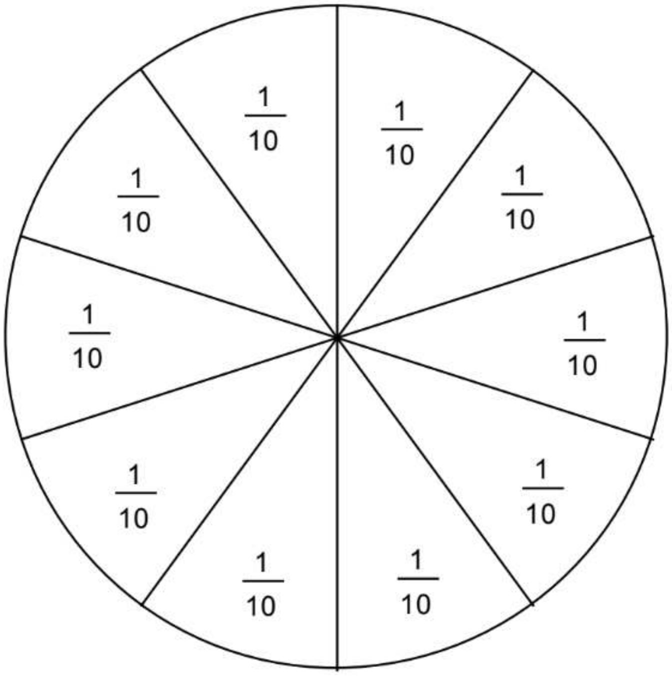
Orange (tenths)



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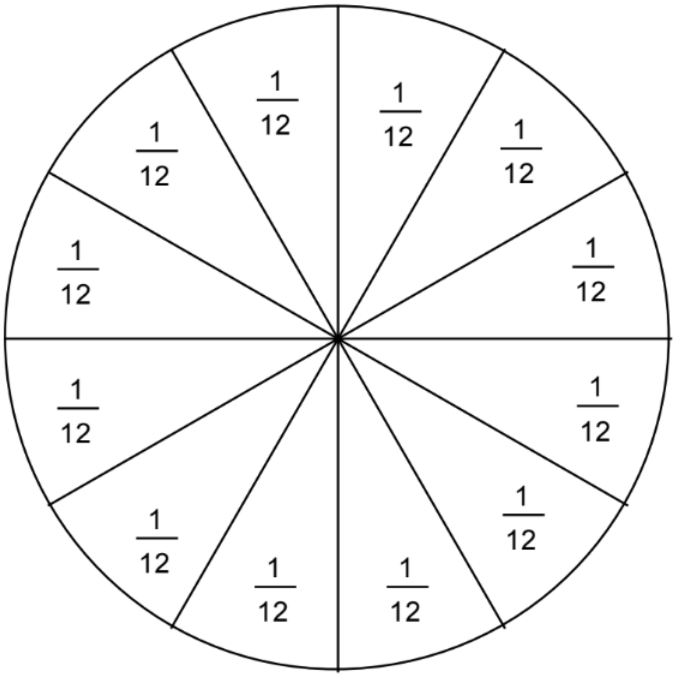
Hot Pink (twelfths)



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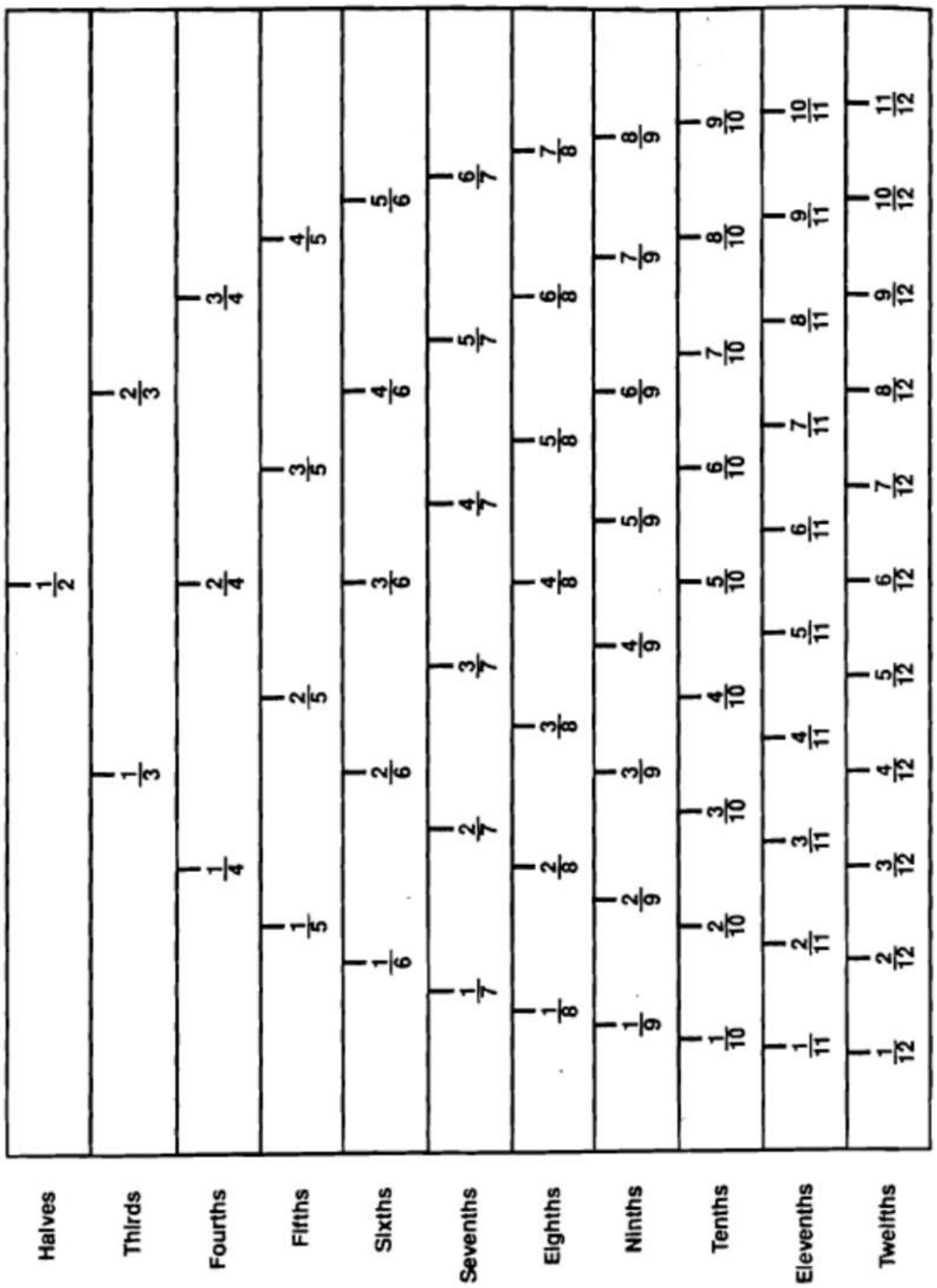
11.22: Strips and Arrays

Fraction Arrays


Halves
1/2
Thirds
1/3
2/3
Fourths
1/4
2/4
3/4
Fifths
1/5
2/5
3/5
4/5
Sixths
1/6
2/6
3/6
4/6
5/6
Sevenths
1/7
2/7
3/7
4/7
5/7
6/7
Eighths
1/8
2/8
3/8
4/8
5/8
6/8
7/8
Ninths
1/9
2/9
3/9
4/9
5/9
6/9
7/9
8/9
Tenths
1/10
2/10
3/10
4/10
5/10
6/10
7/10
8/10
9/10
Elevenths
1/11
2/11
3/11
4/11
5/11
6/11
7/11
8/11
9/11
10/11
Twelfths
1/12
2/12
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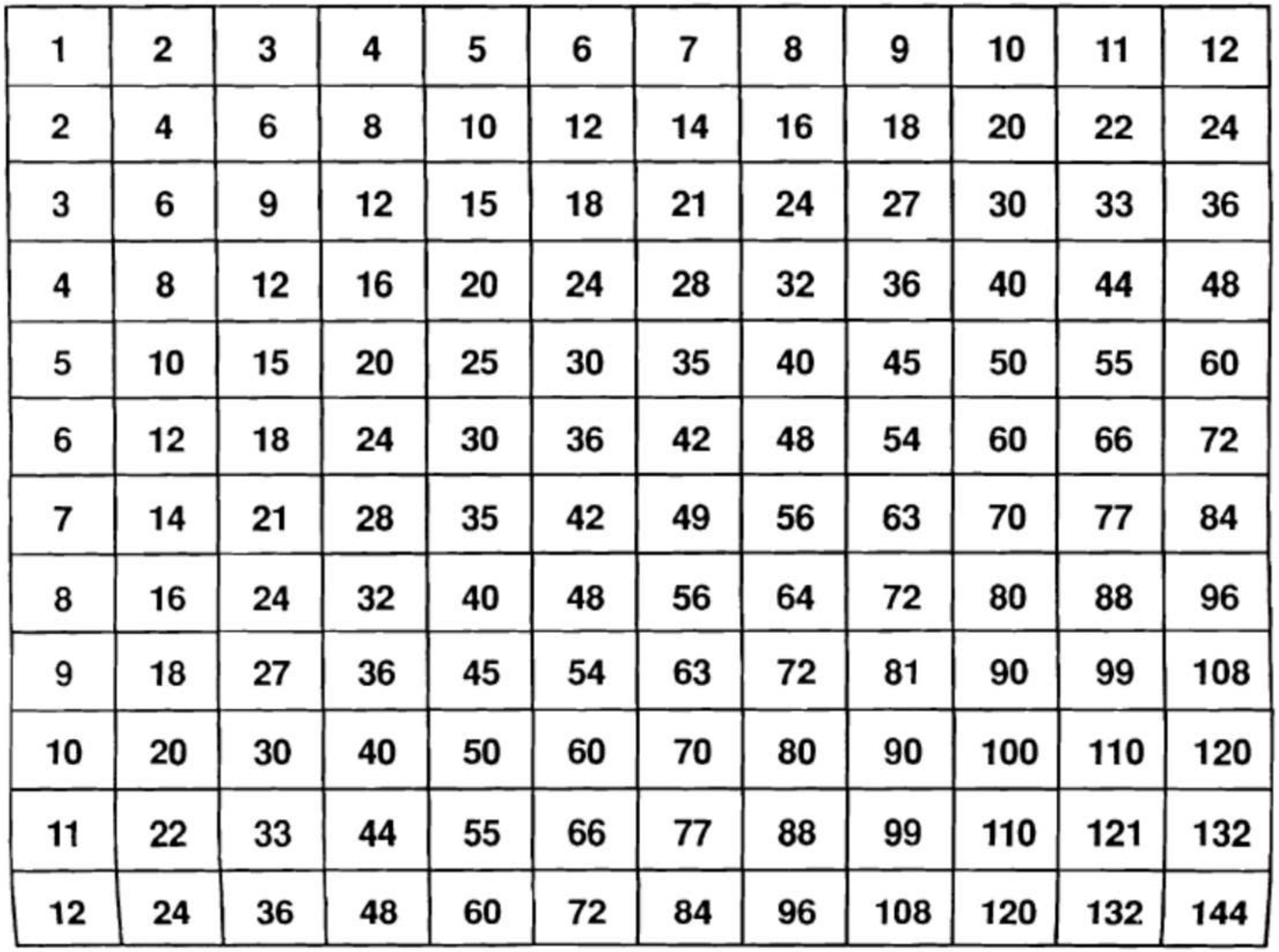
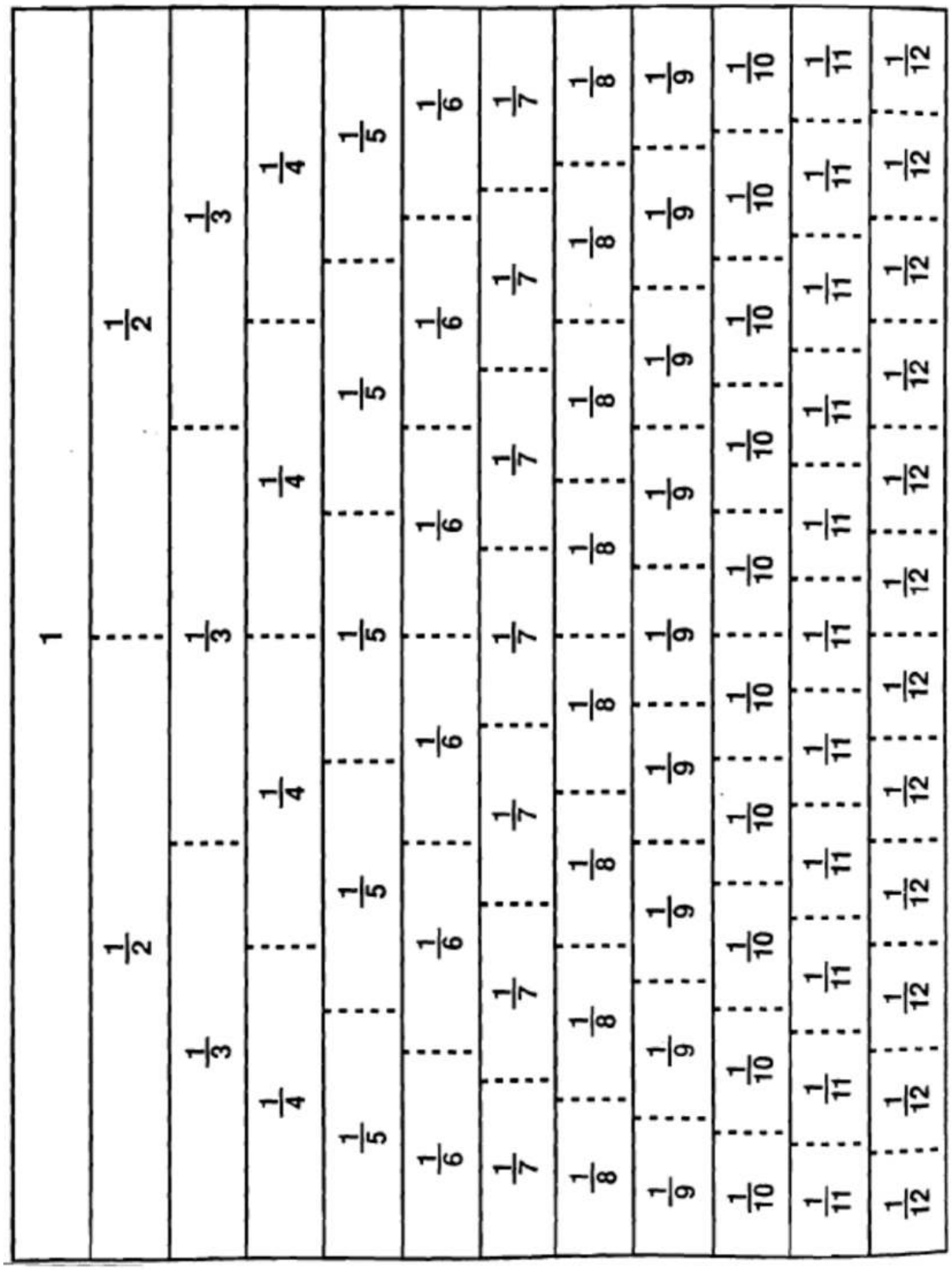
Multiple Strips



1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Fraction Strips


1
1/2
1/3
1/4
1/5
1/6
1/7
1/8
1/9
1/10
1/11
1/12

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Glossary

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Solutions

Module 1 Set Theory

Exercise Set 1 Solutions

Exercise 4

a. 5P, 6P, 7P, 8P

b. 4D, 5D, 6D, 7D, 8D, 9D

c. 5H, 6H, 7H, 9H

d. There are 6 groups. You should have listed 2 of these groups:

1964: 4N, 4D, 4Q
1965: 5P, 5N, 5D, 5Q, 5H
1966: 6P, 6N, 6D, 6Q, 6H
1967: 7P, 7N, 7D, 7Q, 7H
1968: 8P, 8N, 8D, 8Q
1969: 9D, 9Q, 9H

Exercise 5

5. a collection of objects

Exercise 6

6. the objects in a set

Exercise 7

7. The null set is a set that contains no elements that is, it is empty.

Exercise 9

9. Yes.

N = {4N, 5N, 6N, 7N, 8N}
Q = {4Q, 5Q, 6Q, 7Q, 8Q, 9Q}
S = {4N, 4D, 4Q}
W = {6P, 6N, 6D, 6Q, 6H}
Y = {8P, 8N, 8D, 8Q}
T = { }

Exercise 10

10.

a. yes; any element of N will suffice 4N, 5N, 6N, 7N or 8N	b. no	c. no	d. yes

CC BY NC

Exercise 11

11. Answers may vary. Any element in P will suffice 5P, 6P, 7P or 8P

Exercise 12

12.

a. F	b. T	c. T	d. T	e. F
f. T	g. T	h. T	i. F	

Exercise 14

a. 4	b. 5	c. 6	d. 6	e. 3	f. 4
g. 4	h. 0	i. 3	j. 5	k. 5	l. 5
m. 25	n. 0	o. Z ~ S; P~H, H ~ Y; N ~ W;X ~ V; Q ~ D (it's not necessary to include any null subsets)			

Exercise 15

15.

a. Ø	b. {9H}	c. {4H}
d. {6Q}	e. Ø	

Exercise 17

17. Two sets, A and B, are disjoint if $A \cap B = \emptyset$ that is to say that their intersection is empty.

Exercise 18

18. 5D, 6P, 7P, 8P, 5P, 5N, 5Q, 5H. yes, there are 8 coins.

Exercise 19

19.

a. {4N, 5N, 6N, 7N, 8N, 4D, 4Q}
b. {4Q, 5Q, 6Q, 7Q, 8Q, 9Q, 6P, 6N, 6D, 6H}
c. {5P, 6P, 7P, 8P, 4D, 5D, 6D, 7D, 8D, 9D}
d. {7P, 7N, 7D, 7Q, 7H, 8P, 8N, 8D, 8Q}

Exercise 23

23.

a. {2, 4}	b. {3, 5}
c. { } or Ø	d. {1, 2, 3, 4, 5, 6, 8}
e. {1, 2, 3, 4, 5, 7}	f. {2, 3, 4, 5, 6, 7, 8}
g. {6, 7, 8, 9}	h. {1, 3, 5, 7, 9}

i. {1, 2, 4, 6, 8, 9}	p. {1, 3, 5, 7}
j. {1, 3, 5}	q. {3, 5, 6, 7, 8, 9}
k. {6, 8}	r. {2,3,4,5,7}
l. {1, 2, 4}	s. {2, 4}
m. {7}	t. {6, 8, 9}
n. **B** or {2,4,6,8}	u. {7, 9}
o. **C** or {3,5,7}	v. {1, 3, 5, 6, 7, 8, 9}
w. {7, 9}	x. {3, 5, 6, 7, 8, 9}
y. T	z. T
aa. F	bb. F
cc. T	dd. F
ee. T	ff. F
gg. F	hh. T
ii. T	jj. F

? Exercise 25

25.

a. {a, b, c, ..., z}
b. {124, 126, 128, ..., 698}
c. {101, 102, 103, ..., 999}
d. {x|x is a past president of the U.S. up until 1995}
e. {Reagan, Bush, Clinton}

? Exercise 30

30.

a. 3; 8; answers may vary
b. 4; 6; answers may vary
c. 2; 12; answers may vary
d. 12; 2; answers may vary
e. 8; 3; answers may vary
f. 6; 4; answers may vary

? Exercise 31

31. e. union

? Exercise 32

32.

c. no
d. yes; LYC, LYQ, LYT
e. intersection;{LYC, LYQ, LYT}

Exercise 34

34.

a. {}
b. {}, {P}
c. { }, {G},{F}, {F,G}
d. {}, {X}, {Y}, {Z}, {X,Y}, {X,Z}, {Y,Z}, {X,Y,Z}
e. { }, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}

Exercise 35

35.

a. 1	b. 2	c. 4	d. 8
e. 16	f. 32	g. 2^n	

Exercise 36

36. 35

Exercise 37

37. 35

Exercise 38

38. yes

Exercise 39

39.

a. bc	b. b^2	c. c^2

Exercise 40

40. One possibility is let A = {m} and B = {n}. Then $A \times B = (m, n)$ but $B \times A = (n, m)$. Therefore, $A \times B \nleq B \times A$ (be sure to write your own solution)

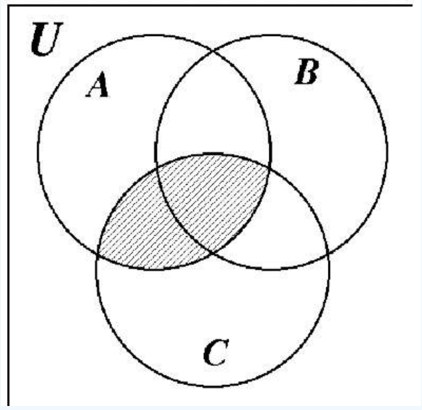
Exercise 41

41.

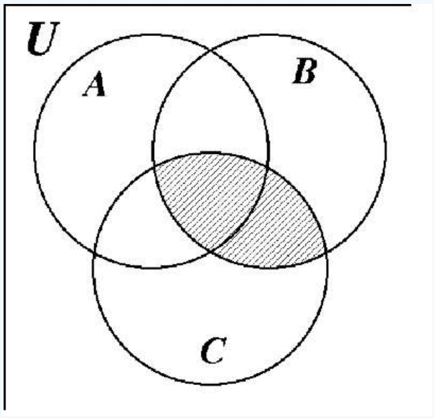
a. {(3,2), (3,6), (4,2), (4,6)}
b. {(6,5), (7,5), (8,5), (9,5)}
c. { }
d. {(a,a)}
e. {(x,x), (x,y), (y,x), (y,y)}
f. {(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)}
g. {((9,4),D), ((9,4), {a,b,c}), (C,D), (C, {a,b,c})}
h. {({5,6,7,8,9}, g), ({5,6,7,8,9},{4,3})}

Exercise Set 2 Solutions

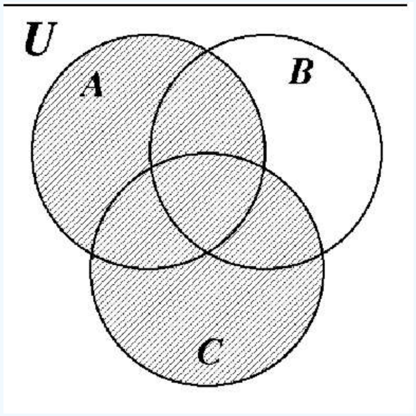
? 1



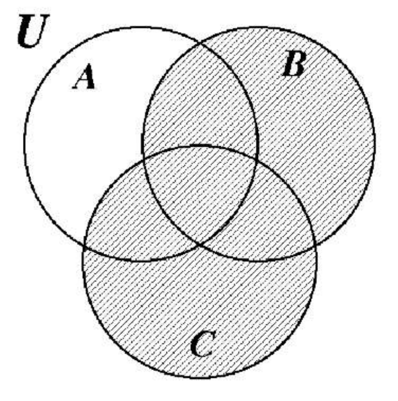
? 2



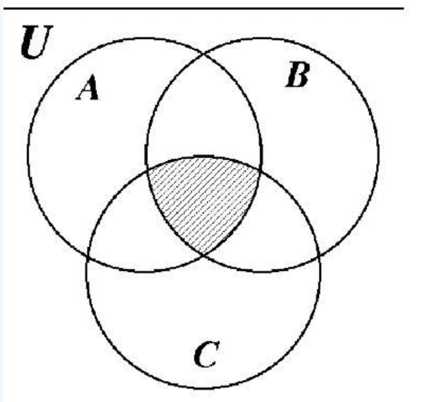
? 3



? 4



? 5



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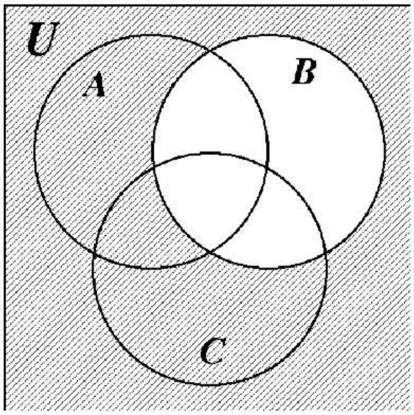

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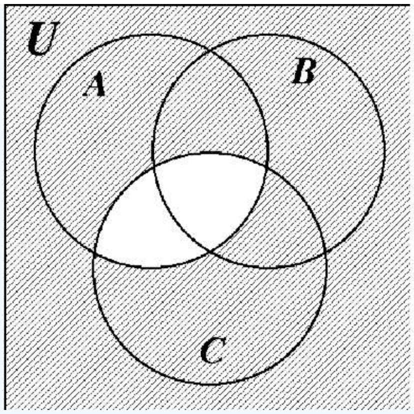
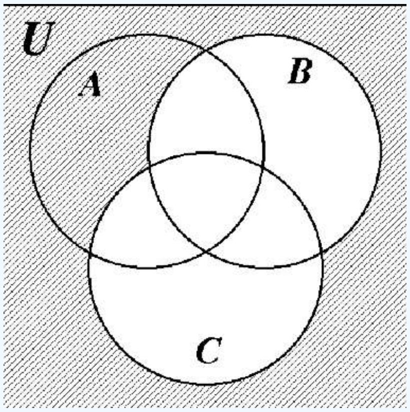
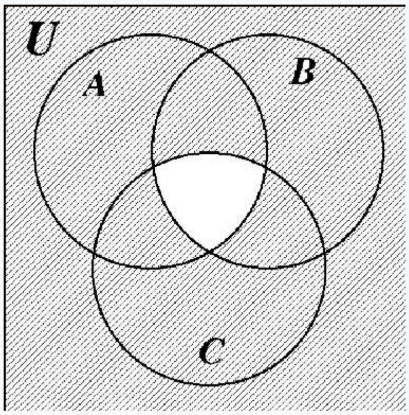


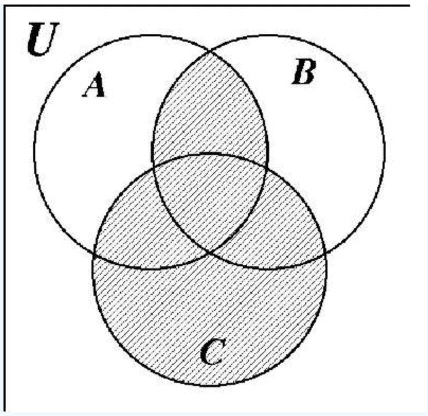
? 8



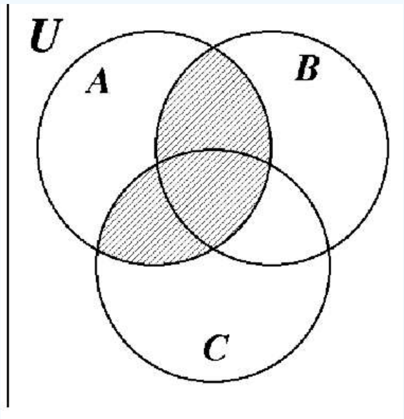
? 9



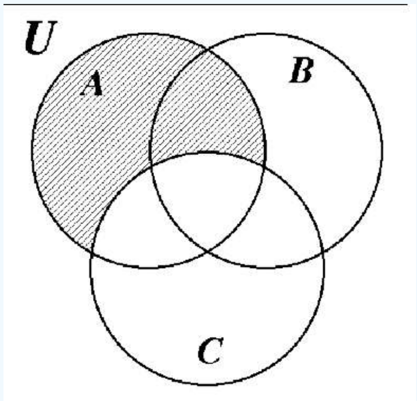
? 10



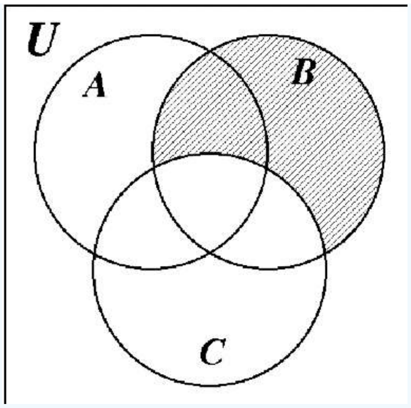
? 11



? 12



? 13



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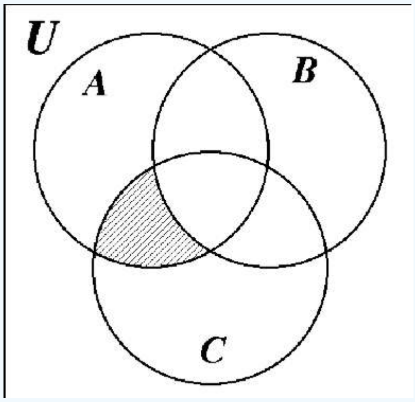
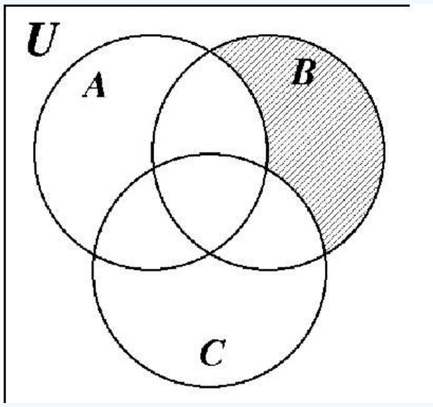


? 15



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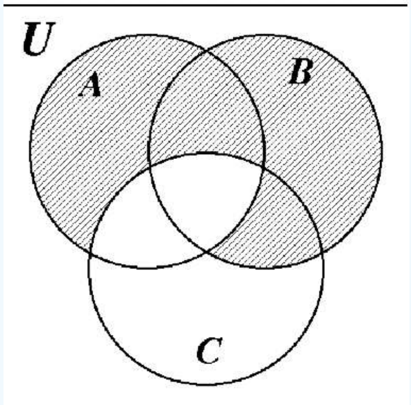


18-56: Answers to these exercises are the Venn diagrams at the end of Exercise Set 2.

Exercise Set 3 Solutions

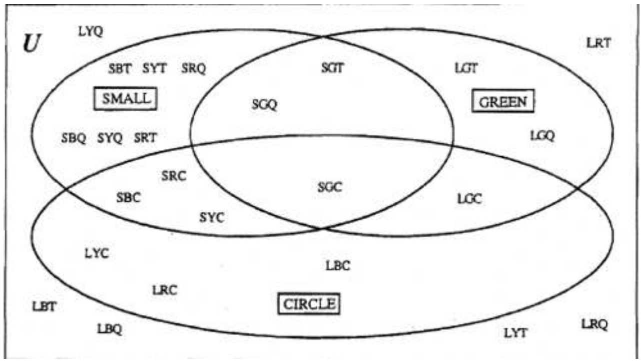
Exercise 1

1. Answers may vary. The overlap is the intersection. There are 8 regions.

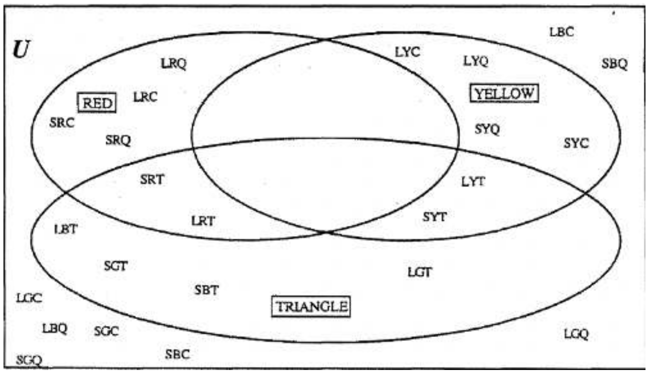
Exercise 2

2.

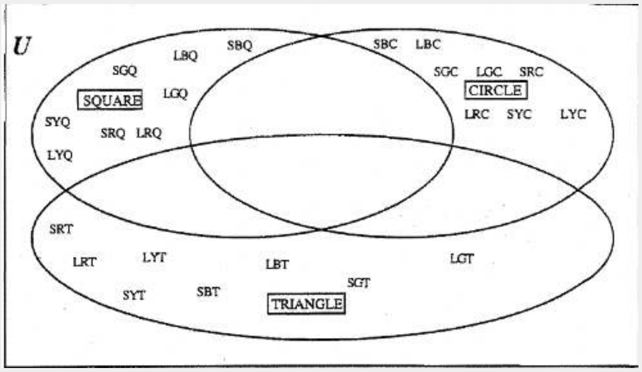
a.


U
LYQ
LRT
SBT SYT SRQ
SGT
LGT
SMALL
SGQ
GREEN
SBQ SYQ SRT
LGQ
SRC
SBC
SGC
LGC
SYC
LYC
LBC
LRC
CIRCLE
LBT
LBQ
LYT
LRQ

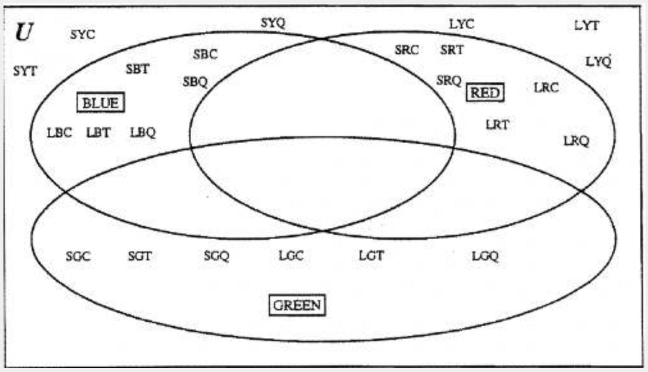
b.


U
LBC
LYC
LRQ
LYQ
SBQ
RED
LRC
YELLOW
SRC
SRQ
SYQ
SYC
SRT
LYT
LRT
SYT
LBT
SGT
LGT
SBT
LGC
TRIANGLE
LBQ
SGC
LGQ
SGQ
SBC

c.


U
SBQ
SBC LBC
LBQ
SGQ
SGC LGC SRC
SQUARE
LGQ
CIRCLE
SYQ
LRC SYC LYC
SRQ LRQ
LYQ
SRT
LRT
LYT
LBT
LGT
SGT
SYT
SBT
TRIANGLE

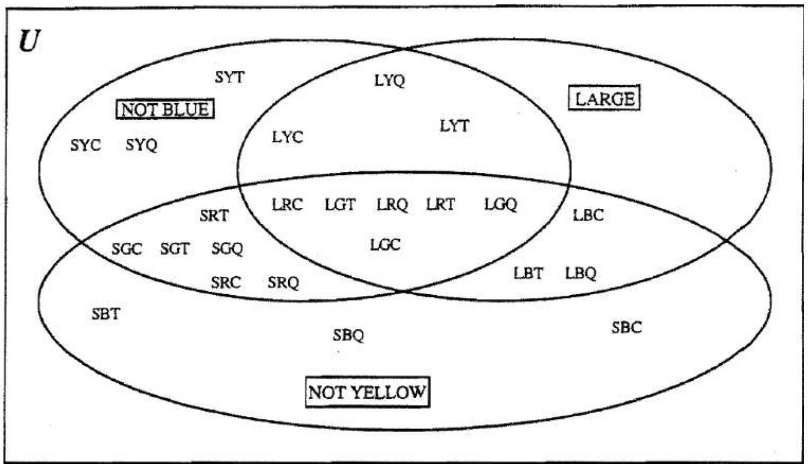
d.


U
SYC
SYQ
LYC
LYT
SBC
SRC SRT
SYT
SBT
LYQ
SBQ
SRQ
RED
LRC
BLUE
LRT
LBC LBT LBQ
LRQ
SGC SGT SGQ LGC LGT LGQ
GREEN

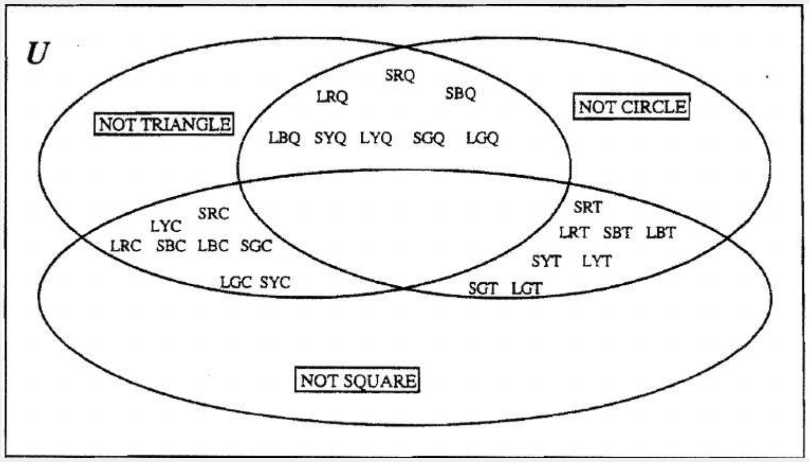
Exercise 4

4.

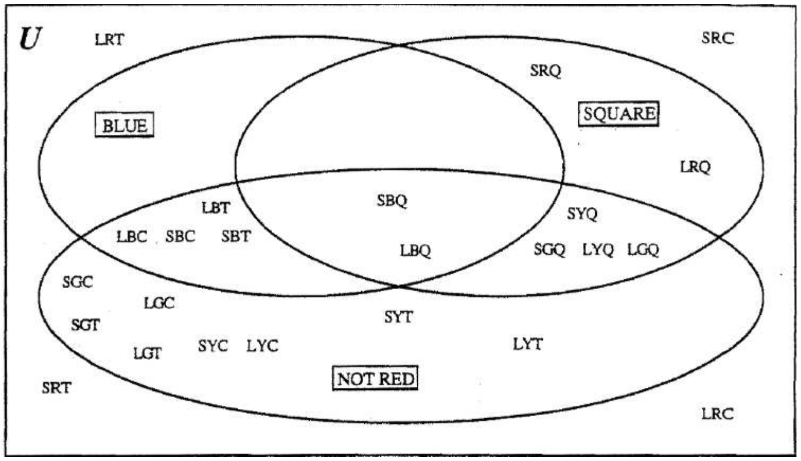
a.


U
SYT
LYQ
NOT BLUE
LARGE
SYC SYQ
LYC
LYT
SRT
LRC LGT LRQ LRT LGQ
LBC
SGC SGT SGQ
LGC
SRC SRQ
LBT LBQ
SBT
SBQ
SBC
NOT YELLOW

b.


U
SRQ
LRQ
SBQ
NOT TRIANGLE
NOT CIRCLE
LBQ SYQ LYQ SGQ LGQ
SRC
LYC
SRT
LRC SBC LBC SGC
LRT SBT LBT
SYT LYT
LGC SYC
SGT LGT
NOT SQUARE

c.


U
LRT
SRC
SRQ
SQUARE
BLUE
LRQ
SBQ
LBT
SYQ
LBC SBC SBT
LBQ
SGQ LYQ LGQ
SGC
LGC
SYT
SGT
SYC LYC
LGT
LYT
NOT RED
SRT
LRC

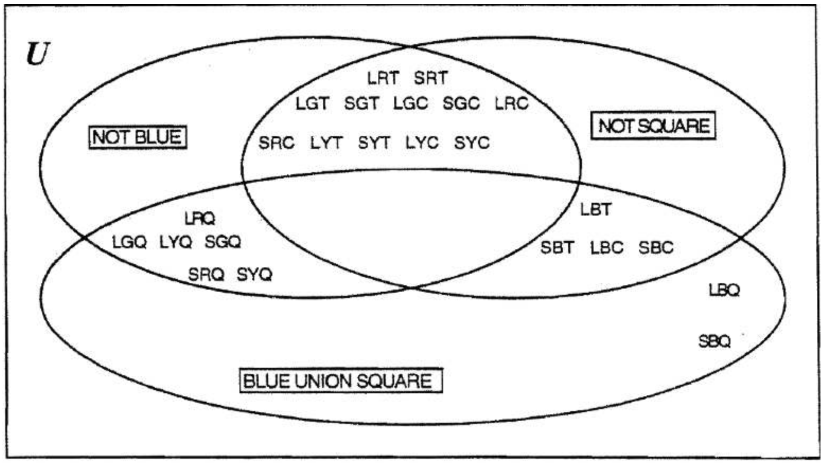
Exercise 5

5. Some possibilities are: triangle, circle, square; large, small and any other value.

? Exercise 8

8.

a.


U
NOT BLUE
NOT SQUARE
LRT SRT
LGT SGT LGC SGC LRC
SRC LYT SYT LYC SYC
LRQ
LGQ LYQ SGQ
SRQ SYQ
LBT
SBT LBC SBC
LBQ
SBQ
BLUE UNION SQUARE

b. SRT, SYT, SGT, LGT, SYC, LYT, SRC, SGC, LRT, LGC, LRC, LYC

c. SRT, SYT, SGT, LGT, SYC, LYT, SRC, SGC, LRT, LGC, LRC, LYC

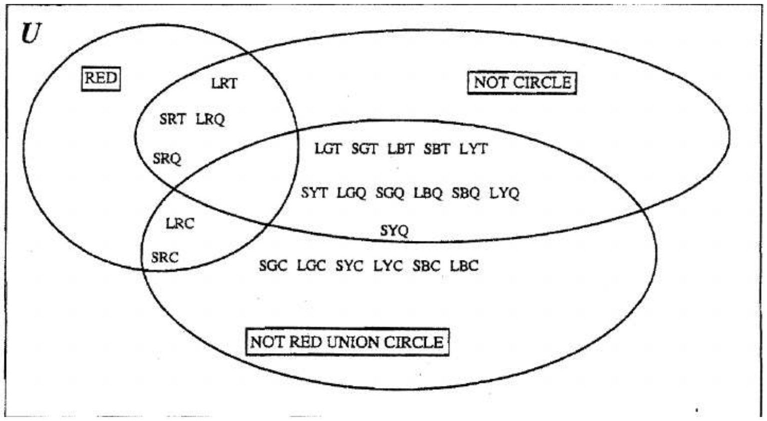
d. The elements are exactly the same

e. $B \cup Q)^c = B^c \cap Q^c$

? Exercise 9

9.

a.


U
RED
NOT CIRCLE
LRT
SRT LRQ
SRQ
LGT SGT LBT SBT LYT
SYT LGQ SGQ LBQ SBQ LYQ
SYQ
LRC
SRC
SGC LGC SYC LYC SBC LBC
NOT RED UNION CIRCLE

b. SRT, LRQ, LRT, SRQ

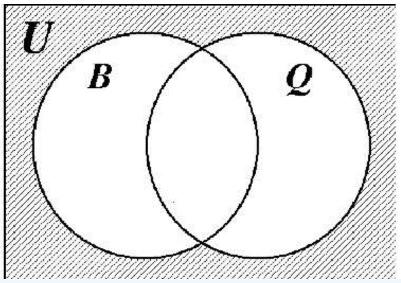
c. SRT, LRQ, LRT, SRQ

d. The elements are exactly the same.

e. $(R^c \cup C)^c = R \cap C^c$

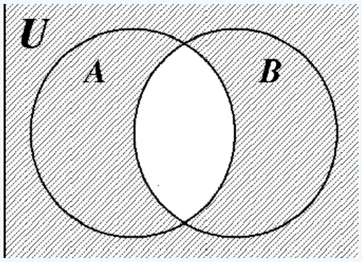
? Exercise 10

10. For part a and b, the final shadings are the same as shown on the Venn diagram below. The equation you can write is: $(B \cup Q)^c = B^c \cap Q^c$



Exercise 11

11. For part a and b, the final shadings are the same as shown on the Venn diagram below. The equation you can write is: $(A \cap B)^c = A^c \cup B^c$



Exercise 12

12.

a. $R \cup S$	b. $(M^c \cup N^c)^c$	c. $F^c \cap G$	d. $(H \cap I^c)^c$	e. $P \cup Q^c$	f. $(S^c \cup T)^c$

Exercise 14

14. Only a proof to 14.c. is provided

Extra close brace or missing open brace
$= 1, 3, 5, 7, 9 \cup 3, 5, 6)^c$
$= (1, 3, 5, 7, 9)^c$
$= \{2, 4, 6, 8\}$.

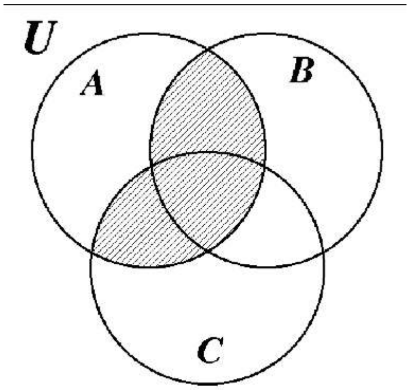
$B \cap C^c = 2, 4, 6, 8 \cap (1, 3, 5, 7, 9)^c$
$= 2, 4, 6, 8 \cap 2, 4, 6, 8$
$= \{2, 4, 6, 8\}$.

Since $(B^c \cup C)^c$ and $B \cap C^C$ have the exact same elements, they are equal. Therefore, $(B^c \cup C)^c = B \cap C^c$.

Exercise 15

15.

a, b: final shadings are the same as shown on the Venn diagram below.



c. {2,3,4,5}

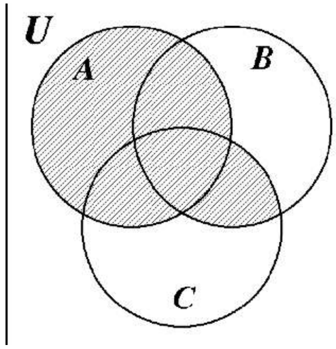
d. {2,3,4,5}

e. They are the same equal.

Exercise 16

16.

a, b: final shadings are the same as shown on the Venn diagram below.



c. {1,2,3,4,5}

d. {1,2,3,4,5}

e. They are the same.

Exercise 17

17.

a. $(X \cap Y) \cup (X \cap Z)$

b. $(P \cup Q^c) \cap (P \cup R)$

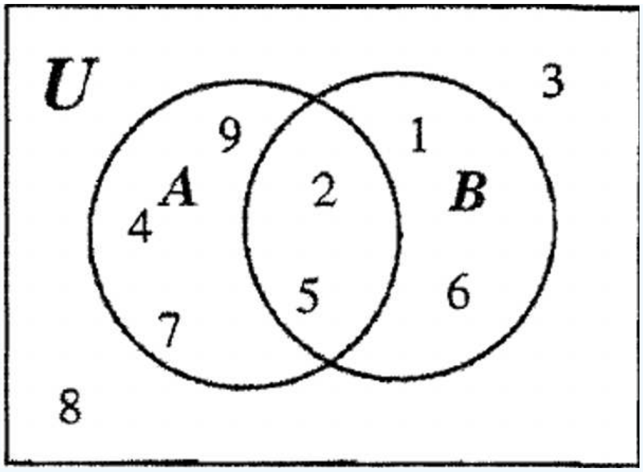
c. $K^c \cap (L \cup M)$

d. $D \cup (E^c \cap F)$

Exercise Set 4 Solutions

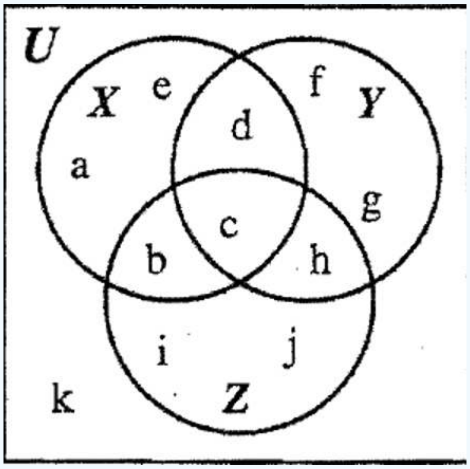
Exercise 1

1.


U
9
A
4
7
2
5
1
B
6
3
8

Exercise 2

2.


U
X
e
a
d
f
Y
g
c
b
h
i
j
k
Z

Exercise 3

3.

a. $(A \cup B \cup C)^c$	b. A - $(B \cup C)$
c. (A \cap B) - C	d. B - $(A \cup C)$
e. $(A \cap B \cap C)$	f. $(A \cap C) - B$
g. $(B \cap C) - A$	h. $C -$ \\($(A \cup B)$

Exercise 4

4.

a. 43	b. 51	c. 46	d. 38
e. 18	f. 25	g. 13	h. 89

Exercise 5

5.

$a.\ n(A - (B \cup C)) = 11$

b. $n(\mathbf{A})$ = 69

c. $n(B\cap C)$ = 51

d. $n((A\cup B)-C)$ = 71

e. $n(C^c)$ = 73

f. $n(B\cup C)$ = 160

g. n(B - C) = 60

h. $n((A\cap B)\cup(B\cap C)\cup(A\cap C))$ = 66

i. $n((A\cap B)\cup(B\cap C)\cup(A\cap C)-(A\cap B\cap C))$ = 23

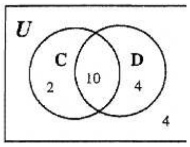
? Exercise 6

6.

a. 34	b. 5	c. 69	d. 37	e. 13
f. 3	g. 21	h. 9	i. 23	

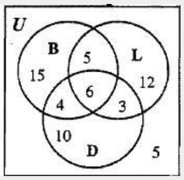
? Exercise 7

7.

a. 12 	b. 2	c. 6

? Exercise 8

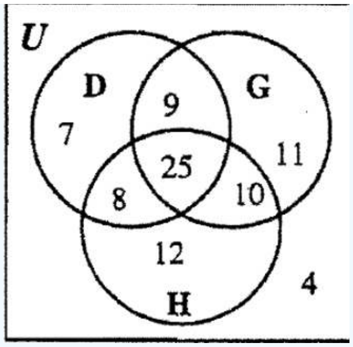
8.

a. 9	b. 5	c. 10	d. 43
e. 37 	f. 18	g. 3	

? Exercise 9

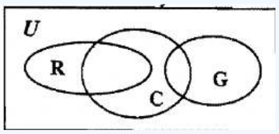
9.

a. 86	b. 49	c. 31
d. 25	e. 4	f. 27


U
D
G
7
9
25
11
8
10
12
4
H

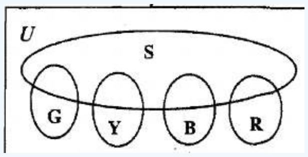
Exercise 10

10. This is one way to show it.



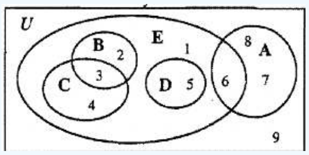
Exercise 11

11. This is one way to show it.



Exercise 12

12. This is one way to show it.


U
B
2
E
1
8
A
C
3
D
5
6
7
4
9

Homework Solutions

Exercise 1

1.

a. {a, c}
c. {e, v, w, z}
e. {r, u, x}
g. {a, c, r, u}
i. {x}
k. Ø or { }
m. 10
n. 2

Exercise 3

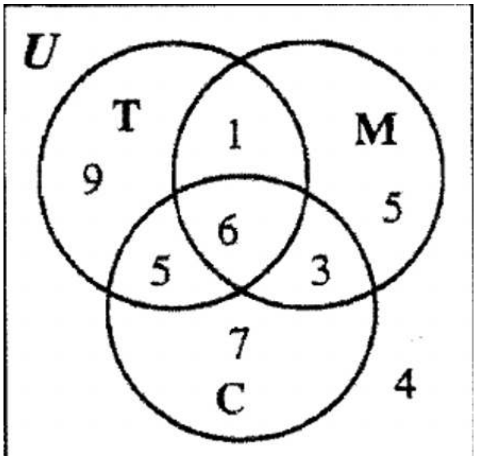
3. a. $(N^c \cap P)^c$

Exercise 4

4. a. $A^c \cup (E \cap F)$

Exercise 5

5.

a. 
c. 15
e. 5
g. 9

Exercise 6

6.

a. 8	b. 3

Exercise 7

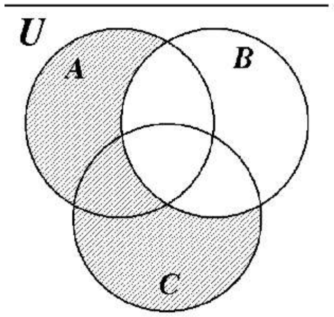
7.

a. T	c. F	e. T

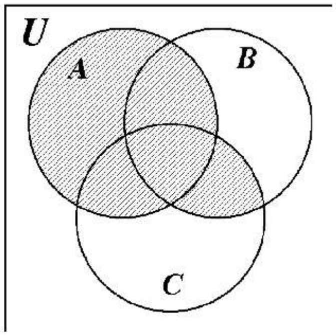
Exercise 8

8.

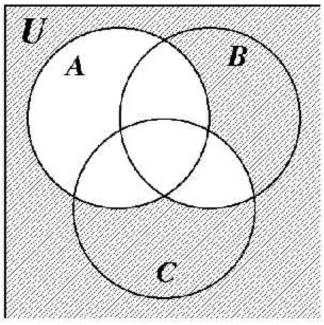
a.



c.



e.



? Exercise 9

9. a. $A \cup (B \cap C)$

? Exercise 10

10.

a. { }

c. { }, {a}, {b}, {a, b}

? Exercise 11

11.

a. {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4)}

c. {({a, c}, {a, c}), ({a, c}, 5), (5, {a, c}), (5, 5)}

e. {(x, x)}

? Exercise 12

12.

a. {LBC, LYC, LGC}

c. {LRC, LBC, LGC, LYC, SRC}

Module 2 Counting and Numerals

Exercise Set 1 Solutions

? Exercise 8

8. The sets have the same number of elements. Cardinality is equal.

? Exercise 10

10.

a)
1) $1 \leftrightarrow A$

b.
1) $2 \leftrightarrow B, 3 \leftrightarrow C$
2) $2 \leftrightarrow C, 3 \leftrightarrow B$

c.
1) $4leftrightarrowD, 5leftrightarrowE, 6leftrightarrowF$
2) $4leftrightarrowD, 5leftrightarrowF, 6leftrightarrowE$
3) $4leftrightarrowE, 5leftrightarrowD, 6leftrightarrowF$
4) $4leftrightarrowE, 5leftrightarrowF, 6leftrightarrowD$
5) $4leftrightarrowF, 5leftrightarrowD, 6leftrightarrowE$
6) $4leftrightarrowF, 5leftrightarrowE, 6leftrightarrowD$

? Exercise 11

11.

a)
1) $M \leftrightarrow M$

b.
1) $x \leftrightarrow x, y \leftrightarrow z$
2) $x \leftrightarrow z, y \leftrightarrow x$

c.
1) $1 \leftrightarrow 1, 2 \leftrightarrow 2$
2) $1 \leftrightarrow 2, 2 \leftrightarrow 1$

d.
1) $1 \leftrightarrow 1, 2 \leftrightarrow 2, 3 \leftrightarrow 3$
2) $1 \leftrightarrow 1, 2 \leftrightarrow 3, 3 \leftrightarrow 2$
3) $1 \leftrightarrow 2, 2 \leftrightarrow 1, 3 \leftrightarrow 3$
4) $1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 1$
5) $1 \leftrightarrow 3, 2 \leftrightarrow 1, 3 \leftrightarrow 2$
6) $1 \leftrightarrow 3, 2 \leftrightarrow 2, 3 \leftrightarrow 1$

e.
1) $1 \leftrightarrow 3, 2 \leftrightarrow 4, 3 \leftrightarrow 5$
2) $1 \leftrightarrow 3, 2 \leftrightarrow 5, 3 \leftrightarrow 4$
3) $1 \leftrightarrow 4, 2 \leftrightarrow 3, 3 \leftrightarrow 5$
4) 1 \leftrightarrow 4, 2 \leftrightarrow 5, 3 \leftrightarrow 3
5) 1 \leftrightarrow 5, 2 \leftrightarrow 3, 3 \leftrightarrow 4
6) 1 \leftrightarrow 5, 2 \leftrightarrow 4, 3 \leftrightarrow 3

? Exercise 12

12.

a. 1	b. 2	c. 6

? Exercise 14

14.

a. | | | | | | | | | | |

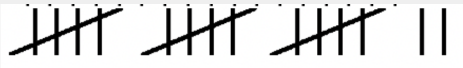
b. Make 512 stroke marks; make 2,000,000 stroke marks.

? Exercise 16

16.

a. | | | | | | | | | | | | | | | | |

b.



Exercise 17

17. 36

Exercise 18

18. Answers may vary. Some possibilities are to make 123 strokes or to make 8 I's and 53 strokes.

Exercise 19

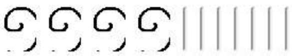
19.

a. 301,020	b. 4,010,507
c. 35,000	d. 110,023

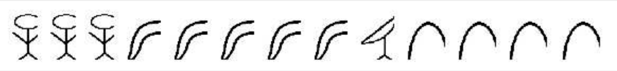
Exercise 20

20.

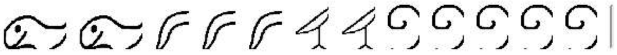
a.



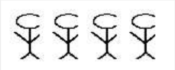
b.



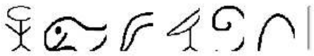
c.



d.



e.



Exercise 21

21. Base Ten. They only have symbols up to a million in Egyptian. This makes writing very large numerals too cumbersome and possibly virtually impossible.

Exercise 24

24.

a. XXXII	b. DLXI
c. DCCVIII	d. MMLIII

Exercise 25

25.

a. 2,687	b. 1,232

Exercise 26

26.

a. $\mathrm{CCC\overline{X}XXDCCCCII}$

b. $\mathrm{L\overline{\overline{X}}XMDCLI}$

Exercise 27

27.

a. CCCCCCCCCXXXXIIII or DCCCCXXXXIIII

b. CCCCXXXXXXXXXIIIIIIIII or CCCCLXXXXVIIII

Exercise 29

29.

a. DCCXLVIII

b. $\mathrm{X\overline{I}VCDLXXX}$

c. CMLXIX

d. CDXLII

Exercise 30

30.

a. $\mathrm{X\overline{I}XCDLIII}$

b. MMDCCCXLIX

c. MCMXCVI

Exercise 32

32. 3, 1, 3, 1, 3, 1

Exercise 33

33.

a. 10	b. 1	c. 2
d. 7	e. 7	

Exercise 34

34.

a. 3

b. 2

c. 9
d. 900 strokes
e. 180 5 stroke tallies

Exercise 35

35.

Hindu-Arabic: 3, 3, 7, 5, 4

Stroke: 143, 400, 1000000, 30009, 2124

Tally: 31, 80, 200000, 6005, 428

Egyptian: 8, 4, 1, 12, 9

Roman: 6, 2, 1, 5, 7

Exercise Set 2 Solutions

Exercise 1

1.

a. 五千九十三	b. 六百一十	c. 四百二十七	d. 八千八

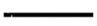
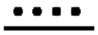
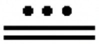
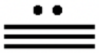
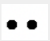
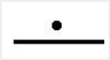
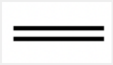
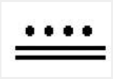
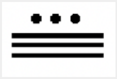
Exercise 2

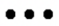
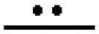
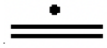
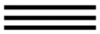
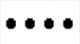
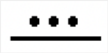
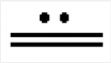
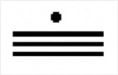
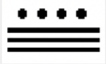
2.

a. 109	b. 6063	c. 40	d. 2815	e.7800

Exercise 3

3.

1.	5. 	9. 	13. 	17. 
2. 	6. 	10. 	14. 	18. 

3. 	7. 	11. 	15. 	
4. 	8. 	12. 	16. 	19. 

Exercise 4

4. A dot represents the number one and a line segment represents the number five. Any combination of 1-3 line segments and/or 1-4 dots forms any numeral up to 19.

Exercise 5

5.

a. 90	b. 320	c. 162

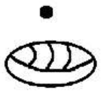
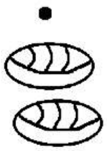
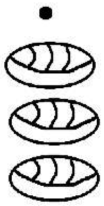
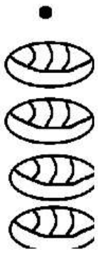
Exercise 6

6.

a. 3974	b. 1946	c. 3300
d. 32454	e. 7319	

Exercise 7

7.

a.	b. 	c. 	d. 	e. 

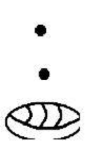
Exercise 8

8.

a. 27 rem. 156
b. 9 rem. 6709
c. 16 rem. 13
d. 2 rem. 142040

Exercise 9

9. A 19 at the second level up is 380. The Mayan numeral for 380 is shown below.

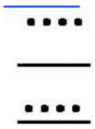
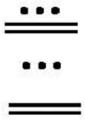
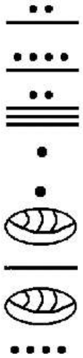
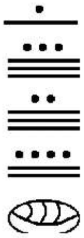


Exercise 10

10. You can't have higher than the 17 at the second level up because 18 at that level represents 360, which should be represented on the third level up. For all other levels, the highest numeral can be 19.

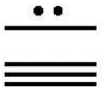
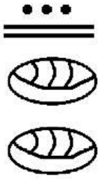
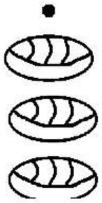
Exercise 11

11.

a. 	b. 	c. 	d. 	e.

Exercise 12

12.

a. 	b. 	c. 

Exercise 13

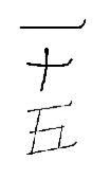
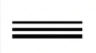
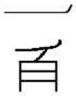
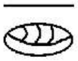
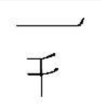
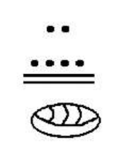
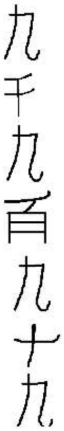
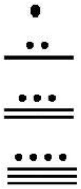
13.

a. 10	b. 12	c. 3

Exercise 14

14.

	Chinese	**Mayan**

a. 15		
b. 100		
c. 1000		
d. 9999		

Exercise Set 3 Solutions

? Exercise 1

1.

a. 2	b. 2	c. 2	d. 2	e. 2

? Exercise 2

2.

a. 27	b. 13, 1, 14
c. 6, 1, 1, 8	d. 3, 0, 1, 1, 5
e. 1, 1, 0, 1, 1, 4	f. 1, 1, 0, 1, 1
g. 27, 4	

? Exercise 3

3.

a. 18	b. 9, 0, 9
c. 4, 1, 0, 5	d. 2, 0, 1, 0, 3

e. 1, 0, 0, 1, 0, 2	f. 1, 0, 0, 1, 0

? Exercise 4

4.

a. 45	b. 22, 1, 23
c. 11, 0, 1, 12	d. 5, 1, 0, 1, 7
e. 2, 1, 1, 0, 1, 5	
f. 1, 0, 1, 1, 0, 1, 4	
g. 1, 0, 1, 1, 0, 1	

? Exercise 5

5.

a. three, zero, two, base six
b. one, zero, one, one, base two
c. four, three, five, base seven

? Exercise 6

6.

a. 5016_{eight}	b. 101001_{two}

? Exercise 7

7.

2048, 1024, 512, 256, ...

? Exercise 8

8.

a. 19	b. 65	c. 63

? Exercise 9

9. 3125, 625, 125, 25, 5, 1

? Exercise 10

10. 8

? Exercise 11

11. 2187, 729, 243, 81, 27, 9, 3, 1

? Exercise 12

12.

a. 500	b. 3711	c. 1093

? Exercise 13

13.

a. Six: 7776, 1296, 216, 36, 6, 1

b. Seven: 16807, 2401, 343, 49, 7, 1

c. Nine: 59049, 6561, 729, 81, 9, 1

d. Eight: 32768, 4096, 512, 64, 8, 1

e. Four: 1024, 256, 64, 16, 4, 1

f. Ten: 100000,10000,1000,100,10,1

g. Twelve: 20736, 1728, 144, 12, 1

h. Eleven: 14641, 1331, 121, 11, 1

? Exercise 14

14. $n^{10}, n^9, n^8, n^6, n^5, n^4, n^2, n^1$

? Exercise 15

15.

a. $3 \times 815 + 6 \times 811 + 2 \times 87$

b. $3 \times 515 + 4 \times 510 + 2 \times 57$

c. $4 \times 1114 + 3 \times 1110 + 2 \times 110$

d. $1 \times 214 + 1 \times 211 + 1 \times 24$

? Exercise 16

16.

a. 9^6	b. 9^0	c. 9^{12}	d. 9^{10}

? Exercise 17

17.

a. 2401000300_{five}

b. $30040001600_{\text{eight}}$

c. $7000804000_{\text{twelve}}$

d. 201010200_{three}

? Exercise 18

18.

a. 6^{17}	b. 9^{19}

? Exercise 19

19. 3: 0, 1, 2

? Exercise 20

20. 4: 0, 1, 2, 3

? Exercise 21

21. 5: 0, 1, 2, 3, 4

? Exercise 22

22. 6: 0, 1, 2, 3, 4, 5

? Exercise 23

23. 8: 0, 1, 2, 3, 4, 5, 6, 7

? Exercise 24

24. No, because the only digits that can occur in a Base Seven numeral are 0, 1, 2, 3, 4, 5 and 6.

? Exercise 25

25.

a. Seven
b. There is no highest base, it can occur in any base higher than Six.

? Exercise 26

26. 11

? Exercise 27

27. 12

? Exercise 28

28. 13

? Exercise 29

29.

a. 671	b. 1291	c. 1830
d. 3496	e. 1358	f. 2231

? Exercise 30

30.

a. 6	b. 12	c. 20	d. 30
e. 42	f. 56	g. 72	h. 90
i. 132	j. 156	k. 182	l. 420

Exercise 31

31. $132011543200501 46_{\text{eleven}}$ is larger. Exercise 30 illustrates that if two numerals have the same exact digits, the one with the higher base has a higher value since each place value is worth more. Explanations may vary.

Exercise 32

32. $4 \times 1213 + 10 \times 129 + 11 \times 123$

Exercise 33

33. $600T000E0000W00_{\text{thirteen}}$

Exercise Set 4 Solutions

Exercise 1

1. Ask a friend how your art work is.

Exercise 2

2.

a. 4, 4, 4, 4, 4, 4
b. 4,16,64,256
c. 5
d. 36

Exercise 3

3. 2, 2, 2, 2, 2

Exercise 4

4. 3, 3, 3

Exercise 5

5. 4, 4, 4

Exercise 6

6.a.

i. 42
ii. 21, 0
iii. 10, 1, 0
iv. 5, 0, 1, 0
v. 2, 1, 0, 1, 0
vi. 1, 0, 1, 0, 1, 0
vii. See first row of part d.

b. See second row of part d.

c. See third row of part d.

d. 101010 Base Two
001120 Base Three
000222 Base Four

Exercise 7

7.

a. 110100_{two}

b. 1221_{three}

c. 310_{four}

Exercise 8

8.

a. 3020_{four}

b. 1212_{seven}

c. 1011001_{two}

d. 100210_{three}

e. $1E8_{\text{thirteen}}$

Exercise 9

9.

a. 12202_{three}

b. $10E_{\text{twelve}}$

c. 10011011_{two}

d. 1110_{five}

Exercise 10

10.

a. $T23T_{\text{eleven}}$

b. $625T_{\text{thirteen}}$

Exercise 11

11.

a. $59T_{\text{twelve}}$

b. 1506_{eight}

c. 2305_{seven}

d. 1131_{nine}

Exercise 13

13. 10

Exercise 14

14. The last digit of the previous numeral must have been a 9 and the last digit of the new numeral will be 0.

Exercise 15

15. 99,999,999

Exercise 16

16. 1,000,000,000

Exercise 17

17. 3

Exercise 18

18. The last digit of the previous numeral must have been a 2 and the last digit of the new numeral will be 0.

Exercise 19

19. 1000000_{three}

Exercise 20

20. 2222222_{three}

Exercise 21

21.

a. 1202011_{three}
b. 2220012_{three}
c. 1010110_{three}
d. 2100220_{three}
e. 2121000_{three}

Exercise 22

22.

a. 1200101_{three}
b. 1202220_{three}
c. 2110012_{three}
d. 2110022_{three}

Exercise 23

23. The digit "3" can't be in the numeral.

Exercise 24

24.
$0_{\text{four}}, 1_{\text{four}}, 2_{\text{four}}, 3_{\text{four}}, 10_{\text{four}}, 11_{\text{four}}, 12_{\text{four}}, 13_{\text{four}}, 20_{\text{four}}, 21_{\text{four}}, 22_{\text{four}}, 23_{\text{four}}, 30_{\text{four}}, 31_{\text{four}}, 32_{\text{four}}, 33_{\text{four}}, 100_{\text{four}}, 101_{\text{four}}, 102_{\text{four}}, 103_{\text{four}}, 110_{\text{four}}, 111_{\text{four}}, 112_{\text{four}}, 113_{\text{four}}$

Exercise 25

25. 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, 10010, 10011, 10100

Exercise 26

26. 75, 76, 77, 100, 101, 102, 103, 104

Exercise 27

27.

a. $3026_{\text{eight}}, 3030_{\text{eight}}$
b. $1234_{\text{five}}, 1241_{\text{five}}$
c. $101011_{\text{two}}, 101101_{\text{two}}$

Exercise 28

28.

a. 12221_{three}, 160
b. 2405_{eight}, 1285
c. 1310_{three}, 205

Exercise Set 5 Solutions

Exercise 1

1.

a. $\frac{1}{4}$	b. $\frac{1}{16}$	c. $\frac{1}{64}$
d. $\frac{1}{3}$	e. $\frac{1}{25}$	f. $\frac{1}{8}$

Exercise 2

2.

a. $35\frac{1}{4}$	b. $168\frac{7}{11}$
c. $15\frac{1}{2}$	d. $27\frac{1}{2}$

Exercise 3

3.

a. $27\frac{13}{36}$	b. $45\frac{27}{64}$
c. $90\frac{9}{49}$	d. $7\frac{5}{8}$
e. $26\frac{26}{27}$	f. $1475\frac{5}{72}$

Exercise 4

4.

a. binary: 00110101; decimal: 53
b. binary: 10011010; decimal: 154
c. binary: 01010100; decimal: 84
d. binary: 00000000; decimal: 0
e. binary: 11111111; decimal: 255

Exercise 5

5.

a. binary: 00110101; hex: 35

b. binary: 10011010; hex: 9A

c. binary: 01010100; hex: 54

d. binary: 00000000; hex: 00

e. binary: 11111111; hex: FF

Exercise 6

6.

a. binary: 01011110; decimal: 94

b. binary: 11100101; decimal: 229

c. binary: 00111001; decimal: 57

d. binary: 00011111; decimal: 31

e. binary: 10011000; decimal: 152

f. binary: 00101010; decimal: 42

g. binary: 00000111; decimal: 7

h. binary: 01000000; decimal: 64

Exercise 7

7.

a. I LOVE MATH!

b. TEACHING IS A CHALLENGING, BUT REWARDING CAREER!

Exercise 8

8.

a. Hex: 48, 45, 4C, 50, 21
Binary: 01001000,01000101, 01001100, 01010000, 00100001

b. Hex: 2,45,20,48,41,50,50,59,2E
Binary: 01000010, 01000101,00100000,01001000,01000001,01010000, 01010000, 01011001, 00101110

c. This will depend on your name!

Exercise 9

9.

a. binary: 1001001_{two}
hex: 49_{sixteen}

b. binary: 1111010_{two}
hex: $7A_{\text{sixteen}}$

c. binary: 110010_{two}
hex: 32_{sixteen}

d. binary: 11111010_{two}
hex: FA_{sixteen}

e. binary: 1111101000_{two}

hex: $3E8_{\text{sixteen}}$

Homework Solutions

Exercise 1

1. b. There is no matching possible because the cardinality of the two sets is different

Exercise 2

2. One possibility is: $1 \leftrightarrow 5, 2 \leftrightarrow 10, 3 \leftrightarrow 15, 4 \leftrightarrow 20$, etc. Give your own answer

Exercise 3

3.

(1) $SBC \leftrightarrow SRC, SBT \leftrightarrow SRT, SBQ \leftrightarrow SRQ$

(2) $SBC \leftrightarrow SRC, SBT \leftrightarrow SRQ, SBQ \leftrightarrow SRT$

(3) $SBC \leftrightarrow SRT, SBT \leftrightarrow SRC, SBQ \leftrightarrow SRQ$

(4) $SBC \leftrightarrow SRT, SBT \leftrightarrow SRQ, SBQ \leftrightarrow SRC$

(5) $SBC \leftrightarrow SRQ, SBT \leftrightarrow SRT, SBQ \leftrightarrow SRC$

(6) $SBC \leftrightarrow SRQ, SBT \leftrightarrow SRC, SBQ \leftrightarrow SRT$

Exercise 4

4.

a. 16

c. 1962

d. 744

f. 3,031,020

h. 50,703

i. 1395

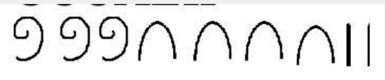
k. 971

Exercise 5

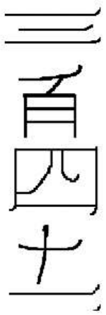
5.

a. CCCXLII

c.

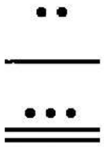


e.



Exercise 6

6.

a. $59T_{\text{twelve}}$	d. 

Exercise 7

7. $7T4E_{\text{twelve}}$

Exercise 10

10. a. $17342575T_{\text{eleven}}$

Exercise 11

11. a. $539100E0_{\text{twelve}}$

Exercise 13

13.

a. 10122_{three}	b. 98

Exercise 14

14.

a. 300640000_{seven}	c. $40T00E00_{\text{thirteen}}$

Exercise 15

15.

a. $2 \times 911 + 5 \times 97 + 3 \times 94$	c. $8 \times 1210 + 11 \times 125 + 10 \times 123$

Exercise 16

16.

a. $39\frac{1}{3}$	b. $16\frac{7}{9}$

Exercise 18

18.

a. 302.301_{four}	c. 101.1001_{two}

Module 3 Addition and Subtraction

Exercise Set 1 Solutions

Exercise 1

1.

b. 3	d. 4	f. 7

Exercise 2

2.

a. {t, u, v, w, x, y, z}	b. 7	c. no	d. no	e. A & B must have no elements in common.

Exercise 4

4.

a. {LBT, LBC, LBQ, LRT, LRC, LRQ,LGT, LGQ, LGC, LYT, LYC, LYQ, SBT, SRT, SGT, SYT}	b. 16	c. no	d. L and T have elements in common.

Exercise 5

5.

a. Let A = {x, y} and B = {a, b, c, d} 2+4 = n(A) + n(B) = $n(A \cup B)$ = n({x, y, a, b, c, d}) = 6 Therefore, 2 + 6 = 6	Since n(A) = 2, n(B) =4, and $A \cap B = \emptyset$, then by substituting n(A) for 2 and n(B) for 4 by substituting n(A) for 2 and n(B) for 4 by computing $A \cup B$ by counting the elements in $A \cup B$
b. Let A = {x, y} and B = {a, b} 2+2 = n(A)+n(B) = $n(A \cup B)$ = n({x, y, a, b}) = 4 Therefore, 2 + 2 = 4	Since n(A) = 2, n(B) =2, and $A \cap B = \emptyset$, then by substituting n(A) for 2 and n(B) for 2 by the set theory definition of addition by computing $A \cup B$ by counting the elements in $A \cup B$

c. Let A = {x, y, z} and B = { } 3 + 0 = n(A) + n(B) = $n(A \cup B)$ = n({x, y, z}) = 3 Therefore, 3 + 0 = 3	Since n(A) = 3, n(B) =0, and $A \cap B = \emptyset$, then by substituting n(A) for 3 and n(B) for 0 by the set theory definition of addition by computing $A \cup B$ by counting the elements in $A \cup B$

Exercise 6

6.

a. K	b. H	c. B	d. O	e. S
f. P	g. B	h. N	i. H	j. B
k. N	l. H	m. each pair is equal.		

Exercise 8

8.

a. Y, B	b. D, B	c. same	d. B, S	e. D, S	f. same

Exercise 11

11.

a. Commutative property of addition
b. Commutative property of addition
c. Associative property of addition
d. Associative property of addition

Exercise 13

13. W + W + W, W + R, R + W, L

Exercise 14

14. 1 + 1 + 1 = 3, 1 + 2 = 3, 2 + 1 = 3, 3 = 3

Exercise 15

15. P,W + W + W + W, W + W + R, W + R + W, R + W + W, W + L, L + W, R + R

Exercise 16

16. 4 = 4, 1 + 1 + 1 + 1 = 4, 1 + 3 = 4, 3 + 1 = 4, 2 + 2 = 4, 1 + 1 + 2 = 4, 1 + 2 + 1 = 4, 2 + 1 + 1 = 4

Exercise 17

17. There are eleven trains: R + P, L+ L, R + R + R, W + W + W + L, D, W + W + W + W + W + W, W + Y, W + W + W + W + R, W + L + R, W + W + P, W + W + R + R

Exercise 18

18. 1 + 1 + 1 + 1 + 1 + 1 = 6, 1 + 5 = 6, 1 + 1 + 4 = 6, 1 + 1 + 1 + 1 + 2 = 6, 1 + 1 + 2 + 2 = 6, 1 + 1 + 1 + 3 = 6, 6 = 6, 2 + 2 + 2 = 6, 1 + 2 + 3 = 6, 3 + 3 = 6, 2 + 4 = 6

Exercise 19

19.

a. R	b. Y	c. N
d. R	e. P	f. W

Exercise 20

20.

b. because 30 = 22 + 8
c. because 156 = 96 + 60
d. because 80 = 0 + 80
e. because 231 = 195 + 36
f. because 987 = 967 + 20

Exercise 21

21.

b. because 30 + 69 = 99
c. because 19 + 51 = 70
d. because 0 + 32 = 32
e. because 489 + 11 = 500
f. because 65 + 136 = 201

Exercise 22

22.

a. left	b. right

Exercise 23

23. yes

Exercise 24

24. yes

Exercise 25

25.

a. <	b. >

Exercise 26

26. {0, 1, 2, 3, . . .}

? Exercise 27

27. yes

? Exercise 28

28. no

? Exercise 29

29.

a. closed (since 0 + 0 = 0)

b. not closed; 1 + 1 = 2 is a counterexample

c. closed; mult of 2: proof is left to you!

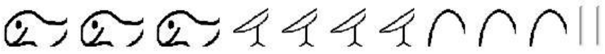
d. not closed; 1 + 3 = 4 is a counterexample

Exercise Set 2 Solutions

? Exercise 1

1.

a.



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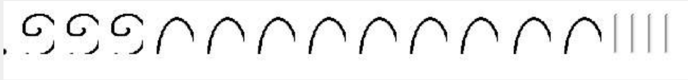
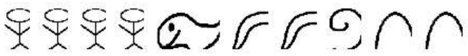
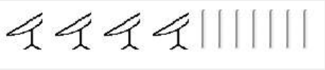


c.



d.



? Exercise 2

2.

a. B

b. D

c. G

? Exercise 3

3.

a. DDDCCCCCCAAAAA

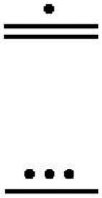
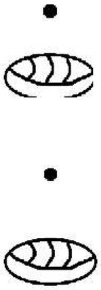
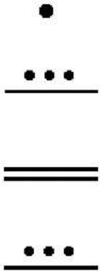
b. EEEDCCCCCBBAAAAA

c. FEB

d. E

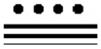
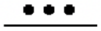
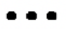
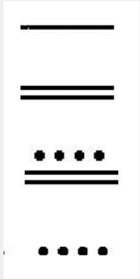
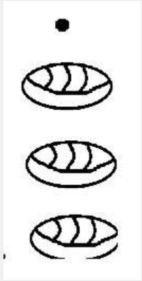
Exercise 4

4.

a. 	b. 	c. 

Exercise 5

5.

a. 	b. 	c. 
d. 	e. 	

Exercise 6

6. $33_{\text{four}} + 31_{\text{four}} = 130_{\text{four}}$

Exercise 7

7. $21_{\text{seven}} + 16_{\text{seven}} = 40_{\text{seven}}$

Exercise 8

8. \(1111_{\text{two}} + 1101_{\text{two}} = 11100_{\text{two}}\)

Exercise 9

9. $16_{\text{nine}} + 14_{\text{nine}} = 31_{\text{nine}}$

Exercise 9

10. $120_{\text{three}} + 111_{\text{three}} = 1001_{\text{three}}$

Exercise 9

11. $17_{\text{eight}} + 15_{\text{eight}} = 34_{\text{eight}}$

Exercise 13

13.

a. 2_{nine}	b. 2_{seven}	c. T_{twelve}
d. 8_{eleven}	e. 4_{six}	f. 1_{two}
g. 1_{three}	h. 3_{four}	i. 2_{five}

Exercise 14

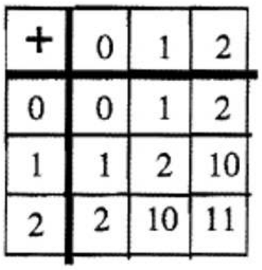
14.

a. 14_{eleven}	b. 12_{five}
c. 11_{eight}	d. 11_{thirteen}
e. 12_{four}	f. 13_{seven}
g. 11_{three}	h. 10_{two}
i. 12_{six}	j. 13_{nine}

Exercise 15

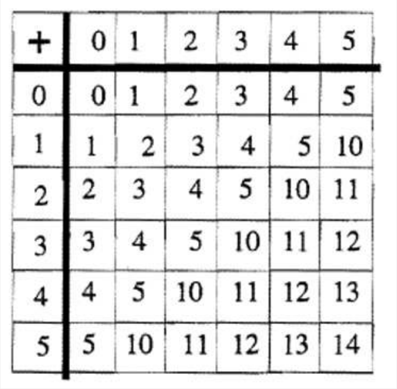
15.

a. Base Three Addition Table



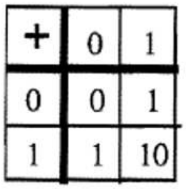
+	0	1	2
0	0	1	2
1	1	2	10
2	2	10	11

b. Base Six Addition Table



+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	11
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

c. Base Two Addition Table



+	0	1
0	0	1
1	1	10

Exercise Set 3 Solutions

? Exercise 1

1.

a. 50	b. 51	c. 47	d. 52

? Exercise 2

2.

a. 69	b. 70	c. 74

? Exercise 3

3.

a. 47	b. 43	c. 53	d. 53

? Exercise 4

4.

a. 1065_{thirteen}	b. 10110_{two}
c. 12222_{five}	d. 1168_{nine}
e. 110100_{two}	f. 10211_{three}
g. $T450_{\text{twelve}}$	h. 11521_{seven}

? Exercise 6

6.

a. 43 + 47 = (40 + 3) + (40 + 7)
= (40 + 40) + (3 + 7)
= 80 + 10 = 90

b. 88 + 54 = (80 + 8) + (50 + 4)
= (80 + 50) + (8 + 4)
= 130 + 12
= 100 + 30 + 10 + 2
= 100 + 40 + 2 = 142

Exercise 7

7. The first student adds the ones (9 + 7 = 16), then the tens (50 + 60 = 110) and thirdly, the hundreds (800 + 400 = 1200). Next those three intermediate sums are added together (16 + 110 + 1200 = 1326). The second student is doing basically the same thing except the hundreds are added first, then the tens and ones are added. The first student is adding right to left whereas the second student is adding left to right.

Exercise 9

9.

a. 12,109	b. 111,463
c. 16,668	d. 128,334

Exercise 10

10.

a. 8,805	b. 10,463
c. 13,029	d. 106,003

Exercise 11

11.

a. 1300_{six}	b. 1392_{eleven}
c. 1166_{eight}	d. 11001_{two}
e. 3201_{five}	f. 1065_{thirteen}
g. 10110_{two}	h. 12222_{five}
i. 1168_{nine}	j. 110100_{two}
k. 10211_{three}	l. $T450_{\text{twelve}}$
m. 11521_{seven}	

Exercise 12

12.

a. seven
b. thirteen
c. twelve
d. three - thirteen
e. four - thirteen

Exercise 13

13.

a. Base Eight; 132_{eight}
b. Base Nine: 82_{nine}
c. Base Three; 110_{three}
d. Base Ten; 123

Exercise 14

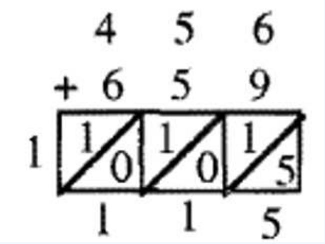
14. She thought of 68 as 70 - 2. So, 47 + (70 - 2) = (47 + 70) - 2 = 117 - 2 = 15

Exercise 15

15. First, he added the tens together, (40 + 60 = 100), then he added the ones together (8 + 7 = 15) and finally he added the two preliminary sums together (100 + 15 = 115).

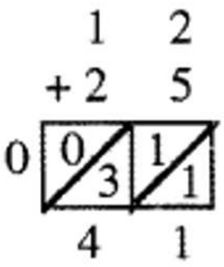
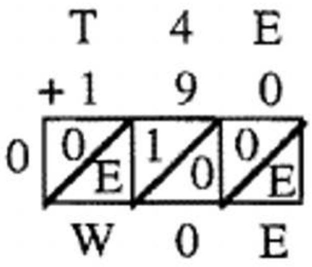
Exercise 18

18. 1115; LATTICE SHOWN BELOW



Exercise 19

19.

a. 41_{six} 	b. WOE_{thirteen} 

Exercise 20

20. Estimate: $46; Actual Sum: $45.18

Exercise 22

22. You'll underestimate if you happen to round down on almost every item. You'll overestimate if you happen to round up on almost every item.

Exercise Set 4 Solutions

Exercise 2

2.

a. 7	b. 3	c. 4

Exercise 4

4.

a. Let A = {t, u, v, w, x, y, z} and B = {w, y, z}. Since n(A) = 7, n(B) = 3 and $B \subseteq A$,
7 - 3 = n(A) - n(B) by substituting n(A) for 7 and n(B) for 3
= n(A \ B) by the set theory definition of substitution
= n({t, u, v, x}) by computing A \ B
= 4 by counting the elements in A \ B
Therefore, 7 - 3 = 4.

a. Let A = {t, u, v, w, x, y, z} and B = {w, y, z}. Since n(A) = 7, n(B) = 3 and $B \subseteq A$,
7 - 3 = n(A) - n(B) by substituting n(A) for 7 and n(B) for 3
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Therefore, 7 - 3 = 4.

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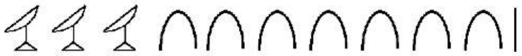
7 - 3 = n(A) - n(B) by substituting n(A) for 7 and n(B) for 3
= n(A \ B) by the set theory definition of substitution
= n({t, u, v, x}) by computing A \ B
= 4 by counting the elements in A \ B
Therefore, 7 - 3 = 4.

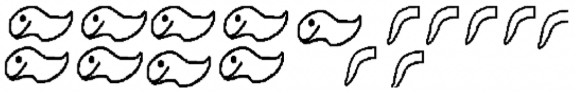
? Exercise 5

5.

a.

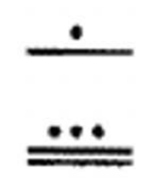


b.



? Exercise 6

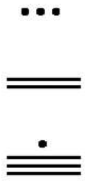
6.



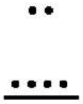
? Exercise 7

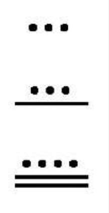
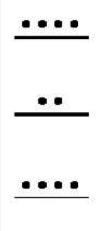
7.

a.



b.



c. 	d. 

Exercise 8

8.

$32_{\text{five}}; 14_{\text{five}}; 13_{\text{five}}$
$32_{\text{five}} - 14_{\text{five}} = 13_{\text{five}}$

Exercise 9

9.

$21_{\text{eight}}; 7_{\text{eight}}; 12_{\text{eight}}$
$21_{\text{eight}} - 7_{\text{eight}} = 12_{\text{eight}}$

Exercise 10

10.

$210_{\text{three}}; 21_{\text{three}}; 112_{\text{three}}$
$210_{\text{three}} - 21_{\text{three}} = 112_{\text{three}}$

Exercise 12

12. L

Exercise 13

13.

a. B	b. Y	c. N	d. N	e. S

Exercise 14

14. This is done with the C-Strips.

Exercise 15

15. No; 7 –10 is not a whole number

Exercise 16

16.

a. closed	b. not closed; 2-4 is a counterexample

Exercise 17

17.

a. 12 – 4 = 8 ; 12 – 8 = 4

b. 170 – 130 = 40 ; 170 – 40 = 130

c. 80 – 62 = 18 ; 80 – 18 = 62

Exercise 18

18.

a. 4 = 7 –3

b. 8 = 9 –1

Exercise 19

19. No. 4 < 7 but it isn't true that 7 < 4

Exercise 21

21.

a. 2; 9 = 7 + 2

b. 7; 13 = 6 + 7

c. 80; 88 = 8 + 80

d. 56; 70 = 14 + 56

Exercise 22

22. 41_{five}

Exercise 23

23. 1_{two}

Exercise 24

24. $4E_{\text{twelve}}$

Exercise 25

25.

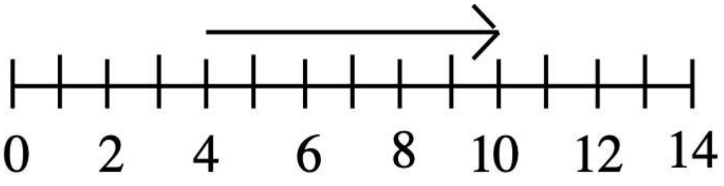
a. -5

b. 3

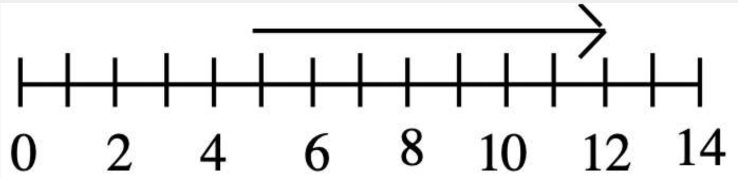
Exercise 26

26.

a. 6



b. 7


0
2
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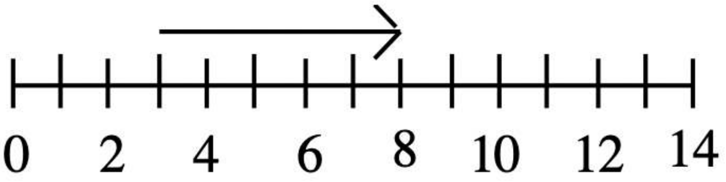
8
10
12
14

c. 5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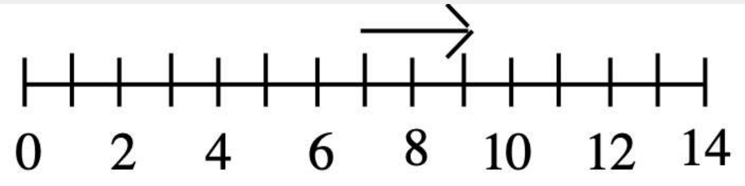
d. 2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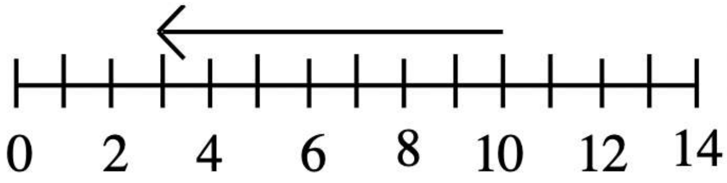
e. -7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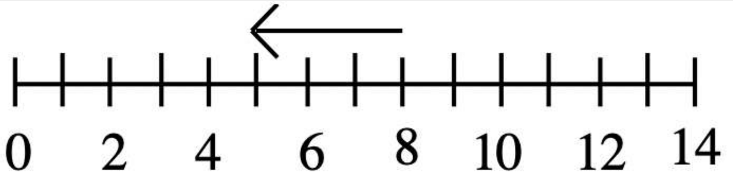
12
14

f. -3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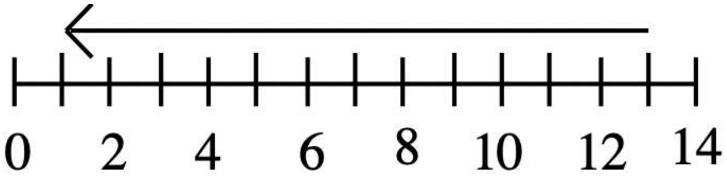
g. -12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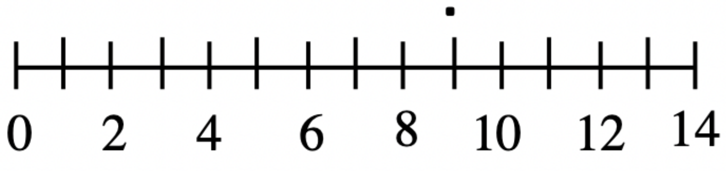
h. 0


0
2
4
6
8
10
12
14

Exercise 27

27. vector a: 6; vector c: -5; vector d: 3

Exercise Set 5 Solutions

Exercise 1

1.

a. 314_{thirteen}	b. 151_{nine}
c. 1225_{six}	d. 134_{five}
e. 12_{three}	

Exercise 2

2.

a. 1618	b. 31_{four}
c. 363_{twelve}	d. 45_{six}
e. 111_{two}	f. 254_{nine}
g. 33_{five}	h. 234_{seven}
i. 2_{three}	j. 236_{eight}

Exercise 3

3. Show the check by adding the answer to the subtrahend and making sure the sum is the minuend.

Exercise 4

4.

a. 6717	b. 1214_{five}
c. 1424_{eight}	d. $49E_{\text{thirteen}}$
e. 1010_{two}	

Exercise 5

5. Show the check by adding the difference (answer) to the subtrahend and making sure the sum is the minuend.

Exercise 6

6.

a. 6697	b. 1144_{five}
c. 1774_{eight}	d. $4WE_{\text{thirteen}}$
e. 111_{two}	

Exercise 7

7. Show the check by adding the difference (answer) to the subtrahend and making sure the sum is the minuend.

Exercise 10

10.

a. 9462	b. 122_{four}	c. 101_{two}

Exercise 11

11. 1_{two}

Exercise 12

12.

a. 4677

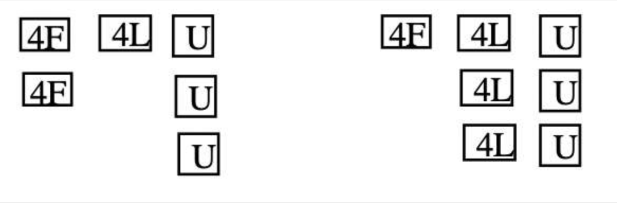
b. 134_{five}

Exercise 13

13.

a. Since the minuend is 120_{three}, the next place value block is a base three block. Since the subtrahend is 12_{three}, we have to figure out what to add to 12_{three} to make a block. 12_{three} is a long and two units, so I have to add 1 unit, 1 long and 2 flats to make a block. Therefore, ass 2 flats, a long a unit to the minuend, 120_{three}, which gives 1 block, 1 flat and 1 unit for the new minuend. If I remove the largest block from this new minuend, I should have the answer of 1 flat and 1 unit.So the answer is 101_{three}. I've left the drawing up to you. See #13 below to see how to start the problem with an abbreviated drawing.

b. Since the minuend is 213_{four}, the next place value block is a base four block. Since the subtrahend is 133_{four}, we have to figure out what to add to 133_{four} to make a block. 133_{four} is a flat, 3 longs and three units, so I have to add 1 unit and 2 flats to make a block. Therefore, add 2 flats and a unit to the minuend, 213_{four}, which gives 1 block and 2 longs for the new minuend. If I remove the largest block from this new minuend, I should have the answer of 2 longs. So the answer is 20_{four}. You can use abbreviations for the drawing and start by representing each original number as shown below. I've left the part to add on and final subtraction for you



Exercise 14

14.

a. She is breaking up the subtrahend into three parts (100 + 50 + 2), subtracting one place value at a time. 634 – 100 = 534; 534 – 50 = 484; 484 – 2 =482.

b. He is using a complementary method. If you add 48 to 152, you get 200. So he adds 48 to the minuend (634 + 48 = 682) and then subtracts 200.

Homework Solutions

Exercise 1

1. Provide your own sets. Here is one solution.

a. First, we must define two sets, A and B, that are disjoint with each other, such that one set has 5 elements and the other has 3 elements.

Let A = {1, 2, 3, 4, 5} and B = {x, y, z}. Since n(A) = 5, n(B) = 3 and $A \cap B = \emptyset$, we can find the sum of 5 and 3 by first forming the union of A and B and then counting the number of elements in their union. $A \cup B$ = {1, 2, 3, 4, 5, x, y, z}. Since n($A \cup B$) = 8, then 5 + 3 = 8.

Exercise 2

2. a. since 79 = 66 + 13

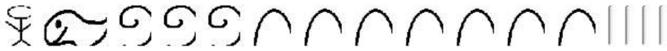
Exercise 3

3. a. closed under both operations

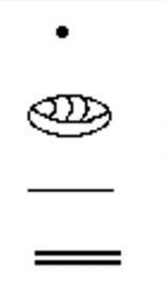
Exercise 4

4.

a.



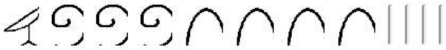
c.



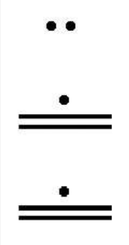
Exercise 5

5.

a.



c.



Exercise 6

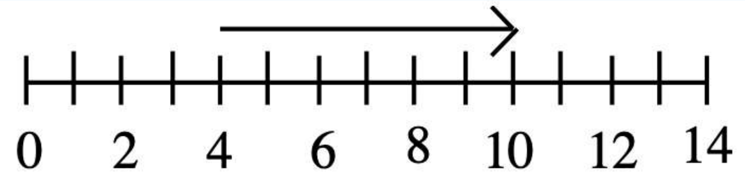
6.

a. 12325_{six}

b. 2535_{five}

Exercise 13

13. a. 6



Exercise 14

14.

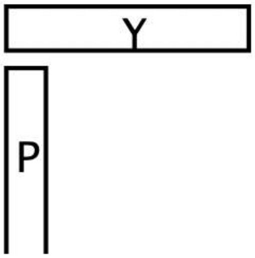
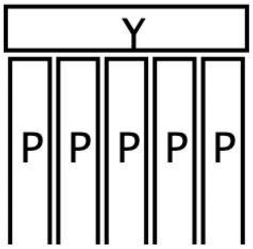
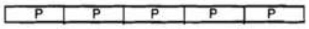
a. Base 9
c. Bases Three through Thirteen

Module 4 Multipication

Exercise Set 1 Solutions

Exercise 1

1.

b. Step 1: 	Step 2: 	c. Step 3: 	d. both trains have the same length

Exercise 2

2.

a. 2 dark green	b. 6 red	c. 12, 12	d. same length	e. $R \times D = D \times R; 2 \times 6 = 6 \times 2$

Exercise 3

3.

a. 7 light green	b. 3 black	c. 21, 21	d. same length	e. $K \times L = L \times K; 7 \times 3 = 3 \times 7$

Exercise 4

4.

a. 12 white	b. 1 hot pink	c. 12, 12	d. same length	e. $H \times W = W \times H; 12 \times 1 = 1 \times 12$

Exercise 5

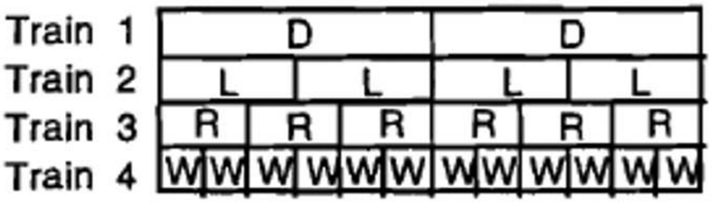
5.

a. D	b. W	c. R	d. P	e. L	f. K	g. L	h. R	i. R

Exercise 6

6.

a. These are the four trains, but they may have been listed them in a different order.



b. Using the trains as listed above,
Train 1: $R \times D = H$, or $2 \times 6 = 12$
Train 2: $P \times L = H$, or $4 \times 3 = 12$
Train 3: $D \times R = H$, or $6 \times 2 = 12$
Train 4: $H \times W = H$, or $12 \times 1 = 12$

Exercise 7

7. Red, R

Exercise 8

8.

a. 3 + 3 + 3 + 3 + 3 + 3 = 18
b. 9 + 9 = 18
c. 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 18

Exercise 9

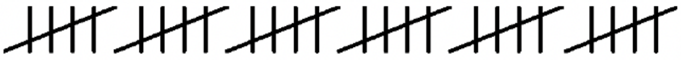
9.

a. $4 \times 3 = 3 + 3 + 3 + 3 = 6 + 3 + 3 = 9 + 3 = 12; 3 \times 4 = 4 + 4 + 4 = 8 + 4 = 12$
b. $2_{\text{five}} \times 3_{\text{five}} = 3_{\text{five}} + 3_{\text{five}} = 11_{\text{five}}; 3_{\text{five}} \times 2_{\text{five}} = 2_{\text{five}} + 2_{\text{five}} + 2_{\text{five}} = 11_{\text{five}}$

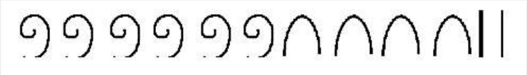
Exercise 10

10. Only the answer, and not the individual steps, are shown

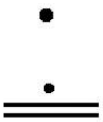
a.



b.

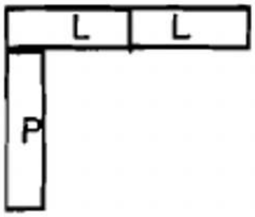
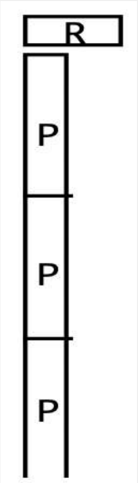


c.



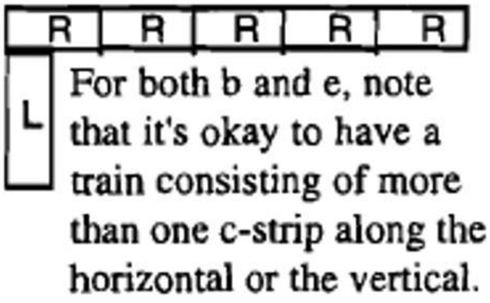
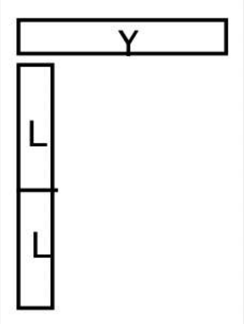
Exercise 11

11.

b.  For both b and e, note that it's okay to have a train consisting of more than one c-strip along the horizontal or the vertical.	c. 6
e. 	f. 6
g. same length	h. $(R \times L) \times P) = R \times (L \times P)$
i. $(2 \cdots 3) \cdots 4 = 2 \cdots (3 \cdots 4)$	j. $(2 \cdots 3) \cdots 4 = 6 \cdots 4 = 24; 2 \cdots (3 \cdots 4) = 2 \cdots 12 = 24$
k. yes	

Exercise 12

12.

b.  For both b and e, note that it's okay to have a train consisting of more than one c-strip along the horizontal or the vertical.	c. 10
e. 	f. 10
g. same length	h. $(Y \times R) \times L) = Y \times (R \times L)$
i. $(5 \cdots 2) \cdots 3 = 5 \cdots (2 \cdots 3)$	j. $(5 \cdots 2) \cdots 3 = 10 \cdots 3 = 30; 5 \cdots (2 \cdots 3) = 5 \cdots 6 = 30$
k. yes	

Exercise 13

13. Do the order of operations to simplify each side of each equation in a - d.

Exercise 14

14. There are various combinations you could use for part a.

Exercise 15

15.

a. 3 purple strips and 3 yellow strips.	b. same as part a
c. $L \times (P + Y) = (L \times P) + (L \times Y)$	d. $4 \times (2 + 3) = (4 \times 2) + (4 \times 3)$
e. $4 \times (2 + 3) = 4 \times 5 = 20$ and $(4 \times 2) + (4 \times 3) = 8 + 12 = 20$	

Exercise 16

16.

a. 2 white strips and 2 dark green strips	b. same as part a
c. $R \times (W + D) = (R \times W) + (R \times D)$	d. $2 \times (1 + 6) = (2 \times 1) + (2 \times 6)$
e. $2 \times (1 + 6) = 2 \times 7 = 14$ and $(2 \times 1) + (2 \times 6) = 2 + 12 = 14$	

? Exercise 17

17. Do the order of operations to simplify each side of each equation in a - d.

? Exercise 18

18. There are various combinations you could use for part a and b.

? Exercise 19

19.

a. R; $1 \cdots 2 = 2$	b. B; $1 \cdots 9 = 9$	c. D; $1 \cdots 6 = 6$
d. S; $1 \cdots 11 = 11$	e. N; $8 \cdots 1 = 8$	f. H; $12 \cdots 1 = 12$

? Exercise 20

20. W, W

? Exercise 21

21.

a. $764 \cdots (1000 - 1) = 764 \cdots 1000 - 764 \cdots 1 = 764000 - 764 = 763236$
b. $324 \cdots (100 + 2) = 324 \cdots 100 + 324 \cdots 2 = 32400 + 648 = 33048$
c. $83 \cdots (74 + 26) = 83 \cdots 100 = 8300$

? Exercise 22

22. Make sure you know all the properties and can provide examples.

? Exercise 23

23.

a. {(3,2), (3,6), (4,2), (4,6)}
b. {(6,5), (7,5),(8,5),(9,5)}
c. { }
d. {(a,a)}
e. {(x,x), (x,y), (y,x), (y,y)}
f. {(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)}

? Exercise 24

24. a. Any two sets, one having 5 elements and one having 2 elements can be used. This is just one possible solution. Let M = {a,b,c,d,e} and let N = {x,y}. Since *n*(M) = 5 and *n*(N) = 2, then $5 \times 2 = n(M \times N) = n((a,x),(a,y),(b,x),(b,y),(c,x),(c,y),(d,x),(d,y),(e,x),(e,y)) = 10$.

? Exercise 25

25.

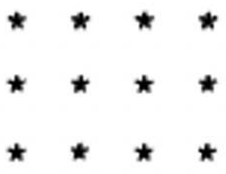
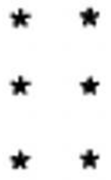
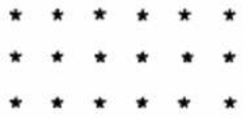
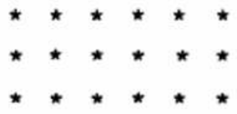
a. 8×6	b. 3×10	c. 8×6	d. 5×5	e. 4×14	f. 3×7

Exercise 28

28. c. 6×4

Exercise 30

30.

a. 	b. 	c. 	d. 	e. yes

Exercise 31

31. The distributive property of multiplication over addition.

Exercise 32

32. This should be done like exercise 30. Explain the steps as well as show the pictures.

Exercise 33

33.

a. 8	b. 8	c. same length

Exercise Set 2 Solutions

Exercise 2

2.

\(13 \times 29\) 1 -----> 29 2 -----> 58 4 ----->116 8 ----->232 ---------------------------- 232 + 116 + 29 = **377**	\(29 \times 13\) 1 ----->13 2 ----->26 4 ----->52 8 ----->104 16 ----->208 ---------------------------- 208 + 104 + 52 + 13 = **377**

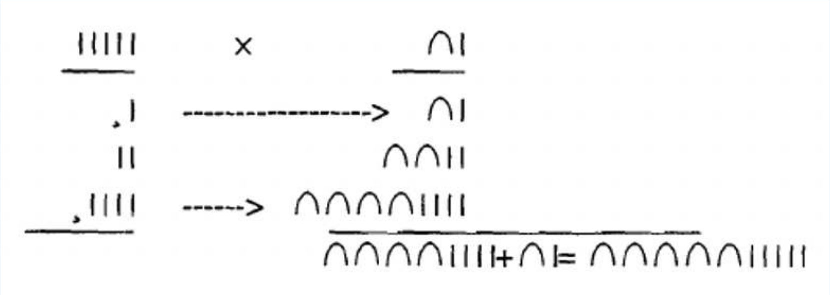
Exercise 3

3.

\(27 \times 14\) 1 -----> 14 2 -----> 28 4 -----> 56 8 -----> 112 16 -----> 224 ---------------------------- 224 + 112 + 28 + 14 = **378**	\(14 \times 27\) 1 -----> 27 2 ----->54 4 ----->108 8 ----->216 ---------------------------- 216 + 108 + 54 = **378**

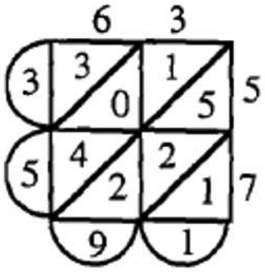
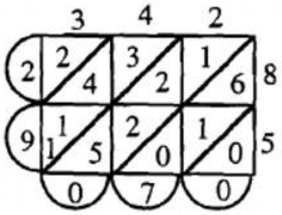
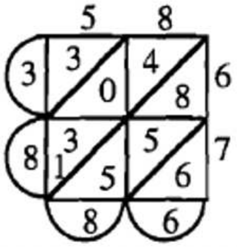
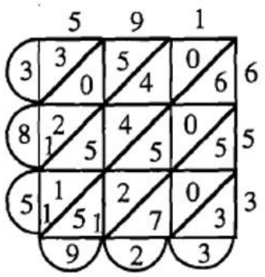
Exercise 4

4.



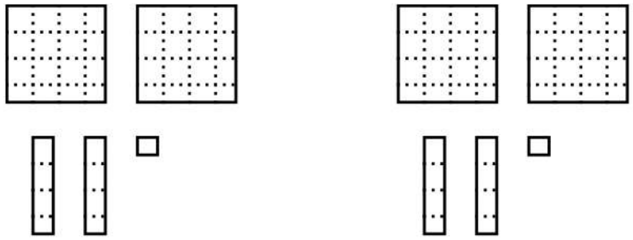
Exercise 5

5.

a. Answer: 3,591 	b. Answer: 29,070 	e. Answer: 3,886 	f. 

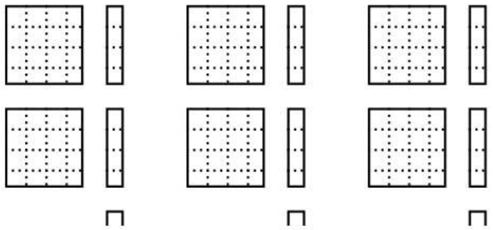
Exercise 6

6.

a. 221_{four}	b. 
c. 1102_{four}	d. $2_{\text{four}} \times 221_{\text{four}} = 1102_{\text{four}}$

Exercise 7

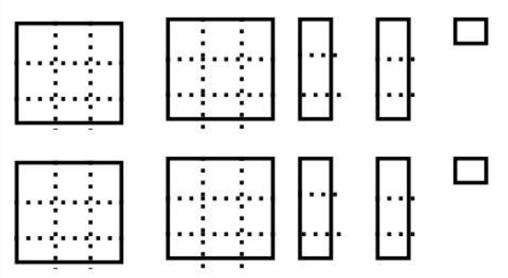
7.

a. 221_{four}	b. 

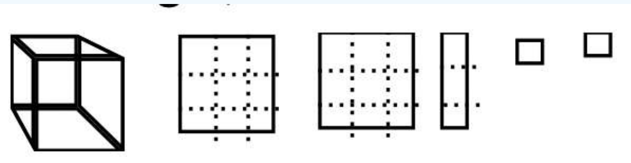
c. 1102_{four}	d. $2_{\text{four}} \times 221_{\text{four}} = 1102_{\text{four}}$

? Exercise 8

8. Make two piles of base three blocks, each having 2 flats, 2 longs and a unit. The two equal piles are shown below.



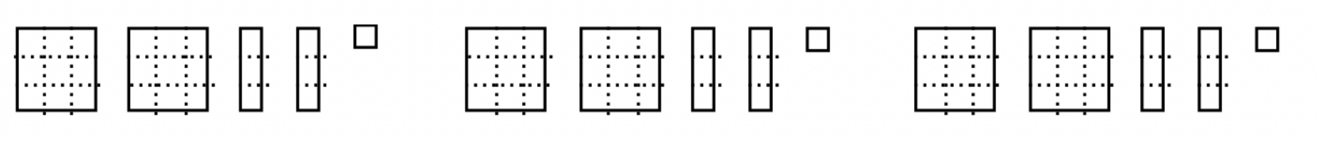
Then, combine the piles together and make appropriate exchanges. After making exchanges, this is what it looks like:



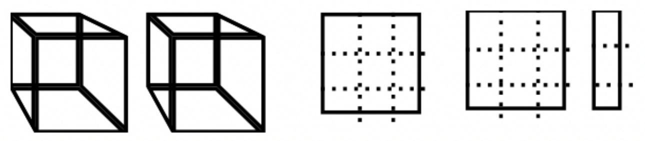
Write the base three numeral represented after making all exchanges. The answer is 1212_{three}.

? Exercise 9

9. Make three piles of base three blocks, each having 2 flats, 2 longs and a unit. The three equal piles are shown below.



Then, combine the piles together and make appropriate exchanges. After making exchanges, this is what it looks like:



Write the base three numeral represented after making all exchanges. The answer is 2210_{three}.

? Exercise 10

10.

a. a flat	b. a flat	c. a flat	d. $L \times L =$ **F**

? Exercise 11

11.

a. a block	a. a block	a. a block	d. $F \times L =$ **B**

? Exercise 12

12.

a. a long block	a. a long block	a. a long block	d. $B \times L =$ **LB**

Exercise 13

13.

a. F	b. B	c. LB	d. B	e. LB

Exercise 15

15.

a. 3, 1 and 2	b. 3 blocks, 1 flat and 2 longs

Exercise 16

16. Only the answers are given here. Make sure you show it with blocks and EXPLAIN.

a. 1032_{four}	b. 10122_{three}

Exercise 17

17.

a. a long block	b. a long block	c. a long block	d. $F \times F = \mathbf{LB}$

Exercise 18

18.

a. U	b. L	c. F	d. B
e. L	f. F	g. B	h. F
i. B	j. LB		

Exercise 20

20. The answer is 20010_{three}. Make sure you show it with blocks and EXPLAIN.

Exercise 21

21. The entire table is shown:

×	U	L	F	B
U	**U**	**L**	**F**	**B**
L	**L**	**F**	**B**	**L**
F	**F**	**B**	**L**	**F**
B	**B**	**L**	**F**	**B**

Exercise 22

22.

a. $23_{\text{five}} \times 32_{\text{five}}$

= (2L + 3U) × (3L + 2U)

= $2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$

= 6F + 4L + 9L + 6U

= 6F + 13L + 6U

= 1B + 1F + 2F + 3L + 1L + 1U

= 1B + 3F + 4L + 1U

= 1341_{five}

$42_{texteight} \times 53_{\text{eight}}$

= (4L + 2U) × (5L + 3U)

= $4L \times 5L + 4L \times 3U + 2U \times 5L + 2U \times 3U$

= 20F + 12L + 10L + 6U

= 20F + 22L + 6U

= 2B + 4F + 2F + 6L + 6U

= 2B + 6F + 6L + 6U

= 2666_{eight}

? Exercise 23

23.

$212_{\text{four}} \times 102_{\text{four}}$

= (2F + 1L + 2U) × (1F + 2U)

= $2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U$

= 2LB + 4F + 1B + 2L + 2F + 4U

= 2LB + 1B + 6F + 2L + 4U

= 2LB + 1B + 1B + 2F + 2L + 1 L

= 2LB + 2B + 2F + 3L

= 22230_{four}

? Exercise 24

24.

a. $361_{\text{nine}} \times 15_{\text{nine}}$

= (3F + 6L + 1U) × (1L + 5U)

= $3F \times 1L + 3F \times 5U + 6L \times 1L + 6L \times 5U + 1U \times 1L + 1U \times 5U$

= 3B + 15F + 6F + 30L + 1L + 5U

= 3B + 21F + 31L + 5U

= 3B + 2B + 3F + 3F + 4L + 5U

= 5B + 6F + 4L + 5U

= 5645_{nine}

b. $111_{\text{two}} \times 11_{\text{two}}$

= (1F + 1L + 1U) × (1L + 1U)

= $1F \times 1L + 1F \times 1U + 1L \times 1L + 1L \times 1U + 1U \times 1L + 1U \times 1U$

= 1B + 1F + 1F + 1L + 1L + 1U

= 1B + 2F + 2L + 1U

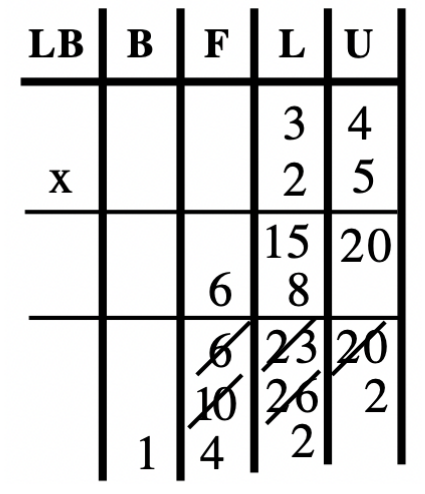
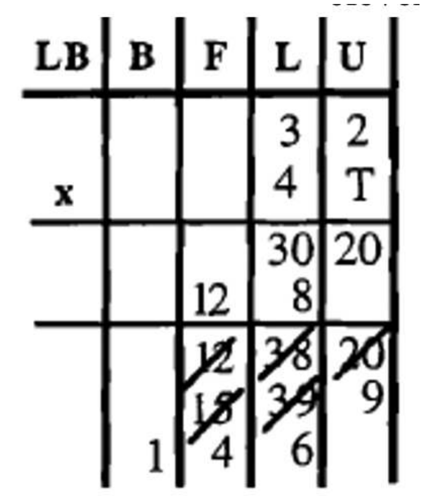
= 1B + 1B + 1F + 1U

= 1LB + 1F + 1U

= 10101_{two}

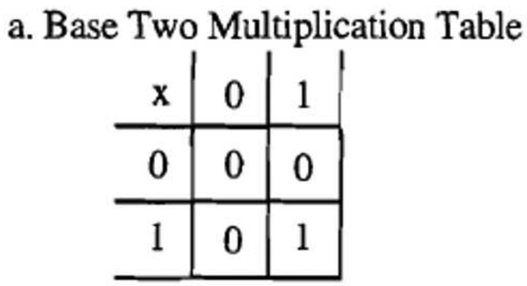
? Exercise 25

25.

a. Answer: 1422_{six}  LB B F L U 3 4 x 2 5 15 20 6 8 ~~6~~ ~~23~~ ~~20~~ ~~10~~ ~~26~~ 2 1 4 2	b. Answer: 1469_{eleven}  LB B F L U 3 2 x 4 T 30 20 12 8 ~~12~~ ~~38~~ ~~20~~ ~~18~~ ~~39~~ 9 1 4 6	c. 100011_{two}
d. $T09_{twelve}$	e. 20211_{three}	f. 2122_{four}

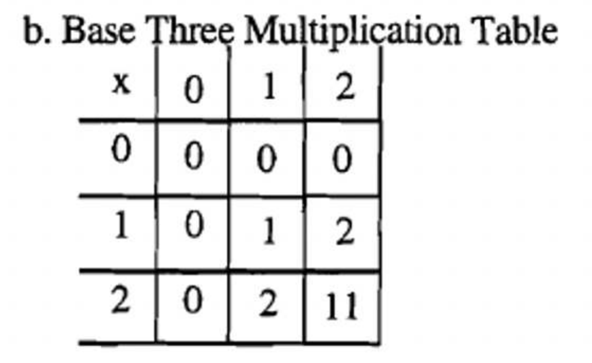
Exercise 26

26. Only a, b and c are shown here. Make sure you do d and e.



a. Base Two Multiplication Table

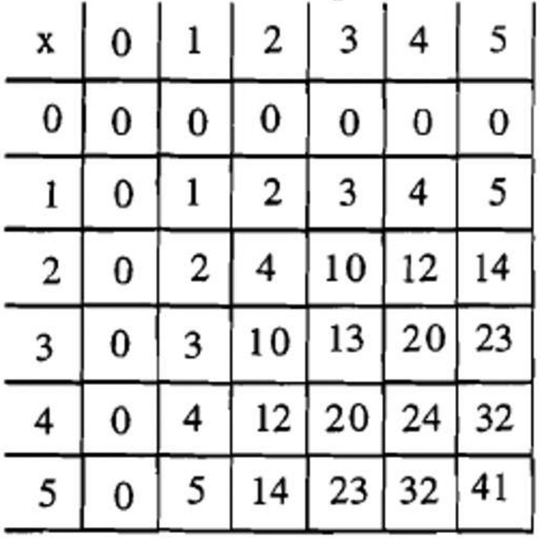
x	0	1
0	0	0
1	0	1



b. Base Three Multiplication Table

x	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

c. Base Six Multiplication Table



x	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	10	12	14
3	0	3	10	13	20	23
4	0	4	12	20	24	32
5	0	5	14	23	32	41

Exercise 27

27.

a. 1103_{five}	b. $1T58_{\text{thirteen}}$	c. 1022_{three}
d. 16015_{seven}	e. 16015_{seven}	f. 101310_{four}

Exercise 29

29. 2144_{five}

Exercise 30

30. 31653_{seven}

Exercise 31

31. 10001111_{two}

Exercise 32

32. 1220212_{three}

Exercise 33

33. 15417_{twelve}

Exercise 34

34. 115503_{six}

Exercise 35

35. 2012220_{three}

Exercise 36

36. 424204_{six}

Exercise 37

37. 101001010_{two}

Exercise 38

38. $T73E8_{\text{twelve}}$

Exercise 39

39. 615032_{eight}

Exercise 40

40. 1033230_{four}

Exercise 41

41. 2321411_{five}

Exercise 42

42. 1563526_{seven}

Exercise 43

43. 414025_{nine}

Exercise 44

44. $8T6697_{\text{eleven}}$

Homework Solutions

Exercise 1

1. a. 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 32

Exercise 3

3. a. P; $4 \cdots 1 = 4$

Exercise 4

4. a. {(3,0), (3,1), (3,6), (x,0), (x,1), (x,6)}

Exercise 6

6.

a. B	c. LB	e. FB

Exercise 10

10.

a. Associative Property of Multiplication

Illustrate this equation is true: $(2_{\text{five}} \cdots 3_{\text{five}}) \cdots 4_{\text{five}} = 2_{\text{five}} \cdots (3_{\text{five}} \cdots 4_{\text{five}})$

Left side: $(2_{\text{five}} \cdots 3_{\text{five}}) \cdots 4_{\text{five}} = 11_{\text{five}} \cdots 4_{\text{five}} = 44_{\text{five}}$

Right side: $2_{\text{five}} \cdots (3_{\text{five}} \cdots 4_{\text{five}}) = 2_{\text{five}} \cdots 22_{\text{five}}) = 44_{\text{five}}$

Both expressions equal 44_{five}, so $(2_{\text{five}} \cdots 3_{\text{five}}) \cdots 4_{\text{five}} = 2_{\text{five}} \cdots (3_{\text{five}} \cdots 4_{\text{five}})$

Distributive Property of Multiplication of Addition:

Illustrate this equation is true: $2_{\text{five}} \cdots (3_{\text{five}} + 4_{\text{five}}) = (2_{\text{five}} \cdots 3_{\text{five}})+(2_{\text{five}} \cdots 4_{\text{five})}$

Left side: $2_{\text{five}} \cdots (3_{\text{five}} + 4_{\text{five}}) = 2_{\text{five}} \cdots (12_{\text{five}}) = 24_{\text{five}}$

Right side: $(2_{\text{five}} \cdots 3_{\text{five}}) + (2_{\text{five}} \cdots 4_{\text{five}}) = 11_{\text{five}} + 13$\trxt$_{five} = 24_{\text{five}}$

Both expressions equal 24_{five}, so $2_{\text{five}} \cdots (3_{\text{five}} + 4_{\text{five}}) = (2_{\text{five}} \cdots 3_{\text{five}})+(2_{\text{five}} \cdots 4_{\text{five}})$

Module 5 Binary Operations

Exercise Set 1 Solutions

? Exercise 2

2.

a. 16	b. 65	c. 2	d. 42	e. 4
f. 18	g. 14	h. 41	i. 20	

? Exercise 3

3.

a. m ! n = n ! m; Yes; Proof: If m ! n =n ! m, then ! is commutative. m ! n = 2 and n ! m = 2. Since both expressions (m ! n and n ! m) equal 2, then ! is commutative.
b. $m \bigoplus n = n \bigoplus m$; Yes; Proof: If m\bogoplus$n = n \bigoplus m$, then $\bigoplus$ is commutative. $m \bigoplus n = 3mn$ and $n \bigoplus m = 3nm$. Since 3mn = 3nm, then $m \bigoplus n = n \bigoplus m.$ Therefore, is commutative.
c. m # n = n # m; Yes; Provide your own an example using numbers.
d. m ⊙ n = n ⊙ m; Yes; Proof: If m ⊙ n = n ⊙ m, then ⊙ is commutative. m ⊙ n = 2m + 2n, and n ⊙ m = 2n + 2m. Since 2m + 2n = 2n + 2m, then m ⊙ n = n ⊙ m. Therefore, ⊙ is commutative.
e. m\XBox$n = n$\XBox$m$; No; Provide your own counterexample using numbers.
f. m , n = n , m; Yes; Proof: If m , n = n , m, then , is commutative. $m, n = m^2 + n^2$ and $n, m = n^2 + m^2$. Since $m^2 + n^2 = n^2 + m^2$, then m , n = n , m. Therefore ,, is commutative.
g. m * n = n * m; No; Provide your own counterexample using numbers.
h. m ♓ n = n ♓ m; No; Provide your own counterexample using numbers.

? Exercise 4

4.

a. 17	b. 4	c. 2	d. 216	e. 36	f. 125

? Exercise 5

5.

a. 270	b. 34	c. 27	d. 6	e. 6
f. 629	g. 6	h. 4	i. 12	

Exercise 6

6.

a. (a ! b)q c = a ! (b! c); yes; Proof: If (a ! b) ! c = a ! (b! c), then ! is associative. (a ! b) !c = 2! c = 2, and a ! (b! c) = a ! 2 = 2. Since both (a ! b) ! c and a ! (b! c) are equal to 2, then ! is associative.

b. $(a \bigoplus b) \bigoplus c = a \bigoplus (b$\bioplus$c)$; yes; Proof: If $(a \bigoplus b) rc = a \bigoplus (b \bigoplus c)$, then $\bigoplus$ is associative. $(a \bigoplus b) \bigoplus c = 3ab \bigoplus c = 3 \cdots 3ab \cdots c = 9abc$, and $a \bigoplus (b \bigoplus c) = a \bigoplus 3bc = 3 \cdots a \cdots 3bc = 9abc$. Since both expressions $[(a \bigoplus b) \bigoplus c$ and $a \bigoplus (b \bigoplus c)]$ are equal to 9abc, then $\bigoplus$ is associative.

c. (a # b)# c = a # (b # c); yes; Provide your own example using numbers.

d. (a ʘ b) ʘ c = a ʘ (b ʘ c); no; Provide your own counterexample using numbers.

e. $(a$\XBox$b)$\XBox$c = a$\XBox$(b$\XBox$c)$; no; Provide your own counterexample using numbers.

f. (a ,b) , c = a , (b, c); no; Provide your own counterexample using numbers.

g. (a * b)* c = a * (b * c); no; Provide your own counterexample using numbers.

Exercise 7

7. a @ (b + c) = (a @ b) + (a @ c)

Exercise 8

8. a + (b @ c) = (a + b) @ (a + c)

Exercise 9

9.

a. a & (b $ c) = (a & b) $ (a & c)

b. a $ (b & c) = (a $ b) & (a $ c)

Exercise 10

10.

a. a ! (b r c) = (a ! b) r (a ! c)

b. no

c. Provide your own counterexample using numbers.

d. $a \bigoplus (b!c) = (a \bigoplus b)!(a \bigoplus c)$

e. no

f. Provide your own counterexample using numbers.

Exercise 11

11.

a. $a + (b \cdots c) = (a + b) \cdots (a + c)$

b. no

c. Provide your own counterexample using numbers.

Exercise 12

12.

a. $a \cdots (b \cdots c) = (a \cdots b) \cdots (a \cdots c)$

b. yes

c. Provide your own example using numbers.

Exercise 13

13.

a. a , (b @ c) = (a , b) @ (a , c)

b. no

c. Provide your own counterexample using numbers.

d. a @ (b , c) = (a @ b) , (a @ c)

e. no

f. Provide your own counterexample using numbers.

Exercise 14

14. You are on your own here. Be creative.

Exercise 15

15.

a. m ∮ n = n ∮ m; no; provide your own counterexample using numbers

b. m ▢ n = n ▢ m; yes; provide your own proof

c. (m ∮ n) ∮ x = m ∮ (n ∮ x); no; provide your own counterexample using numbers

d. (m ▢ n) ▢ x = m y (n ▢ x); no; provide your own counterexample

e. a ∮ (b + c) = (a ∮ b) + (a ∮ c); no; provide your own counterexample

f. a ▢ (b + c) = (a ▢ b) + (a ▢ c); no; provide your own counterexample

g. a ∮ (b y c) = (a ∮ b) y (a ∮ c); no; provide your own counterexample

h. a ▢ (b ∮ c) = (a ▢ b) ∮ (a ▢ c); no; provide your own counterexample

Exercise 16

16. $(a+b)\cdots c=(a\cdots c)+(b\cdots c)$; yes; provide your own example

Exercise 17

17. $(a\cdots b)+c=(a+c)\cdots(b+c)$; no; provide your own counterexample

Exercise 18

18. $(a+b)\div c=(a\div c)+(b\div c)$; yes; provide your own example

Exercise 18

19. $a\div(b+c)=(a\div b)+(a\div c)$; no; provide your own counterexample

Exercise 20

20.

a. (b + c) ∮ a = (b ∮ a) + (c ∮ a); no; provide your own counterexample

b. (b + c) ▢ a = (b ▢ a) + (c ▢ a); no; provide your own counterexample

c. (b ⌈^⌉ c) ∮ a = (b ∮ a) ⌈^⌉ (c ∮ a); no; provide your own counterexample

d. (b ∮ c) ⌊^⌋ a = (b ⌊^⌋ a) ∮ (c ⌈^⌉ a); no; provide your own counterexample

Homework Solutions

Exercise 1

1.

a. 23	b. 20	c. 2a + b	d. 26	g. 2
j. 90	m. 1	p. 10	s. 26	v. 17
y. 45				

Exercise 2

2. a. 19

Exercise 3

3.

a. 20	c. 36	e. 21

Module 6 Integers

Exercise Set 1 Solutions

Exercise 1

1. {..., -3, -2, -1}

Exercise 2

2. {..., -3, -2, -1, 0, 1, 2, 3, ...}

Exercise 3

3.

a. 4	b. 8	c. 2	d. 1	e. 0

Exercise 4

4.

a. 7	b. 13	c. 4	d. $\frac{3}{7}$	e. $\frac{3}{7}$	f. 6

Exercise 5

5. a. 6, -6

Exercise 6

6. a. 19, -19

? Exercise 7

7. 0

? Exercise 8

8. none

? Exercise 9

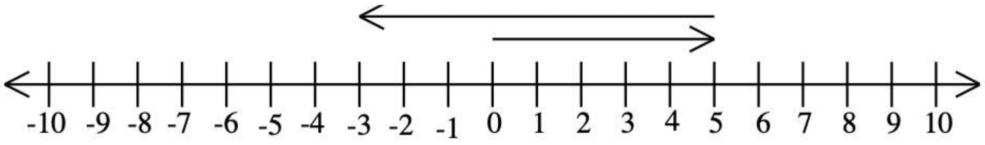
9.

a. -5	b. +3

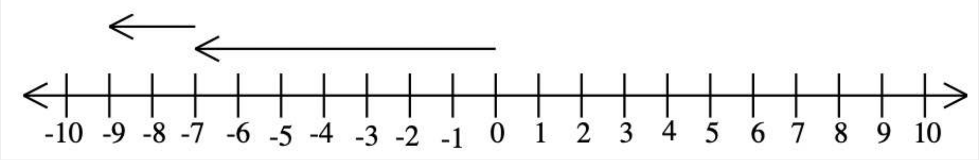
? Exercise 10

10.

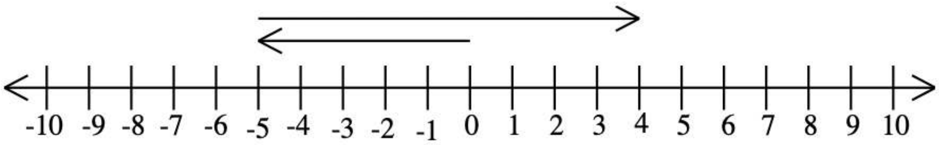
a. -3 since the terminal point of the last vector landed on -3.


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

b. -9 since the terminal point of the last vector landed on -9.


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

c. 4 since the terminal point of the last vector landed on 4.


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
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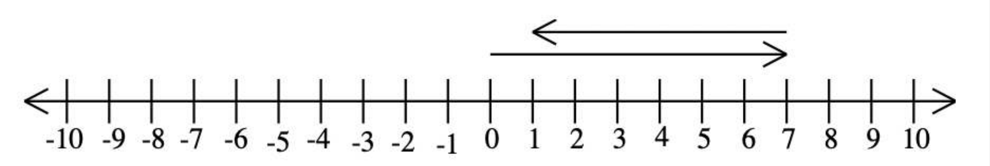
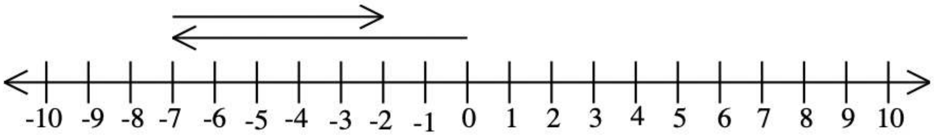
6
7
8
9
10

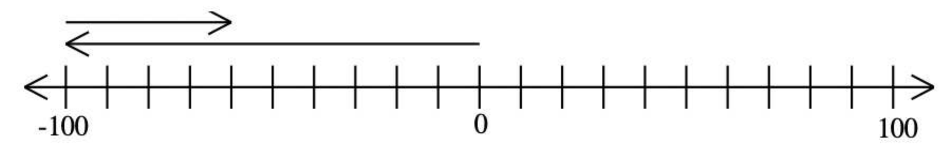
d. 1 since the terminal point of the last vector landed on 1.


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

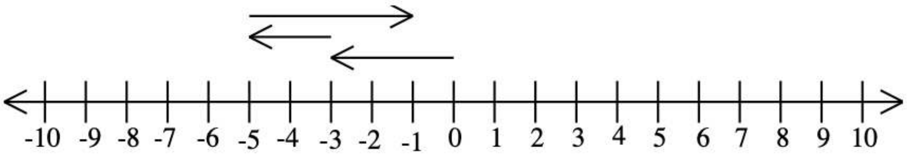
e. -2 since the terminal point of the last vector landed on -2.


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

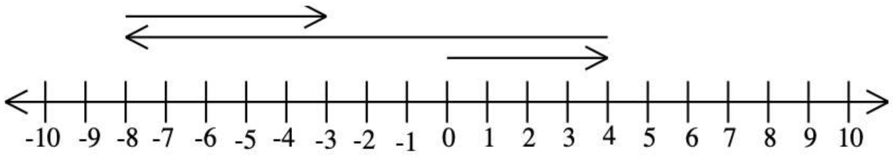
f. -60 since the terminal point of the last vector landed on -60.



g. -1 since the terminal point of the last vector landed on -1.



h. -3 since the terminal point of the last vector landed on -3.



? Exercise 11

11. thermometer

? Exercise 12

12. walk two blocks east

? Exercise 13

13. The answers for these are the same as for those in exercise 10.

? Exercise 14

14.

a. 5	b. -9	c. -3	d. 6

? Exercise 15

15.

a. RRRRR	b. GG	c. RRRR

? Exercise 16

16. 0; reasons may vary

? Exercise 17

17. Answers will vary, but there will always be the same number of negative counters as positive counters in each representation.

? Exercise 18

18.

a. -2	b. -2	c. -2	d. -2	e. -3	f. +3	g. +3	h. +4

? Exercise 19

19. Answers will vary, but there will be 4 more positive counters than negative counters in each representation.

Exercise 20

20. Answers will vary, but there will be 4 more negative counters than positive counters in each representation.

Exercise 21

21.

a. (1) Combine 5 reds and 3 reds: RRRRR + RRR
(2) There are 8 reds, RRRRRRRR, which represents -8.
(3) Therefore, -5 + -3 = -8.

b. (1) Combine 6 reds and 9 greens: RRRRRR + GGGGGGGGG = R R R R R R G G G G G G G G G
(2) Remove 6 red-green pairs (zero), leaving GGG, which represents -3.
(3) Therefore, -6 + 9 = 3.

c. (1) Combine 8 reds and 6 greens: RRRRRRRR + GGGGGG = R R R R R R R R G G G G G G
(2) Remove 6 red-green pairs (zero), leaving RR, which represents -2.
(3) Therefore, -8 + 6 = -2.

d. (1) Combine 5 reds, 3 reds and 6 greens: RRRRR + RRR + GGGGGG = R R R R R R R R G G G G G G
(2) Remove 6 red-green pairs (zero), leaving RR, which represents -2.
(3) Therefore, -5 + -3 + 6 = -2.

e. (1) Combine 6 reds, 9 greens and 1 red: RRRRRR + GGGGGGGGG + R = R R R R R R R G G G G G G G G G
(2) Remove 7 red-green pairs (zero), leaving GG which represents +2.
(3) Therefore, -6 + 9 + -1 = +2.

f. (1) Combine 4 greens, 6 greens and 5 reds: GGGG + GGGGGG + RRRRR = G G G G G G G G G G R R R R R
(2) Remove 5 red-green pairs (zero), leaving GGGGG, which represents +5.
(3) Therefore, +4 + +6 + -5 = +5.

Exercise 22

22.

a. (1) 6 – 3 means remove 3 greens from a collection of counters representing 6.
(2) Represent 6 with 6 greens: GGGGGG
(3) Remove 3 greens, leaving GGG, which represents +3.
(4) Therefore, 6 – 3 = 3

b. (1) 4 – 6 means remove 6 greens from a collection of counters representing 4.
(2) Represent 4 with 4 greens: GGGG
(3) Add 2 red-green pairs (zero) to the collection:
G G G G G G R R
(4) Remove 6 greens, leaving RR, which represents -2.
(5) Therefore, 4 – 6 = -2

c. (1) -7 – -6 means remove 6 reds from a collection of counters representing -7.
(2) Represent -7 with 7 reds: RRRRRRR
(3) Remove 6 reds, leaving R, which represents -1.
(4) Therefore, -7 – (-6) = -1

d. (1) -3 – -7 means remove 7 reds from a collection of counters representing -3.
(2) Represent -3 with 3 reds: RRR
(3) Add 4 red-green pairs (zero) to the collection:
R R R R R R R G G G G
(4) Remove 7 reds, leaving GGGG, which represents +4.
(5) Therefore, -3 – (-7) = +4

e. (1) 4 – -3 means remove 3 reds from a collection of counters representing +4.
(2) Represent +4 with 4 greens: GGGG
(3) Add 3 red-green pairs (zero) to the collection:
G G G G G G G R R R
(4) Remove 3 reds, leaving GGGGGGG, which represents +7.
(5) Therefore, 4 – (-3) = +7

f. (1) -2 – 5 means remove 5 greens from a collection of counters representing -2.
(2) Represent -2 with 2 reds: RR
(3) Add 5 red-green pairs (zero) to the collection:
R R R R R R R G G G G G
(4) Remove 5 greens, leaving RRRRRRR, which represents -7.
(5) Therefore, -2 – 5 = -7

g. (1) 5 – 5 means remove 5 greens from a collection of counters representing 5.
(2) Represent 5 with 5 greens: GGGGG
(3) Remove 5 greens, leaving nothing, which represents 0.
(5) Therefore,5 – 5 = 0

h. (1) 5 – (-5) means remove 5 reds from a collection of counters representing 5.
(2) Represent 5 with 5 greens: GGGGG
(3) Add 5 red-green pairs (zero) to the collection:
G G G G G G G G G G. R R R R R
(4) Remove 5 reds, leaving GGGGGGGGGG, which represents 10.
(5) Therefore, 5 – (-5) = +10

i. (1) -4 – (- 4) means remove 4 reds from a collection of counters representing -4.
(2) Represent -4 with 4 reds: RRRR
(3) Remove 4 reds, leaving nothing, which represents 0..
(5) Therefore, -4 – -4 = 0.

j. (1) -6 – 6 means remove 6 greens from a collection of counters representing -6.
(2) Represent -6 with 6 reds: RRRRRR
(3) Add 6 red-green pairs (zero) to the collection:
R R R R R R R R R R R R G G G G G G
(4) Remove 6 greens, leaving RRRRRRRRRRRR, which represents -12.
(5) Therefore, -6 – 6 = -12

? Exercise 23

23.

a. (1) 5 – 8 means remove 8 greens from a collection of counters representing 5.
(2) Represent 5 with 5 greens: GGGGG
(3) Add 3 red-green pairs (zero) to the collection:
G G G G G G G G R R R
(4) Remove 8 greens, leaving RRR, which represents -3.
(5) Therefore, 5 – 8 = -3

b. (1) 5 + -8 means combine 5 greens and 8 reds: GGGGG + RRRRRRRR = G G G G G R R R R R R R R
(2) Remove 5 red-green pairs (zero), leaving RRR, which represents -3.
(3) Therefore, 5 + -8 = -3.

c. The answers are the same.

? Exercise 24

24. There are various ways to do this problem. But in any case, the final collection for each should give the same answer.

Exercise 25

25.

a. 1st way: Remove 3 greens from a collection representing -5.
2nd way: Combine a collection of 5 reds and 3 reds together.

b. 1st way: Remove 8 greens from a collection representing +4.
2nd way: Combine a collection of 4 greens and 8 reds together.

c. 1st way: Remove 7 reds from a collection representing -9.
2nd way: Combine a collection of 9 reds and 7 greens together.

d. 1st way: Remove 10 reds from a collection representing -10.
2nd way: Combine a collection of 10 reds and 10 greens together.

Exercise 26

26.

a. 4 + (-5)	b. -9 + (-7)	c. -3 + (+4)

Exercise 27

27.

a. 4 + -5 + -7 + -6 + +9	b. -43 + (+75) + 12 + 63 + (-9)

Exercise 28

28.

a. -6	b. +8	c. negative, negative	d. positive, positive	e. opposite

Exercise 29

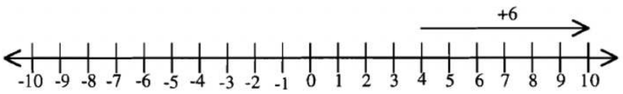
29.

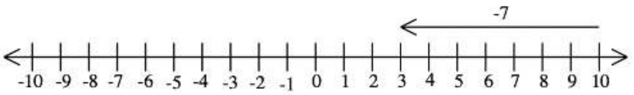
a. When the minuend and subtrahend are the same number	b. 0

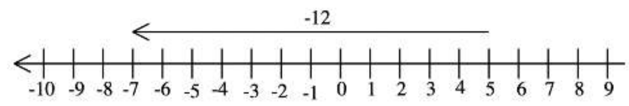
Exercise 30

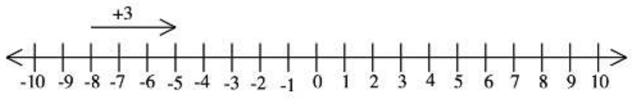
30.

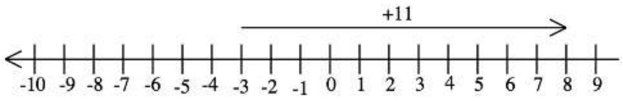
<table>
<tr>
<td>
a. 10 – 4 = +6

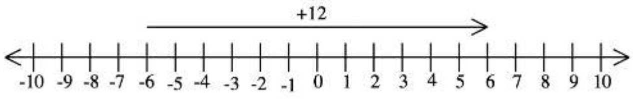

+6
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
</td>
<td>
e. 3 – 10 = -7

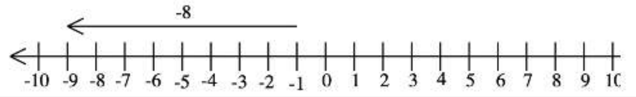
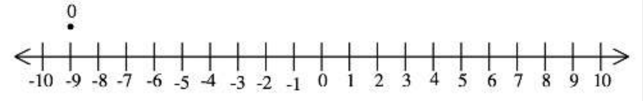

-7
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
</td>
</tr>
<tr>
<td>
b. -7 – 5 = -12


-12
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
</td>
<td>
f. -5 – (-8) = +3


+3
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
</td>
</tr>
<tr>
<td>
c. 8 – (-3) = +11


+11
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
</td>
<td>
g. 6 – (-6) = +12


+12
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
</td>
</tr>
</table>

d. -9 – -1 = -8  	h. 9 – 9 = 0 

Exercise 31

31.

a. 1 – (-5) = +6	b. 13 – 10 = +3	c. 3 – 8 = -5

Exercise 32

32.

a. -92 – -71 = -21	d. -122 – 76 = -198
b. 92 – -81 = +173	e. 208 – 389 = -181
c. -110 – -200 = +90	a. 46 + -76 + 92 = 138 + -76 = 62

Exercise 33

33.

a. 46 + -76 + 92 = -30 + 92 = 62

b. -63 + 94 + 45 + -71 = 31 + 45 + -71 = 76 + -71 = 5

Exercise 34

34.

a. 46 + -76 + 92 = 138 + -76 = 62

b. -63 + 94 + 45 + -71 = 139 + -134 = 5

Exercise 35

35.

a. Closed

a. Closed

a. Closed

d. Not Closed; one counterexample is: 7 – 10 = -3

e. Closed

f. Not Closed; one counterexample is: -11 – -20 = +9

g. Not closed; one counterexample is: 1 + 1 = 2

h. Not closed; one counterexample is: 1 – -1 = 2

Exercise Set 2 Solutions

Exercise 1

1.

a. R R R R R G

b. R R R R R R R R R R R R R R R G G G

c. RRRRRRRRRRRR = -12

Exercise 2

2.

a. R R R R R R G G

b. R R R R R R R R R R R R R R R R R R G G G G G G

c. RRRRRRRRRRRR = -12

Exercise 3

3.

a. R R R R

b. R R R R R R R R R R R R

c. -12

Exercise 4

4.

a. 4×-2 means to combine 4 sets of 2 reds: RR + RR + RR + RR = RRRRRRRR = -8

b. 3×5 means to combine 3 sets of 5 greens: GGGGG + GGGGG + GGGGG = GGGGGGGGGGGGGGG = +15

c. 5×-3 means to combine 5 sets of 3 reds: RRR + RRR + RRR + RRR + RRR = RRRRRRRRRRRRRRR = -15

d. 7×2 means to combine 7 sets of 2 greens: GG + GG + GG + GG + GG + GG + GG = GGGGGGGGGGGGGG = +14

e. 0×-3 means to combine 0 sets of 3 reds = 0

Exercise 5

5.

a. R R R R R R R R R R R R R R G G G G G G G G G G G G G G

b. R R R R R R R R R R R R R R G G

c. R R R R R R R R R R R R = -12

Exercise 6

6.

a. R R R R R R R R R R R R G G G G G G G G G G G G

b. R R R R R R R R R R R R

c. -12

Exercise 7

7.

a. -5×3 means to remove 5 sets of 3 greens from a collection representing zero.
(1) Let zero be represented by 15 reds and 15 greens:
R R R R R R R R R R R R R R R
G G G G G G G G G G G G G G G
(2) Remove 5 sets of 3 greens from the above collection, which leaves
RRRRRRRRRRRRRRR, which represents -15. Therefore, $-5 \times 3 = -15$.

b. -3×2 means to remove 3 sets of 2 greens from a collection representing zero.
(1) Let zero be represented by 6 reds and 6 greens:
R R R R R R
G G G G G G
(2) Remove 3 sets of 2 greens from the above collection, which leaves
RRRRRR, which represents -6. Therefore, $-3 \times 2 = -6$

c. 2×-3 means to combine 2 sets of 3 reds: RRR + RRR = RRRRRR = -6. Therefore $2 \times -3 = -6$

d. -2×3 means to remove 2 sets of 3 greens from a collection representing zero.
(1) Let zero be represented by 6 reds and 15 greens:
R R R R R R
G G G G G G
(2) Remove 2 sets of 3 greens from the above collection, which leaves RRRRRR, which represents -6. Therefore, $-2 \times 3 = -6$.

e. 3×2 means to combine 3 sets of 2 greens: GG + GG + GG = GGGGGG = 6. Therefore $3 \times 2 = 6$

f. 0×-4 means to combine 0 sets of 4 reds = 0

g. -4×0 means to remove 4 sets of 0 counters from a collection representing zero.
(1) Choose any representation of zero. One possibility is to let zero be represented by 6 reds and 6 greens.
R R R R R R
G G G G G G
(2) Remove 0 sets of 0 counters from the above collection, which leaves the original collection, which represents 0. Therefore, $-4 \times 0 = 0$.

? Exercise 8

8.

-3×6: Remove 3 sets of 6 greens from zero; 5G and 23R; -18

-3×6: Remove 3 sets of 6 greens from zero; 18R; -18

-4×4: Remove 4 sets of 4 greens from zero; 2G and 18R; -16

-4×-4: Remove 4 sets of 4 reds from zero; 18G and 2R; 16

-5×2: Remove 5 sets of 2 greens from zero; 4G and 14R; -10

-2×5: Remove 2 sets of 5 greens from zero; 3G and 13R; -10

-5×-2: Remove 5 sets of 2 reds from zero; 11G and 1R; 10

-2×-5: Remove 2 sets of 5 reds from zero; answers will vary; 10

-3×3: Remove 3 sets of 3 greens from zero; 1G and 10R; -9

-3×-3: Remove 3 sets of 3 reds from zero; 12G and 3R; 9

-7×2: Remove 7 sets of 2 greens from zero; answers will vary; -14

-2×8: Remove 2 sets of 8 greens from zero; 2G and 18R; -16

-2×-8: Remove 2 sets of 8 reds from zero; 18G and 2R; 16

? Exercise 9

9.

a. positive	b. negative	c. zero	d. positive

Exercise 10

10.

a. Closed
b. Closed
c. Not Closed: one counterexample is: $-2 \times -8 = 16$
d. Not Closed: one counterexample is: $-1 \times -1 = 1$
e. Closed
f. Closed

Exercise 11

11.

a. true
b. true
c. true
d. false; 2 < 3; Multiply both sides by 5: $2 \times 5 > 3 \times 5$ is false.
e. true
f. false: 2 < 3; Multiply both sides by -5: $2 \times -5 > 3 \times -5$ is false.

Exercise 12

12.

a. If a > b and b > c, then a > c
b. If a > b, then a + c > b + c
c. If a > b, then ap > bp
d. If a > b, then an < bn

Homework Solutions

Exercise 2

2.

a. 16	c. 2	e. 8

Exercise 3

3. a. 4 and -4

Exercise 4

4.

a. +7	c. -9

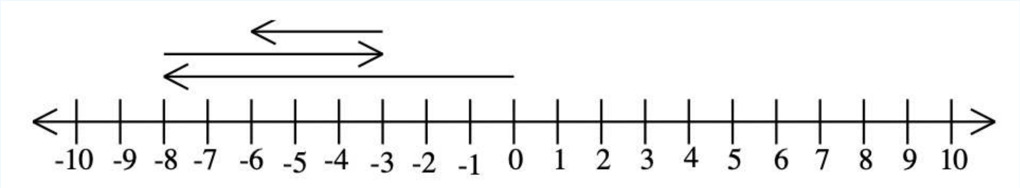
Exercise 5

5.

a. 2 – (-5)	c. 3 – 12

Exercise 6

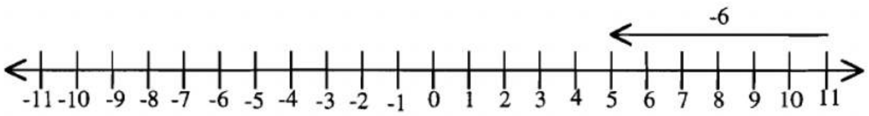
6. -8 + 5 + (-3) = -6, since the terminal point of the last vector landed on -6.


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

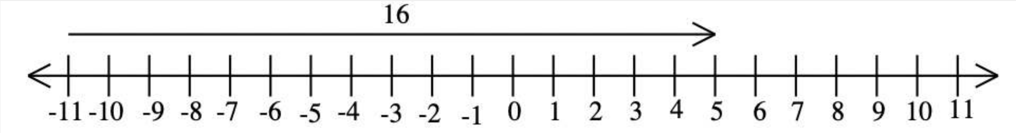
Exercise 8

8.

a. 5 – 11 = -6


-6
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
11

c. 5 – (-11)= 16


16
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
11

Exercise 9

9.

a. (1) -4 + (-3) means to combine 4 reds and 3 reds: RRRR + RRR = RRRRRRR, which represents -7

(2) Therefore, -4 + -3 = -7

Exercise 10

10.

a. (1) 6 – 8 means remove 8 greens from a collection of counters representing +6.

(2) Represent +6 with 6 greens: GGGGGG

(3) Add 2 red-green pairs (zero) to the collection:

G G G G G G G G

R R

(4) Remove 8 greens, leaving RR, which represents -2.

(5) Therefore, 6 – 8 = -2

Exercise 11

11.

a. Closed

b. Not closed; a counterexample is 3 – 10 = -7

c. Closed

Exercise 12

12.

a. -5×3 means to remove 5 sets of 3 greens from a collection representing zero.

(1) Let zero be represented by 15 reds and 15 greens:

R R R R R R R R R R R R R R R

G G G G G G G G G G G G G G G

(2) Remove 5 sets of 3 greens from the above collection, which leaves

RRRRRRRRRRRRRRR, which represents -15. Therefore, $-5 \times 3 = -15$.

Homework Solutions

HW #2

2.

a. 16	c. 2	e. 8

HW #3

3. a. 4 and -4

HW #4

4. a. +7

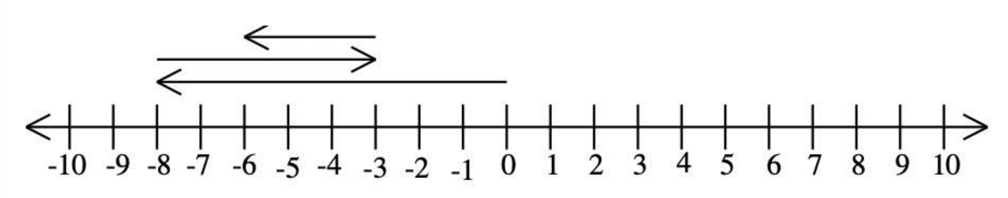
HW #5

5.

a. 2 – (-5)	c. 3 – 12

HW #6

6. -8 + 5 + -3 = -6, since the terminal point of the last vector landed on -6.

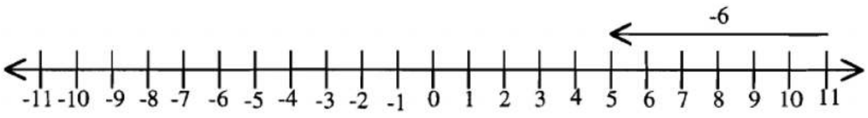


-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

HW #8

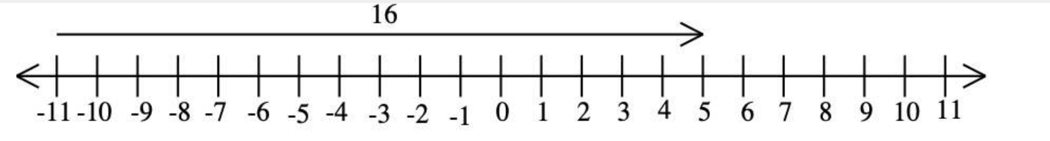
8.

a. 5 – 11 = -6



-6
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
11

c. 5 – (-11) = 16



HW #9

9.

a. (1) -4 + -3 means to combine 4 reds and 3 reds: RRRR + RRR = RRRRRRR, which represents -7

(2) Therefore, -4 + -3 = -7

HW #10

10.

a. (1) 6 – 8 means remove 8 greens from a collection of counters representing +6.

(2) Represent +6 with 6 greens: GGGGGG

(3) Add 2 red-green pairs (zero) to the collection:

G G G G G G G G

R R

(4) Remove 8 greens, leaving RR, which represents -2.

(5) Therefore, 6 – 8 = -2

HW #11

11.

a. Closed
b. Not closed; a counterexample is 3 – 10 = -7
c. Closed

HW #12

12.

a. -5×3 means to remove 5 sets of 3 greens from a collection representing zero.

(1) Let zero be represented by 15 reds and 15 greens:

R R R R R R R R R R R R R R R

G G G G G G G G G G G G G G G

(2) Remove 5 sets of 3 greens from the above collection, which leaves RRRRRRRRRRRRRRR, which represents -15. Therefore, $-5 \times 3 = -15$.

Module 7 Division

Exercise Set 1 Solutions

Exercise 1

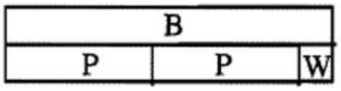
1.

a. 2	b. R	c. R, R

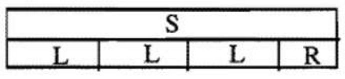
Exercise 2

2.

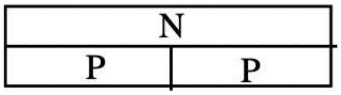
a. P, W, P, W 9, 2, 4, 1, 9, 4, 2, 1



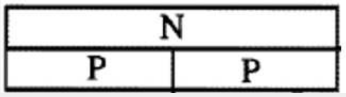
b. L, R, L, R, 11, 3, 3, 2, 11, 3, 3, 2



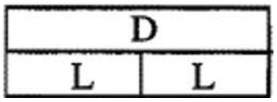
c. R, blank (or 0) , R, blank (or 0), 8, 4, 2, blank (or 0), 8, 2, 4, blank (or 0)



d. W, L, W, L, 7, 4, 1, 3, 7, 1, 4, 3

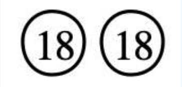


e. R, blank (or 0), R, blank (or 0), 6, 3, 2, blank (or 0), 6, 2, 3, blank (or 0)



Exercise 3

3. 2 equal piles of 18 units each



Exercise 4

4.

a. \(210_{\text{four}\)

b. Methods will vary.

c. 102_{four}

d. 210_{four}, 102_{four}

Exercise 5

5.

a. 1100_{three}

b. Methods will vary.

c. 200_{three}

d. 1100_{three}, 200_{three}

Exercise 7

7.

a. 210_{four}

b. Methods will vary.

c. 30_{four}

d. $210_{\text{four}}, 30_{\text{four}}$

Exercise 9

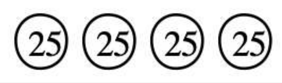
9.

a. $100 \div 4$

b. Partitioning into subsets

c. Disburse 100 marbles into 4 equal subsets.

d.



e. $4 \cdots 25$

f. Count how many marbles were placed in one of the equal subsets.

g. 25

Exercise 10

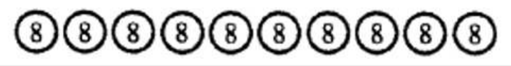
10.

a. $80 \div 8$

b. Repeated Subtraction

c. Put 8 ounces in a subset at a time.

d.



e. $10 \cdots 8$

f. Count how many subsets, each having 8 ounces, were made.

g. 10

Exercise 11

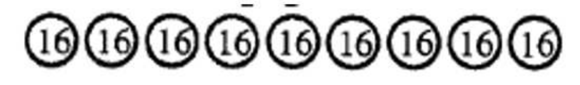
11.

a. $144 \div 16$

b. Repeated Subtraction

c. Count out 16 pages at a time.

d.



e. $9 \cdots 16$

f. Count how many short stories, each having 16 pages, were made.

g. 9

Exercise 12

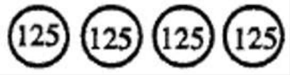
12.

a. $500 \div 4$

b. Partitioning into subsets

c. Disburse $500 into 4 equal subsets.

d.



e. $4 \cdots 125$

f. Count how much money was placed in one of the equal subsets.

g. $125

Exercise 13

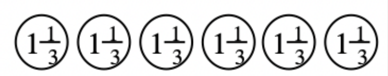
13.

a. $8 \div 1\frac{1}{3}$

b. Repeated Subtraction

c. Put $1\frac{1}{3}$ cups into a subset at a time.

d.



e. $6 \cdots 1\frac{1}{3}$

f. Count how many subsets were formed.

g. 6

Exercise 14

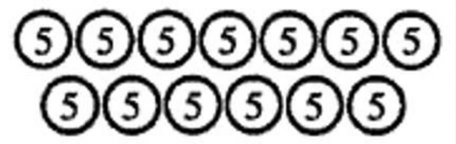
14.

a. $65 \div 13$

b. Partitioning into subsets

c. Disburse 65 baseballs into 13 equal subsets.

d.



e. $13 \cdots 5$

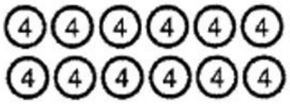
f. Count how baseballs were placed in one of the equal subsets.

g. 5

Exercise 15

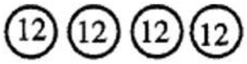
15.

d.


4 4 4 4 4 4
4 4 4 4 4 4

e. $12 \cdots 4$

g.

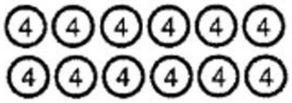


h. $4 \cdots 12$

Exercise 16

16.

d.


4 4 4 4 4 4
4 4 4 4 4 4

e. $4 \cdots 50$

g. Put 4 into a subset at a time.

h. $50 \cdots 4$

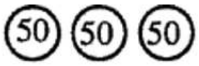
Exercise 17

17.

d. Draw 50 subsets, then disburse 150 amongst the 50 subsets.

e. $50 \cdots 3$

g.



h. $3 \cdots 50$

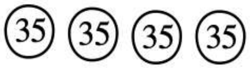
Exercise 18

18.

d. Draw 35 subsets, then disburse 140 amongst the 35 subsets.

e. $35 \cdots 4$

g.



h. $4 \cdots 35$

Exercise 19

19.

d.

⑲ ⑲ ⑲ ⑲ ⑲

e. $5 \cdots 19$

g. Put 5 into a subset at a time.

h. $19 \cdots 5$

Exercise 19

19.

d.

⑲ ⑲ ⑲ ⑲ ⑲

e. $5 \cdots 19$

g. Put 5 into a subset at a time.

h. $19 \cdots 5$

Exercise 19

19.

d.

⑲ ⑲ ⑲ ⑲ ⑲

e. $5 \cdots 19$

g. Put 5 into a subset at a time.

h. $19 \cdots 5$

Exercise 22

22.

a. $56 \div 8$

b. $40 \div 5$

Exercise 23

23.

a. $18 \div 9$

b. $10 \div 2$

Exercise 24

24. a represents how many subsets and b represents how many are in each subset.

Exercise 25

25.

a. Since 32 = 8 ··· 4, then $32 \div 8$ = 4

b. Since 56 = 8 ··· 7, then 56 ÷ 8 = 7

c. Since 32 = 2 ··· 16, then 32 ÷ 2 = 16

d. Since 0 = 13 ··· 0, then 0 ÷ 13 = 0

e. Since 12 = 1 ··· 12, then 12 ÷ 1 = 12

f. Since X = 1 ··· X, then X ÷ 1 = X

g. Since 0 = Y ··· 0, then 0 ÷ Y = 0

Exercise 26

26.

c. Since 48 = 6 ··· 8, then 48 ÷ 6 = 8

d. Since there is no whole number solution to make the equation 35= 4 ··· __ true, 35 ÷ 4 is not defined under whole numbers.

e. Since 48 = 1 ··· 48, then 48 ÷ 1 = 48

f. Since there is no whole number solution to make the equation 55 = 7 ··· __ true, 55 ÷ 7 is not defined under whole numbers.

g. Since 0 = 8 ··· 0, then 0 ÷ 8 = 0

e. 203 r. 314

f. 248 r. 14

Exercise 27

27.

a. There is no number that will make the equation, 6 = 0 ··· ____ true, since any number put in the blank will make the right hand side of the equation zero, which can never equal the left-hand side of the equation, which is 6. Therefore, $6 \div 0$ is not defined.

b. There is no number that will make the equation, 18 = 0 ···____ true, since any number put in the blank will make the right hand side of the equation zero, which can never equal the left-hand side of the equation, which is 18. Therefore, $18 \div 0$ is not defined.

c. There is no number that will make the equation, M = 0 ···____ true, since any number put in the blank will make the right handside of the equation zero, which can never equal the left-hand side of the equation, which is M (since it is assumed that M is not equal to zero). Therefore, $M \div 0$ is not defined.

Exercise Set 2 Solutions

Exercise 1

1.

a. 6 r. 43

b. 7 r. 20

c. 8 r. 8

d. 3 r. 194

e. 5 r. 55

f. 5 r. 165

g. 9 r. 239

Exercise 2

2.

a. 45 r. 15

b. 12 r. 27

c. 32 r. 34

d. 81 r. 29

Exercise 3

3.

a. 45 r. 15

b. 12 r. 27

c. 32 r. 34

d. 81 r. 29

e. 203 r. 314

f. 248 r. 14

Exercise Set 3 Solutions

Exercise 1

1.

a. F F L L U U

b. i. LUU, LUU, LUU, LUU, LUU
ii. U

c. UUUUU

d. quotient: LUU, remainder: U

e. $12_{\text{three}} r.1_{\text{three}}$

f. $12_{\text{three}} \cdots 12_{\text{three}} + 1_{\text{three}} = 221_{\text{three}} + 1_{\text{three}} = 222_{\text{three}}$

Exercise 2

2.

a. $13_{\text{five}} r.3_{\text{five}}$

b. $32_{\text{six}} r.21_{\text{six}}$

c. $11_{\text{four}} r.10_{\text{four}}$

d. $110_{\text{two}} r.1_{\text{two}}$

Exercise 3

3.

a. $23_{\text{seven}} r.3_{\text{seven}}$

b. 102_{three}

Exercise Set 3 Solutions

? Exercise 1

1.

a. F F L L U U

b. i. LUU, LUU, LUU, LUU, LUU
ii. U

c. UUUUU

d. quotient: LUU, remainder: U

e. $12_{\text{three}} r . 1_{\text{three}}$

f. $12_{\text{three}} \cdots 12_{\text{three}} + 1_{\text{three}} = 221_{\text{three}} + 1_{\text{three}} \ = 222_{\text{three}}$

? Exercise 2

2.

a. $13_{\text{five}} r . 3_{\text{five}}$

b. $32_{\text{six}} r . 21_{\text{six}}$

c. $11_{\text{four}} r . 10_{\text{four}}$

d. $110_{\text{two}} r . 1_{\text{two}}$

? Exercise 3

3.

a. $23_{\text{seven}} r . 3_{\text{seven}}$

b. 102_{three}

? Exercise 4

4. $12_{\text{three}} r . 1_{\text{three}}$

? Exercise 5

5. $13_{\text{five}} r . 3_{\text{five}}$

? Exercise 6

6. $11_{\text{four}} r . 10_{\text{four}}$

? Exercise 7

7. $32_{\text{six}} r . 21_{\text{six}}$

? Exercise 8

8. $110_{\text{two}} r . 1_{\text{two}}$

? Exercise 9

9. $23_{\text{seven}} r . 3_{\text{seven}}$

Exercise 10

10. 102_{three}

Exercise 11

11. $31_{\text{five}} r.2_{\text{five}}$

Exercise 12

12. $111_{\text{two}} r.101_{\text{two}}$

Exercise 13

13. $61_{\text{eight}} r.6_{\text{eight}}$

Exercise 14

14. $T2_{\text{thirteen}}\ r.E_{\text{thirteen}}$

Exercise 15

15. $23_{\text{twelve}} r.1E_{\text{twelve}}$

Homework Solutions

Exercise 1

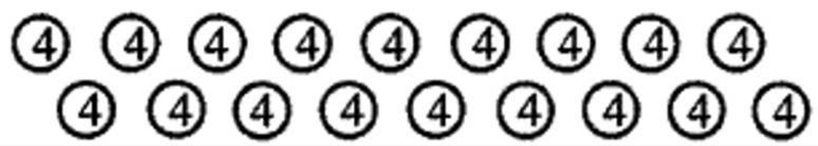
1.

a. $72 \div 4$

b. Repeated subtraction

c. Put 4 lollipops in a bag at a time by repeatedly subtracting 4 from 72 until you run out of lollipops.

d.



e. $18 \cdots 4$

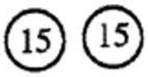
f. Count how many subsets of 4 you made, how many bags of lollipops there are.

g. 18

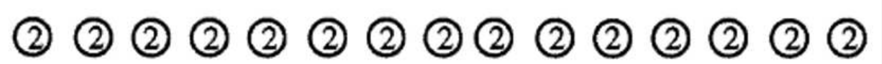
Exercise 7

7.

a.



b.



c. Partitioning into subsets is easier, because you only need to make 2 equal subsets as opposed to subtracting 2 at a time.

Exercise 11

11.

a. the number of subsets
b. amount in each subset

Exercise 13

13. $56_{\text{seven}} r.22_{\text{seven}}$

Exercise 15

15. $285_{\text{eleven}} r.167_{\text{eleven}}$

Exercise 17

17. $2111_{\text{three}} r.210_{\text{three}}$

Exercise 18

18.

a. Not closed; $-1 \div -1 = 1$
c. Not closed; $1 \div 0$ is undefined

Exercise 19

19. No; $10 \div 2 = 5$, but $2 \div 10 = 1/2$. Therefore, $10 \div 2 \not\leq 2 \div 10$. Use a different counterexample in your answers.

Module 8 Number Theory

Exercise Set 1 Solutions

Exercise 1

1.

a. 7	b. 3
c. 8	d. 1
e. 2	f. 8
g. 2	h. 8
i. 0	j. 2
k. 4	l. 0
m. 7	n. 8

Exercise 2

2. same answers as for exercise 1

Exercise 3

3. Yes.

? Exercise 4

4.

a. Ck: $6 + 7 = 13 \rightarrow 4$
$4 \not\leq 2$; mistake

b. Ck: $5 + 1 = 6$
$6 = 6$; correct

c. Ck: $2 + 3 = 5$
$5 \not\leq 4$; mistake

d. Ck: $8 + 2 + 5 = 15 \rightarrow 6$
$6 = 6$; correct

e. Ck: $5 + 2 + 2 = 9 \rightarrow 0$
$0 \not\leq 3$; mistake

f. Ck: $1 + 7 + 0 = 8$
$8 = 8$; correct

? Exercise 5

5.

a. $2 \cdots 8 = 16 \rightarrow 7$
$7 = 7$; correct

b. $2 \cdots 0 = 0$
$0 \not\leq 8$; mistake

c. $5 \cdots 1 = 5$
$5 = 5$; correct

d. $2 \cdot \cdots 7 = 14 \rightarrow 5$
$5 \not\leq 4$; mistake

e. $0 \cdots 1 = 0$
$0 = 0$; correct

f. $3 \cdot 7 = 21 \rightarrow 3$
$3 \not\leq 4$; mistake

g. $2 \cdots 7 = 14 \rightarrow 5$
$5 = 5$; correct

h. $5 \cdots 2 = 10 \rightarrow 1$
$1 \not\leq 0$; mistake

i. $2 \cdots 6 = 12 \rightarrow 3$
$3 = 3$; correct

? Exercise 6

6. Do the problems, then check your answers using digital roots.

? Exercise 7

7.

a. $0 + 5 = 5$
$5 = 5$; correct

b. $8 + 0 = 8$
$8 = 8$; correct

c. $5 + 3 = 8$
$8 \not\leq 6$; mistake

d. 8 + 6 = 14 → 5
$5 \nleq 4$; mistake

e. 5 + 4 = 9 → 0
0 = 0; correct

f. 1 + 4 = 5
5 = 5; correct

? Exercise 8

8. Do the problems, then check your answers using digital roots.

? Exercise 9

9.

a. $1 \cdots 4 = 4; 4 + 7 = 11 \to 2$
2 = 2; correct

b. $6 \cdots 0 = 0; 0 + 0 = 0$
0 = 0; correct

c. $8 \cdots 8 = 64 \to 10 \to 1; 1 + 1 = 2$
$2 \nleq 4$; mistake

d. $6 \cdots 3 = 18 \to 0; 0 + 5 = 5$
5 = 5; correct

? Exercise 10

10. Do the problem, then check your answer using digital roots.

? Exercise 11

11.

a. This is a division problem. The answer is 5.

b. This is a false statement since 35 is not a factor of 7.

c. This is a true statement since 7 is a factor of 35; ($7 \cdots 5 = 35$)

d. This is a division problem. The answer is 5 r. 5.

e. This is a false statement since 56 is not a factor of 8.

f. This is a false statement since 7 is not a factor of 40.

g. This is a true statement since 12 is a factor of 60; ($12 \cdots 5 = 60$)

h. This is a division problem. The answer is 2 r. 20.

i. This is a division problem. The answer is 14.

j. This is a false statement since 42 is not a factor of 3.

k. This is a true statement since 6 is a factor of 42; ($6 \cdots 7 = 42$)

l. This is a division problem. The answer is 8.

m. This is a division problem. The answer is 50.

n. This is a true statement since 4 is a factor of 100; ($4 \cdots 25 = 100$)

o. This is a false statement since 4 is not a factor of 90.

p. This is a false statement since 25 is not a factor of 5.

Exercise 13

13. The proof is written exactly like Example 1 shown above this exercise if a is replaced by x, b is replaced by y, and c is replaced by z.

Exercise 18

18. If a is a factor of b, then am = b for some whole number, m. If a is a factor of c, then an = c for some whole number n. Using substitution, bc = (am)(an) = a(amn), which shows a is a factor of bc.

Exercise 19

19. The proof is written exactly like the example shown above this exercise if a is replaced by c, b is replaced by a, and c is replaced by b.

Exercise 20

20.

a. 2|9,712 since the last digit is even.
b. 5,643 is odd, so it is not divisible by 2.
c. 5,690 is not divisible by 4 since 4 is not a factor of 90.
d. 63,868 is divisible by 4 since 4 is a factor of 68.
e. 854,100 is divisible by 4 since 4 is a factor of 0.
f. 8 is a factor of 12,345,248 since 8 is a factor of 248.
g. 54,094,422 is not divisible by 8 since 8 is not a factor of 422.

Exercise 21

21.

a. 5|9,750 since the last is 0.
b. 5|5,645 since the last is 5.
c. 5,696 is not divisible by 5 since the last digit is not 0 or 5.
d. 10|63,860 since the last digit is 0.
e. 854,105 is not divisible by 10 since the last digit is not 0.

Exercise 22

22.

a. 3|9,750 since 3|3, where 3 is the digital root of 9,750.
b. 3|5,645 is false since 3 is not a factor of 2, which is the d.r. of 5,645.
c. 3|5,696 is false since 3 is not a factor of 8, the d.r. of 5,696.
d. 3|63,860 is false since 3 is not a factor of 5, the d.r. of 63,860.
e. 3|854,115 since 3|6, where 6 is the d.r. of 854,115.

Exercise 23

23.

a. 9|9,753 is false since the digital root of 9,753 is 6, not 0.
b. 9|5,646 is false since the d.r. of 5,646 is 3, not 0.
c. 9|5,697 since the digital root of 5,697 is 0.
d. 9|63,576 since the digital root of 63,576 is 0.

e. 9|854,103 is false since the d.r. of 854,103 is 3, not 0.

? Exercise 24

24.

a. 6|9,753 is false since 9,753 is not even, therefore not divisible by 2.
b. 6|5,645 is false since 5,645 is not even, therefore not divisible by 2.
c. 6|5,696 is false since 3 is not a factor of 8, which is the d.r. of 5,696.
d. 6|63,876 since it is even and is also divisible by 3, since the d.r. is 3.
e. 6|854,103 is false since 854,103 is not even, therefore not divisible by 2.

? Exercise 25

25.

a. 15|9,753 is false since it is not divisible by 5.
b. 15|6,645 since both 3 and 5 are factors.
c. 15|1,690 is false since 3 is not a factor.
d. 15|63,872 is false since 5 is not a factor.
e. 15|654,105 since it is divisible by both 3 and 5.

? Exercise 26

26. Include justification

a. true
b. false
c. true
d. true

? Exercise 27

27. Include justification

a. true
b. false
c. false
d. true
e. false

Exercise Set 2 Solutions

? Exercise 1

1. c. 1, 2, 3, 4, 6, 12

? Exercise 2

2.

a. 1, 2
b. 1, 3

c. 1, 2, 4

d. 1, 5

e. 1, 2, 3, 6

g. 1, 2, 4, 8

h. 1, 3, 9

i. 1, 2, 5, 10

j. 1, 11

k. 1, 13

l. 1, 2, 7, 14

m. 1, 3, 5, 15

n. 1, 2, 4, 8, 16

Exercise 3

3.

a. 2, 3, 5, 7, 11, 13

b. The only factors are 1 and the number itself.

c. 4, 9, 16

d. They are perfect squares.

Exercise 4

4.

a. $45 = 3 \cdots 3 \cdots 5$

b. $65 = 5 \cdots 13$

c. $200 = 2 \cdots 2 \cdots 2 \cdots 5 \cdots 5$

d. $91 = 7 \cdots 13$

e. $76 = 2 \cdots 2 \cdots 19$

f. $350 = 2 \cdots 5 \cdots 5 \cdots 7$

g. $189 = 3 \cdots 3 \cdots 3 \cdots 7$

h. $74 = 2 \cdots 37$

i. $512 = 2 \cdots 2 \cdots 2 \cdots 2 \cdots 2 \cdots 2 \cdots 2 \cdots 2 \cdots 2$

j. $147 = 3 \cdots 7 \cdots 7$

Exercise 5

5. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Exercise 6

6. 26, 27, 28, 30, 32, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48

Exercise 7

7. 50, 51, 52, 54, 55, 56, 57, 58, 60, 62, 63, 64, 65, 66, 68, 69, 70, 72, 74, 75, 76, 77, 78, 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 102, 104, 105, 106, 108, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120

? Exercise 8

8. $11 \cdots 47$

? Exercise 9

9. 22.7

? Exercise 10

10.

a. 11; 149
b. 13; 3 · 7 · 13
c. 19; 3 · 127
d. 19; 19 · 23
e. 19; 509
f. 23; 613
g. 23; 787

? Exercise 11

11.

a. $2 \cdots 2 \cdots 2 \cdots 5 \cdots 7$
b. 281
c. $2 \cdots 3 \cdots 47$
d. 283
e. $2 \cdots 2 \cdots 71$
f. $5 \cdots 3 \cdots 19$
g. $2 \cdots 11 \cdots 13$
h. $7 \cdots 41$
i. $25 \cdots 3 \cdots 3$
j. $17 \cdots 17$
k. $2 \cdots 5 \cdots 29$
l. $3 \cdots 97$
m. $2 \cdots 2 \cdots 73$
n. 293
o. $2 \cdots 3 \cdots 7 \cdots 7$
p. $5 \cdots 59$

? Exercise 12

12. 281 and 283

? Exercise 13

13. No; at least one will be a multiple of 3.

Exercise 14

14.

a. 1, 2, 3, 6, 7, 14, 42
b. 1, 2, 5, 7, 10, 14, 35, 70
c. 1, 2, 7, 14
d. 14
e. 14

Exercise 15

15.

a. 1, 2, 4, 23, 46, 92
b. 1, 5, 23, 115
c. 1, 23
d. 23
e. 23

Exercise 16

16.

a. 1, 2, 3, 4, 6, 8, 12, 16, 24, 48
b. 1, 2, 3, 6, 9, 18, 27, 54
c. 1, 3, 7, 9, 21, 63
d. 1, 3
e. 3
f. 3

Exercise 17

17.

a. 14	b. 23	c. 3
d. 34	e. 25	f. 42

Exercise 18

18. a. $24 \cdots 3 \cdots 132$

Exercise 19

19.

a. $2^4 \cdots 3^2 \cdots 7^6 \cdots 13^2$
b. $3^4 \cdots 5^2$
c. $a^4 \cdots c^3 \cdots d^3 \cdots e^2$
d. $a^2 \cdots d^3$

? Exercise 22

22. a. 2, 7

? Exercise 23

23. 1

? Exercise 24

24.

a. m	b. 2m	c. m
d. 1	e. m	

? Exercise 25

25. a - g: 1, 2, 3, 6

h. same factors

? Exercise 26

26. a - f: 6

? Exercise 27

27.

a. 13	b. 18	c. 36

? Exercise 26

28.

a. 38 r. 186
b. 9 r. 33
c. 7 r. 11
d. 8 r. 25

? Exercise 29

29.

a. 13	b. 18	c. 36

? Exercise 30

30. d. 22

? Exercise 31

31. d. 31

Exercise 32

32. d. 1

Exercise 34

34. 14|n if 2|n and 7|n.

Exercise 35

35.

a. True since 2|742 and 7|742
b. False –7 is not a factor of 968
c. False –2 is not a factor of 483

Exercise 36

36. Think about why YOU think it doesn't work. This is a question only you can answer.

Exercise 37

37. Think about why YOU think it works. This is a question only you can answer.

Exercise 38

38.

a. 4|n and 3|n
b. 9|n and 2|n

Exercise 39

39.

a. 35|n if 5|n and 7|n
b. 28|n if 4|n and 7|n
c. 75|n if 3|n and 25|n
d. 56|n if 7|n and 8|n
e. c|n if 4|n, 27|n, 25|n and 11|n
f. d|n if 16|n, 3|n, 5|n and 11|n

Exercise 40

40.

a - e. 1
f. No; 1 is the smallest factor of every number.

Exercise Set 3 Solutions

Exercise 1

1.

a. 8, 16, 24, 32, 40, 48, 56, 64, 72, 80

b. 12, 24, 36, 48, 60, 72, 84, 96, 108, 120
c. 24, 48, 72
d. 24
e. no

Exercise 2

2.

a. 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60
b. 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 66, 72, 78, 84, 90
c. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150
d. 60
e. 60

Exercise 5

5.

a. X, Y
b. Y, Z
c. X, Y

Exercise 7

7. $2^2 \cdots 3^5 \cdots 5 \cdots 7^3 \cdots 11^2 \cdots 1^3$

Exercise 8

8. $2^2 \cdots 3^4 \cdots 7^3 \cdots 11^2 \cdots 13^2 \cdots 1^9$

Exercise 9

9. $2^4 \cdots 3^6 \cdots 5^4 \cdots 7^6 \cdots 11^2 \cdots 1^9 \cdots 23^2$

Exercise 10

10.

a. GCF: $2^2 \cdots 3^2 \cdots 1^3$
LCM: $2^2 \cdots 3^5 \cdots 5 \cdots 7^3 \cdots 1^3$

b. GCF: 2
LCM: $2^2 \cdots 3^4 \cdots 7^3 \cdots 11^2 \cdots 13^2 \cdots 1^9$

c. GCF: $c^3 \cdots d$
LCM: $a^5 \cdots b^4 \cdots c^5 \cdots d^3 \cdots e^2 9$.

d. GCF: $c \cdots d$
LCM: $a^6 \cdots b^4 \cdots c^4 \cdots d^3 \cdots e^7$

Exercise 11

11.

a. 120
b. 1400

c. 1274

d. 840

e. 1870

? Exercise 12

12.

a. $3 \cdot 2 \cdot 2 \cdot 7 \cdot 3 \cdot 5 = 1260$

b. $2 \cdot 3 \cdot 3 \cdot 11 \cdot 7 \cdot 5 \cdot 2 \cdot 2 \cdot 2 \cdot 17 = 942{,}480$

c. $5 \cdot 2 \cdot 3 \cdot 7 \cdot 2 \cdot 2 \cdot 17 \cdot 3 \cdot 5 = 214{,}200$

d. $3 \cdot 5 \cdot 2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 = 6{,}300$

? Exercise 13

13. 14,112

? Exercise 14

14. 2,835

? Exercise 15

15. 3,705

? Exercise 16

16. 2 and 20; 4 and 10

? Exercise 17

17.

a. 21	b. 18,018

? Exercise 18

18.

a. 73	b. 74,095

? Exercise 19

19.

a. 37	b. 395,641

? Exercise 20

20.

a. 6	b. 58,344,300

Exercise 25

25.

a. 2(4n + 10) is an even number.
b. 2(5k + 4) + 1 is an odd number.
c. 5x + 2 can be even or odd

Exercise 26

26. Let 2n = one even number, and let 2m = another even number. The sum is: 2n+2m = 2(n+m), which is in the form of an even number. Therefore, the sum of 2 even numbers is even.

Exercise 27

27. Let 2n + 1 = one odd number, and let 2m + 1 = another odd number. The sum is: 2n + 1 + 2m + 1 = 2n + 2m + 2 = 2(n + m + 1), which is in the form of an even number. Therefore, the sum of 2 odd numbers is even

Exercise 28

28. Let 2n = one even number, and let 2m + 1 = one odd number. The sum is: 2n + 2m + 1 = 2n + 2m + 1 = 2(n + m) + 1, which is in the form of an odd number. Therefore, the sum of an even number and an odd number is odd.

Exercise 29

29. Let 2n = one even number, and let 2m = another even number. The product is: 2n · 2m = 4mn = 2(2mn), which is in the form of an even number. Therefore, the product of two even numbers is even.

Exercise 30

30. Let 2n + 1 = one odd number, and let 2m + 1 = one odd number. The product is: (2n + 1)(2m +) = 2nm + 2m + 2n + 1 = 2(nm + m + n) + 1, which is in the form of an odd number. Therefore, the product of two odd numbers is odd.

Exercise 31

31. Let 2n = one even number, and let 2m + 1 = one odd number. The product is: 2n(2m + 1) = 4nm + 2n = 2(2nm + n), which is in the form of an even number. Therefore, the product of an even number and an odd number is even.

Exercise 32

32. 3240

Exercise 33

33. 3240

Exercise 34

34. 31,375

Exercise 35

35.

a. 3,77
b.148,181
c.195,650

Exercise 36

36.

a. 3,775
b. 148,181
c. 195,650
d.[n –(k –1)]$(n+k)\div 2$ or (n –k + 1)$(n+k)\div 2$

Exercise 37

37. multiples of 7

Exercise 38

38. 1 + 2 + 3 + . . . + 99 + 100

Exercise 39

39.

a. 5,050
b. 5,050
c. 35,350

Exercise 40

40.

a. 63,000
b. 41,151
c. 226,500

Exercise 41

41. multiples of 4

Exercise 42

42. 28 + 29 + 30 + . . . + 131 + 132

Exercise 43

43. 33,600

Exercise 44

44.

a. 54,450
b. 96,019

? Exercise 45

45. 6

? Exercise 46

46. 4, 8, 14

? Exercise 47

47. 4, 6

? Exercise 48

48. 4

? Exercise 49

49. 8

? Exercise 50

50. 3, 7

? Exercise 51

51. 7

? Exercise 52

52. 68, 178, 288, 398, 508, 618, 728, 838, 948

? Exercise 53

53. 5

? Exercise 54

54. 5

? Exercise 55

55. 5 or 26

? Exercise 56

56. mult. of 7

? Exercise 57

57. 9 or 23

? Exercise 58

58. 3, 5, or 9

? Exercise 59

59. 9

Exercise 60

60. Yes; explanation not provided here

Homework Solutions

Exercise 1

1. GCF:14; LCM: 6,300

Exercise 3

3. a. GCF: 47; LCM: 49,350

Exercise 4

4. a. Since the number is even, we only need to make the resulting number divisible by 3. The digital root so far is 8. So, decide which single digits could be added to 8 to get a number that is still divisible by 3. Adding 1, 4 or 7 to 8 will work. So the possibilities are 4, 7 or 8.

Exercise 5

5. The examples and counterexamples are not provided by these solutions. Make sure you provide them!

a. false
b. true

Exercise 6

6.

a. $7 \cdots 53$
d. $7 \cdots 41$

Exercise 7

7.

a. m
b. m
c. m
d. 1

Exercise 9

9.

a. 49,770
b. 35,800

Exercise 10

10.

a. 39
b. 80

Exercise 11

11.

a. 1, 2, 3; 6; perfect
b. 1; 1; deficient
c. 1, 2, 4; 7; deficient

Exercise 13

13. Examples and counterexamples are not provided here. Make sure you provide them.

a. False
b. True
c. False
d. True

Exercise 15

15. a. 84

Module 9 Rational Numbers

Warm Up Solutions

Exercise 1

1. R

Exercise 2

2. Y

Exercise 3

3. P

Exercise 4

4. D

Exercise 5

5. O

Exercise 6

6. D

Exercise 7

7. H

Exercise 8

8. B

Exercise 9

9. D

Exercise 10

10. O

Exercise 11

11. Y

Exercise 12

12. B

Exercise 13

13. B

Exercise 14

14. L

Exercise Set 1 Solutions

Exercise 1

1. $\frac{1}{12} < \frac{1}{10} < \frac{1}{9} < \frac{1}{8} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2}$

Exercise 2

2.

a. $\frac{1}{90}$	b. $\frac{1}{32}$

Exercise 3

3. $\frac{5}{12} < \frac{5}{10} < \frac{5}{9} < \frac{5}{8} < \frac{5}{6}$

Exercise 4

4.

a. $\frac{15}{37}$	b. $\frac{89}{100}$

Exercise 5

5. $\frac{4}{12} < \frac{4}{11} < \frac{4}{10} < \frac{4}{9} < \frac{4}{8} < \frac{4}{7} < \frac{4}{6} < \frac{4}{5}$

Exercise 6

6. If you are comparing two fractions that have the same numerator, the one with the smaller denominator has the larger value.

Exercise 7, 8

7, 8. $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{7}{8} < \frac{8}{9} < \frac{9}{10} < \frac{11}{12}$

Exercise 9

9.

a. $\frac{94}{95}$

b. $\frac{89}{100}$

Exercise 10

10. $\frac{13}{14} < \frac{25}{26} < \frac{34}{35} < \frac{45}{46} < \frac{51}{52} < \frac{71}{72} < \frac{99}{100}$

Exercise 11

11. If you are comparing two fractions where in each fraction the numerator is one less than the denominator, the fraction with the bigger numerator and denominator has the larger value.

Exercise 12

12. Answers will vary. Try drawing a pie for each fraction, shading in 2 of 3 equal parts for 2/3 and 4 of 5 equal parts for 4/5. Which pie is more shaded?

Exercise 13

13. Answers will vary

Exercise 14

14. $\frac{2}{4}$ and $\frac{3}{6}$ and $\frac{4}{8}$ and $\frac{5}{10}$ and $\frac{6}{12}$

Exercise 15

15. $\frac{4}{6}$ and $\frac{6}{9}$ and $\frac{8}{12}$

Exercise 16-17

16 - 17: There are many possibilities using multiple strips.

Exercise 18

18. Multiply the numerator and denominator by the same number to get an equivalent fraction.

Exercise 19-27

19 - 27. Only the final answers are shown below. Make your own models for exercises 22 - 28 by following the examples making sure you show all the steps, define units, rows, columns, etc.

Exercise 19

19. $\frac{3}{4} < \frac{4}{5}$

Exercise 20

20. $\frac{13}{20}$

Exercise 21

21. $\frac{5}{24}$

Exercise 22

22.

a. $\frac{8}{15}$	b. $\frac{15}{24}$

Exercise 23

23.

a. $\frac{2}{3} \cdots \frac{5}{6} = \frac{10}{18}$	b. $\frac{1}{2} \cdots \frac{7}{8} = \frac{7}{16}$

Exercise 24

24.

a. $\frac{4}{5} \cdots \frac{3}{5} = \frac{12}{25}$	b. $\frac{1}{7} \cdots \frac{3}{4} = \frac{3}{28}$

Exercise 25

25. Answers may vary. One possibility: Let 1 = H; $\frac{3}{4}$ of H = B; $\frac{1}{3}$ of B = L; $\frac{L}{H} = \frac{3}{12}$

Exercise 26`

26.

a. H; B; L; L/H; 3/12

b. N; R; W; W/N; 1/8

c. N; R; L; L/N; 3/8

d. D; L; R; R/D; 2/6

Exercise 27-30

27 - 30. Make your own models. One possibility for each using the C-strip model is given, but there are other choices.

Exercise 27

27. Let 1 unit = O, then 1/10 = W and 1/5 = R. There are 2 W in R, so the answer is 2.

Exercise 28

28. Let 1 unit = B, then 1/9 = W and 1/3 = L. There are 3 W in L, so the answer is 3.

Exercise 29

29. Let 1 unit = H, then 1/6 = R and 2/3 = N. There are 4 R in N, so the answer is 4.

Exercise 30

30. Let 1 unit = P, then 1/4 = W and 3 = H. There are 12 W in H, so the answer is 12.

Exercise 31

31.

a. equivalent	b. equivalent	c. not equivalent

? Exercise 32

32. There are infinitely many possibilities - you can use the multiple strips to find some.

? Exercise 33

33. same as #31

? Exercise 34

34.

a. $\frac{14}{25}$	b. $\frac{13}{17}$	c. $\frac{18}{35}$	d. $\frac{15}{22}$

? Exercise 35

35. same as #31

? Exercise 36

36.

a. $\frac{4}{5} > \frac{5}{8}$	b. $\frac{12}{35} < \frac{11}{18}$	c. $\frac{13}{15} > \frac{14}{17}$

? Exercise 37

37. $89 \cdots 95 < 90 \cdots 94$ since 8455 < 8460. Cross products check.

? Exercise 38

38. It checks, show work

? Exercise 39

39.

a. -4/70	b. 71/72

? Exercise 40

40. Answers may vary. Some possibilities: 21/50, 22/50, 23/50, . . . , 29/50

? Exercise 41, 42

41,42. 1/2

? Exercise 43

43. Some possibilities: 51/150, 52/150, 53/150, . . . , 59/150

? Exercise 44

44. Some possibilities: 21/60, 22/60, 23/60, . . . , 29/60

? Exercise 45

45. 241/300, 242/300, 243/300, . . . , 249/300

? Exercise 46

46. 83/112

? Exercise 47

47. 75/112

? Exercise 48

48. Draw 12 circles. 11 of the circles add up to 66, so put 6 in each circle. Answer: 6 students.

? Exercise 49

49. Draw 7 circles, Since 3 circles add up to 36, put 12 in each circle. The other 4 circles represent students not buying lunch. Answer: 48 students

? Exercise 50

50. Draw 8 circles. Since all 8 circles add up to 120, put 15 in each circle. 5 of the circles represent the females, the other three represent the males. Answer: 75 females and 45 males.

Exercise Set 2 Solutions

? Exercise 1

1.

a. four tenths; $\frac{4}{10} = \frac{2}{5}$

b. twenty-six hundredths; $\frac{26}{100} = \frac{13}{50}$

c. three and eight hundredths; $3\frac{8}{100} = 3\frac{2}{25}$; $\frac{388}{100} = \frac{77}{25}$

d. nine and eighty-five hundredths; $9\frac{85}{100} = 9\frac{17}{20}$; $\frac{985}{100} = \frac{197}{20}$

e. seventeen and three hundred five thousandths; $17\frac{305}{1000} = 17\frac{61}{200}$; $\frac{17305}{1000} = \frac{3461}{200}$

? Exercise 2

2.

a. 0.14	b. 008
c. $4\frac{35}{100} = 4.35$	d. $563\frac{8}{10} = 563.8$
e. $3\frac{5}{100} = 3.05$	

? Exercise 3

3.

a. 0.028	b. 0.32
c. 0.075	d. 0.66

? Exercise 4

4.

a. 1.9; 1.90000

b. 4.0340

Exercise 5

5.

a. 3.5 > .9

b. 35.06 = 3.0600

c. 0.089 < 0.0908

Exercise 6

6.

a. 3.51 > 3.488

b. 35.061 < 35.35

c. 0.8933 < 0.0894

Exercise 7

7.

a. $2 \cdots 5$	b. $2 \cdots 5 \cdots 2 \cdots 5$
c. $2 \cdots 5 \cdots 2 \cdots 5 \cdots 2 \cdots 5$	d. $2 \cdots 5 \cdots 2 \cdots 5 \cdots 2 \cdots 5 \cdots 2 \cdots 5$
e. $2 \cdots 5 \cdots 2 \cdots 5 \cdots 2 \cdots 5 \cdots 2 \cdots 5 \cdots 2 \cdots 5$	

Exercise 8

8. 2 and 5

Exercise 9

9. 3

Exercise 10

10. 2

Exercise 11

11. Well, what do **you** think?

Exercise 12

12.

a. 0.0875	b. 0.875

Exercise 13

13. Write a response in your own words.

Exercise 14

14.

a. $\frac{3}{4} = \frac{3}{2 \cdots 2} \cdots \frac{5 \cdots 5}{5 \cdots 5} = \frac{75}{100} = 0.75$

b. $\frac{9}{20} = \frac{3 \cdots 3}{2 \cdots 2 \cdots 5} \cdots \frac{5}{5} = \frac{45}{100} = 0.45$

c. $\frac{9}{15} = \frac{3 \cdots 3}{3 \cdots 5} = \frac{3}{5} \cdots \frac{2}{2} = \frac{6}{10} = 0.6$

d. Extra close brace or missing open brace

e. $\frac{5}{14} = \frac{5}{2 \cdots 7}$ This can't be written as a terminating decimal because the reduced factor has a prime factor of 7 (which is other than a 2 or 5) in the denominator.

Exercise 15

15. Do three of your own.

Exercise 16

16.

a. 0, 1, 2, 3, 4, 5

b. 0, 1, 2, 3, 4, 5, 6

c. 0, 1, 2, 3, 4, 5, 6, 7, 8

d. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

e. 0, 1, 2

Exercise 17

17. Do three of your own.

Exercise 18

18. a, b, d

Exercise 19

19.

a. There are 33 (including 0 as a remainder). A maximum of 32 digits could be in a sequence without repeating.

b. If you do the long division, you get .939393...

c. 2

d. 13 and 31

Exercise 20

20.

a. $0.\bar{5}$ or 0.555...

b. $0.71\bar{4}285$ or 0.714285714285...

c. $0.1\bar{6}$ or 0.1666...

d. $0.\bar{6}$ or 0.666..

e. $0.\bar{6}3$ or 0.636363...

f. $0.41\bar{6}$ or 0.41666..

g. $0.5\bar{3}$ or 0.5333...

h. $0.3\bar{5}$ or 0.3555...

i. $0.0\bar{7}5$ or 0.0757575...

Exercise 21

21. x - 1

Exercise 22

22.

10x = 7.272727...;

100x = 72.727272...;

1000x = 727.272727...

Exercise 23

23.

a. 99x

b. 72

c. 99x = 72; x = $\frac{72}{99}$

d. x = $\frac{8}{11}$

Exercise 24

24.

a. $\frac{4}{10} = \frac{2}{5}$	b. $\bar{4} = \frac{4}{9}$
c. $\frac{6}{100} = \frac{3}{50}$	d. $0.\bar{0}6 = \frac{6}{99} = \frac{2}{33}$
e. $\frac{9}{10}$	f. $0\bar{.}9 = \frac{9}{9} = 1$ Surprise!
g. $\frac{45}{100} = \frac{9}{20}$	h. $0.\bar{4}5 = \frac{45}{99} = \frac{5}{11}$
i. $\frac{84}{1000} = \frac{21}{250}$	j. $0.0\bar{8}4 = \frac{84}{999} = \frac{28}{333}$

Exercise 25

25.

a. $\frac{2.8}{99} = \frac{28}{99 \cdots 10} = \frac{14}{495}$

b. $\frac{2.6}{9} = \frac{26}{9 \cdots 10} = \frac{13}{45}$

c. $\frac{0.06}{9} = \frac{6}{9 \cdots 100} = \frac{1}{150}$

d. $\frac{101}{999}$

e. $\frac{3.6}{9} = \frac{36}{9 \cdots 10} = \frac{2}{5}$

Exercise 26

26. Be creative!

Exercise 27

27. No, since 9 is a perfect square. So, $\sqrt{9} = 3$

Exercise 28-30

28-30. Be creative!

Exercise 31

31.

a. rational (repeating decimal).
b. irrational (it has a pattern, but does not repeat!)
c. rational
d. irrational (since 80 is not a perfect square)
e. rational (since 100 is a perfect square; so $\sqrt{100} = 10$
f. irrational

Homework Solutions

Exercise 1

1.

a. K	c. D	e. D	g. H

Exercise 2

2.

a. H	c. P	e. R

Exercise 3

3. a. L

Exercise 7

7. $\frac{1}{3} < \frac{3}{8}$

Exercise 8

8. $\frac{17}{24}$

Exercise 9

9. $\frac{7}{20}$

Exercise 10

10. a. $\frac{3}{20}$

Exercise 11

11. a. $\frac{5}{6} \cdots \frac{2}{3} = \frac{10}{18}$

Exercise 12

12. a. $\frac{2}{3} \cdots \frac{3}{8} = \frac{6}{24}$

Exercise 13

13. a. B; D; R; R/B; 2/9

Exercise 14

14. a. 15

Exercise 16

16. a. $\frac{18}{35}$

Exercise 18

18. Some possibilities: 31/80, 32/80, 33/80, . . . , 49/80

Exercise 20

20. a. Draw 10 circles. Since 7 of the circles add up to 21, put 3 in each circle. All 10 circles add up to 30, so there are 30 students.

Exercise 21

21.

a. seven tenths

b. three and twenty-eight hundredths

Exercise 22

22.

a. $\frac{3}{10} \cdots \frac{8}{10} = \frac{14}{100} = 0.14$

b. $\frac{122}{100} \cdots \frac{23}{10} = \frac{2086}{1000} = 2.806$

Exercise 23

23.

a. $\frac{11}{16} = \frac{11}{2 \cdots 2 \cdots 2 \cdots 2} \cdots \frac{5 \cdots 5 \cdots 5 \cdots 5}{5 \cdots 5 \cdots 5 \cdots 5} = \frac{6875}{10000} = 0.6875$

c. $\frac{1}{12} = \frac{1}{2 \cdots 2 \cdots 3}$

This can't be written as a terminating decimal because the reduced factor has a prime factor of 3 (which is other than a 2 or 5. in the denominator).

Exercise 24

24.

a. $\frac{7}{9}$

c. $\frac{235}{999}$

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